Fluid models for complex systems

P. Degond

Toulouse Institute of Mathematics (MIP group)
CNRS and Université Paul Sabatier,
118 route de Narbonne, 31062 Toulouse cedex, France

degond@mip.ups-tlse.fr  (see http://mip.ups-tlse.fr)

Joint work with:
R. Bon, J. Gautrais, M-H. Pillot, G. Théraulaz (Cognition lab.)
S. Motsch, L. Navoret, D. Sanchez, A. Frouvelle (Toulouse, Math)
Summary

1. Introduction

2. From particle to mean-field model

3. From mean-field to 'hydrodynamics'

4. Properties of the hydro model

5. Conclusion
1. Introduction
Complex system

- System with interacting agents without leaders
  - Spontaneous emergence of spatio-temporal coordination
  - Morphogenesis
Complex system

- System with interacting agents without leaders
  - Spontaneous emergence of spatio-temporal coordination
  - Morphogenesis
Elementary interactions

- Difficult to access from experiments
  - complex (e.g. not a sum of pair interactions)
  - mostly unknown
Elementary interactions

- Difficult to access from experiments
  - complex (e.g. not a sum of pair interactions)
  - mostly unknown

- Classical micro-macro approach is bottom-up
  - From the knowledge of elementary interactions
  - build macro models for large systems
Elementary interactions

- Difficult to access from experiments
  - complex (e.g. not a sum of pair interactions)
  - mostly unknown

- Classical micro-macro approach is bottom-up
  - From the knowledge of elementary interactions
  - build macro models for large systems

- Complex systems require top-down approach
  - From macro models build macro observables
  - and test hypotheses about micro interactions
  - use model and data together to extract information
Importance of micro-macro passage

- Link micro interactions to macro model
  - in a (formally) rigorous way
Importance of micro-macro passage

- Link micro interactions to macro model
  - in a (formally) rigorous way

- Macro models are more efficient for large systems
  - particle models scale polynomially with $\# \text{ of particles}$
Importance of micro-macro passage

- Link micro interactions to macro model
  - in a (formally) rigorous way

- Macro models are more efficient for large systems
  - particle models scale polynomially with \# of particles

- Morphogenesis easier with macro models
  - Phase transitions can be encoded more easily
Importance of micro-macro passage

- Link micro interactions to macro model
  - in a (formally) rigorous way

- Macro models are more efficient for large systems
  - particle models scale polynomially with \# of particles

- Morphogenesis easier with macro models
  - Phase transitions can be encoded more easily

- This talk: micro-macro passage for two models
  - Vicsek (alignment interaction)
  - Persistent Turning Walker
2. From particles to mean-field model
Couzin-Vicsek model

- Alignement interaction (‘moving spins’)

- Discrete model

- $X^n_k$: position of $k$-th individual at time $t^n = n\Delta t$

- $\omega^n_k$: velocity with $|\omega^n_k| = 1$
### Couzin-Vicsek model

- **Alignement interaction (‘moving spins’)**
  - **Discrete model**
  - $X^n_k$: position of $k$-th individual at time $t^n = n \Delta t$
  - $\omega^n_k$: velocity with $|\omega^n_k| = 1$

- **During each $\Delta t$:**
  - Particle moves a distance $\omega^n_k \Delta t$
  - $\omega^n_k$ changed to $\omega^{n+1}_k$
    - = direction $\bar{\omega}^n_k$ of average neighbours’ velocity
    + noise
  - Noise accounts for inaccuracy of the perceptive system
Couzin-Vicsek algorithm

[<Vicsek et al, PRL 95>]:

\[ X_{k}^{n+1} = X_{k}^{n} + \omega_{k}^{n} \Delta t \]
\[ \omega_{k}^{n+1} = \bar{\omega}_{k}^{n} + \text{noise} \]
\[ \bar{\omega}_{k}^{n} = \frac{J_{k}^{n}}{|J_{k}^{n}|}, \quad J_{k}^{n} = \sum_{j, |X_{j}^{n} - X_{k}^{n}| \leq R} \omega_{j}^{n} \]

noise = uniform for angle in interval \([-\sigma, \sigma]\) in 2D
Phase transition

- Model shows 2 regimes [Vicsek et al, PRL 95]
  - Disorganized / Aligned
  - Phase transition to disorder
Two time scales are collapsed

- Discretization step $\Delta t$ and Mean interaction time $\tau$
Time scale separation

- Two time scales are collapsed
- Discretization step $\Delta t$ and Mean interaction time $\tau$

After separating theses two time scales:

\[
\frac{\omega_{k}^{n+1} - \omega_{k}^{n}}{\Delta t} = \frac{1}{\tau} (\text{Id} - \omega_{k}^{n+1/2} \otimes \omega_{k}^{n+1/2}) (\bar{\omega}_{k}^{n} - \omega_{k}^{n}) + \text{noise}
\]

\[
\omega_{k}^{n+1/2} = \frac{\omega_{k}^{n+1} + \omega_{k}^{n}}{|\omega_{k}^{n+1} + \omega_{k}^{n}|}
\]

\[
\bar{\omega}_{k}^{n} = \frac{J_{k}^{n}}{|J_{k}^{n}|}, \quad J_{k}^{n} = \sum_{j, |X_{j}^{n} - X_{k}^{n}| \leq R} \omega_{j}^{n}
\]
Letting $\Delta t \to 0$ gives

$$\dot{X}_k(t) = \omega_k(t)$$

$$d\omega_k(t) = (\text{Id} - \omega_k \otimes \omega_k)(\nu(\bar{\omega}_k - \omega_k)dt + \sqrt{2D}dB_t)$$

$$\bar{\omega}_k = \frac{J_k}{|J_k|}, \quad J_k = \sum_{j, |X_j - X_k| \leq R} \omega_j$$

$$\nu = \tau^{-1} = \text{interaction frequency}$$
Letting $\Delta t \to 0$ gives

$$
\dot{X}_k(t) = \omega_k(t)
$$

$$
d\omega_k(t) = (\text{Id} - \omega_k \otimes \omega_k)(\nu \bar{\omega}_k dt + \sqrt{2}DdB_t)
$$

$$
\bar{\omega}_k = \frac{J_k}{|J_k|}, \quad J_k = \sum_{j,|X_j-X_k|\leq R} \omega_j
$$
Mean-field model

\[ f(x, \omega, t) \, dx \, d\omega = \text{probability of finding a particle in } dx \, d\omega \text{ at time } t \]

\[ \Rightarrow \text{satisfies a Fokker-Planck equation} \]
Mean-field model

\[ f(x, \omega, t) \, dx \, d\omega = \text{probability of finding a particle in } dx \, d\omega \text{ at time } t \]

\[ \text{satisfies a Fokker-Planck equation} \]

\[
\partial_t f + \omega \cdot \nabla_x f + \nabla_\omega \cdot (F f) = D \Delta_\omega f
\]

\[ F = \nu (I_d - \omega \otimes \omega) \bar{\omega} \]

\[ \bar{\omega} = \frac{J}{|J|}, \quad J = \int_{|y-x| \leq R, |\nu| = 1} \nu f(y, \nu, t) \, dy \, d\nu \]
Mean-field model

\[ f(x, \omega, t) \, dx \, d\omega = \text{probability of finding a particle in } \, dx \, d\omega \, \text{at time } t \]

satisfies a Fokker-Planck equation

\[
\begin{align*}
\partial_t f + \omega \cdot \nabla_x f + \nabla \omega \cdot (Ff) &= D \Delta \omega f \\
F &= \nu (\text{Id} - \omega \otimes \omega) \bar{\omega} \\
\bar{\omega} &= \frac{J}{|J|}, \quad J = \int_{|y-x| \leq R, |\nu|=1} \nu f(y, \nu, t) \, dy \, d\nu
\end{align*}
\]

Choice of time scale: \( \nu = 1 \)
Rescaled mean-field model

 Passage to macroscopic time and space scales

\[ \tilde{x} = \varepsilon x, \quad \tilde{t} = \varepsilon t \quad \text{with} \quad \varepsilon \ll 1 \]

Interaction radius is microscopic:

\[ \tilde{R} = \varepsilon R \]
Passage to macroscopic time and space scales

\[ \tilde{x} = \varepsilon x, \quad \tilde{t} = \varepsilon t \quad \text{with} \quad \varepsilon \ll 1 \]

Interaction radius is microscopic: \( \tilde{R} = \varepsilon R \)

\[
\begin{align*}
\varepsilon \left( \partial_t f^\varepsilon + \omega \cdot \nabla_x f^\varepsilon \right) + \nabla \omega \cdot (F^\varepsilon f^\varepsilon) &= D \Delta \omega f^\varepsilon \\
F^\varepsilon &= (\mathbf{1} - \omega \otimes \omega) \bar{\omega}^\varepsilon \\
\bar{\omega}^\varepsilon &= \frac{J^\varepsilon}{|J^\varepsilon|}, \quad J^\varepsilon = \int_{|y-x| \leq \varepsilon R, |v|=1} v f^\varepsilon (y, v, t) \, dy \, dv
\end{align*}
\]
Expansion gives

\[ \bar{\omega}^\varepsilon = \Omega^\varepsilon + O(\varepsilon^2) \]

\[ \Omega^\varepsilon = \frac{j^\varepsilon}{|j^\varepsilon|}, \quad j^\varepsilon = \int_{|\nu|=1} \nu f^\varepsilon(x, \nu, t) \, d\nu \]

\[ \Omega^\varepsilon \] is the direction of the local flux
Equivalent mean-field model

Expansion gives

\[ \bar{\omega}^\varepsilon = \Omega^\varepsilon + O(\varepsilon^2) \]
\[ \Omega^\varepsilon = \frac{j^\varepsilon}{|j^\varepsilon|}, \quad j^\varepsilon = \int_{|v|=1} v f^\varepsilon(x, v, t) \, dv \]

\[ \Omega^\varepsilon \] is the direction of the local flux

Rescaled model equivalent (up to HOT) to

\[ \varepsilon (\partial_t f^\varepsilon + \omega \cdot \nabla_x f^\varepsilon) + \nabla \omega \cdot (F_0^\varepsilon f^\varepsilon) = D \Delta \omega f^\varepsilon \]
\[ F_0^\varepsilon = (I_d - \omega \otimes \omega) \Omega^\varepsilon \]
3. From mean-field model to 'hydrodynamics'
Collision operator

Model can be written

$$\partial_t f^\varepsilon + \omega \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} Q(f^\varepsilon)$$

with 'collision operator'

$$Q(f) = -\nabla_\omega \cdot (F_f f) + D \Delta_\omega f$$

$$F_f = (\text{Id} - \omega \otimes \omega) \Omega_f$$

$$\Omega_f = \frac{j_f}{|j_f|}, \quad j_f = \int_{|v|=1} v f(x, v, t) \, dv$$
Collision operator

Model can be written

$$\partial_t f^\varepsilon + \omega \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} Q(f^\varepsilon)$$

with 'collision operator’

$$Q(f) = -\nabla_\omega \cdot (F_f f) + D \Delta_\omega f$$

$$F_f = (\text{Id} - \omega \otimes \omega) \Omega_f$$

$$\Omega_f = \frac{j_f}{|j_f|}, \quad j_f = \int_{|\nu|=1} \nu f(x, \nu, t) \, d\nu$$

Problem: find the formal limit $\varepsilon \to 0$ of this model
1st step: find the equilibria

At leading order, dynamics takes place on the manifold of equilibria

\[ \mathcal{E} = \{ f \mid Q(f) = 0 \} \]
1st step: find the equilibria

- At leading order, dynamics takes place on the manifold of equilibria \( \mathcal{E} = \{ f \mid Q(f) = 0 \} \).

- Rewrite

\[
Q(f) = \nabla_\omega \cdot \left[ -F f f + D \nabla_\omega f \right]
\]

- Introduce the solution of \([\ldots] = 0\).

- For any arbitrary \( \Omega \), \( \exists \) a unique normalized solution \( f = M_\Omega \) s.t. \( \Omega_f = \Omega \).
1st step: find the equilibria

At leading order, dynamics takes place on the manifold of equilibria \( \mathcal{E} = \{ f \mid Q(f) = 0 \} \).

Rewrite

\[
Q(f) = \nabla_\omega \cdot [ -F f f + D \nabla_\omega f ]
\]

Introduce the solution of \([\ldots] = 0\).

For any arbitrary \( \Omega \), \( \exists \) a unique normalized solution \( f = M_\Omega \) s.t. \( \Omega_f = \Omega \).

\[
M_\Omega(\omega) = C_D \exp \left( \frac{\omega \cdot \Omega}{D} \right), \quad \int M_\Omega(\omega) \, d\omega = 1
\]
Equilibria

$Q(f)$ can be written

$$Q(f) = D \nabla_\omega \cdot \left[ M_{\Omega_f} \nabla_\omega \left( \frac{f}{M_{\Omega_f}} \right) \right]$$
Equilibria

$Q(f)$ can be written

$$Q(f) = D \nabla \omega \cdot \left[ M_{\Omega_f} \nabla \omega \left( \frac{f}{M_{\Omega_f}} \right) \right]$$

Entropy inequality

$$H(f) = \int Q(f) \frac{f}{M_{\Omega_f}} \, d\omega = -D \int M_{\Omega_f} \left| \nabla \omega \left( \frac{f}{M_{\Omega_f}} \right) \right|^2 \leq 0$$
Equilibria

- $Q(f)$ can be written

$$Q(f) = D \nabla_\omega \cdot \left[ M_{\Omega_f} \nabla_\omega \left( \frac{f}{M_{\Omega_f}} \right) \right]$$

- Entropy inequality

$$H(f) = \int Q(f) \frac{f}{M_{\Omega_f}} \, d\omega = -D \int M_{\Omega_f} \left| \nabla_\omega \left( \frac{f}{M_{\Omega_f}} \right) \right|^2 \leq 0$$

- $\mathcal{E} = \{ \rho M_{\Omega}(\omega) \text{ for arbitrary } \rho \in \mathbb{R}_+ \text{ and } \Omega \in S^2 \}$

  (or $S^1$ in dim 2)

- $\dim \mathcal{E} = 3$  \( (= 2 \text{ in dim 2}) \)
Particular cases:

- \( D = 0 \) (no noise): all particles concentrate on velocity
  \[ \omega = \Omega : \quad M_\Omega(\omega) = \delta(\omega, \Omega) \]

- \( D = \infty \) (large noise): velocity distribution is isotropic:
  \[ M_\Omega(\omega) = \frac{1}{4\pi} \quad (= \frac{1}{2\pi} \text{ in dim } = 2) \]
Particular cases:

- $D = 0$ (no noise): all particles concentrate on velocity
  \[ \omega = \Omega : \quad M_{\Omega}(\omega) = \delta(\omega, \Omega) \]

- $D = \infty$ (large noise): velocity distribution is isotropic:
  \[ M_{\Omega}(\omega) = \frac{1}{4\pi} \quad (= \frac{1}{2\pi} \text{ in dim } = 2) \]

When $\varepsilon \to 0$:

\[ f^\varepsilon(x, \omega, t) \to \rho(x, t) M_{\Omega(x,t)}(\omega) \]
Particular cases:

- $D = 0$ (no noise): all particles concentrate on velocity
  \[ \omega = \Omega : \quad M_\Omega(\omega) = \delta(\omega, \Omega) \]

- $D = \infty$ (large noise): velocity distribution is isotropic:
  \[ M_\Omega(\omega) = \frac{1}{4\pi} \quad (= \frac{1}{2\pi} \text{ in dim = 2}) \]

When $\varepsilon \to 0$:

\[
f_\varepsilon(x, \omega, t) \to \rho(x, t) M_{\Omega(x,t)}(\omega)
\]

Problem: find the dependence of $\rho$ and $\Omega(x, t)$ upon $(x, t)$
Collision invariant (conserved quantity)

Function $\psi(\omega)$ such that

$$\int Q(f) \psi \, d\omega = 0, \quad \forall f$$

Form a vector space $\mathcal{C}$
Collision invariant (conserved quantity)

- Function $\psi(\omega)$ such that
  \[ \int Q(f)\psi \, d\omega = 0, \quad \forall f \]

- Form a vector space $\mathcal{C}$

- Use:
  - Multiply eq. by $\psi$: $\varepsilon^{-1}$ term disappears
  - Find a conservation law
  - Problem fully determined if $\dim \mathcal{C} = \dim \mathcal{E}$
Lack of collision invariants

- Here $\dim \mathcal{C} = 1$ because $\mathcal{C} = \text{Span}\{1\}$
- $\dim \mathcal{E} = 3 > \dim \mathcal{C} = 1$
- Only conservation of mass

$$\partial_t \rho + \nabla_x \cdot (c_1 \rho \Omega) = 0, \quad c_1 = |j_{M\Omega}| < 1$$
Lack of collision invariants

Here \( \dim \mathcal{C} = 1 \) because \( \mathcal{C} = \text{Span}\{1\} \)

\[ \dim \mathcal{E} = 3 > \dim \mathcal{C} = 1 \]


Only conservation of mass

\[ \partial_t \rho + \nabla_x \cdot (c_1 \rho \Omega) = 0, \quad c_1 = |j_{M\Omega}| < 1 \]

Is the limit problem ill-posed?

\[ \text{Answer} = \text{no} \]

\[ \text{find eq. for } \Omega \text{ by weekening the concept of collision invariant} \]
Given $\Omega$, find $\psi_\Omega$ a GCI, such that

$$\int Q(f) \psi_\Omega \, d\omega = 0, \quad \forall f \text{ such that } \Omega_f = \Omega$$
Given $\Omega$, find $\psi_\Omega$ a GCI, such that

$$\int Q(f) \psi_\Omega \, d\omega = 0, \quad \forall f \text{ such that } \Omega_f = \Omega$$

Thm: given $\Omega$, the GCI form a 3-dim vector space spanned by $1$ and $\vec{\psi}_\Omega(\omega)$
Given $\Omega$, find $\psi_\Omega$ a GCI, such that

$$\int Q(f)\psi_\Omega \, d\omega = 0, \quad \forall f \text{ such that } \Omega_f = \Omega$$

Thm: given $\Omega$, the GCI form a 3-dim vector space spanned by $1$ and $\vec{\psi}_\Omega(\omega)$

$$\vec{\psi}_\Omega(\omega) = \frac{\Omega \times \omega}{|\Omega \times \omega|} g(\Omega \cdot \omega) \quad \text{with } g(\mu) \text{ sol. of an elliptic eq.:}$$

$$-(1 - \mu^2) \partial_\mu (e^{\mu/d}(1 - \mu^2) \partial_\mu g) + e^{\mu/d} g = -(1 - \mu^2)^{3/2} e^{\mu/d}$$
Multiply eq. by $\vec{\psi}_{\Omega f\varepsilon}$

$O(\varepsilon^{-1})$ terms disappear

Let $\varepsilon \to 0$: $\vec{\psi}_{\Omega f\varepsilon} \to \vec{\psi}_{\Omega}$

Get eq.

$$\int (\partial_t (\rho M_{\Omega}) + \omega \cdot \nabla_x (\rho M_{\Omega})) \vec{\psi}_{\Omega} \ d\omega = 0$$
Use of generalized collision invariant

- Multiply eq. by $\vec{\psi}_{\Omega_f \varepsilon}$
  - $O(\varepsilon^{-1})$ terms disappear
- Let $\varepsilon \to 0$: $\vec{\psi}_{\Omega_f \varepsilon} \to \vec{\psi}_{\Omega}$
- Get eq.

$$\int (\partial_t (\rho M_\Omega) + \omega \cdot \nabla_x (\rho M_\Omega)) \vec{\psi}_\Omega d\omega = 0$$

- Not a conservation equation because of dependence of $\vec{\psi}_\Omega$ upon $\Omega$
Macro model of Couzin-Vicsek dynamics

\[ \rho(x, t) \text{ and } \Omega(x, t) \text{ evolve according to} \]

\[
\partial_t \rho + \nabla_x \cdot (c_1 \rho \Omega) = 0
\]

\[
\rho \left( \partial_t \Omega + c_2 (\Omega \cdot \nabla) \Omega \right) + D (\text{Id} - \Omega \otimes \Omega) \nabla_x \rho = 0
\]

\[ |\Omega| = 1 \]
Macro model of Couzin-Vicsek dynamics

\( \rho(x, t) \) and \( \Omega(x, t) \) evolve according to

\[
\begin{align*}
\partial_t \rho + \nabla_x \cdot (c_1 \rho \Omega) &= 0 \\
\rho \left( \partial_t \Omega + c_2 (\Omega \cdot \nabla) \Omega \right) + D (I - \Omega \otimes \Omega) \nabla_x \rho &= 0 \\
|\Omega| &= 1
\end{align*}
\]

\( c_2 \) defined as a particular moment of the GCI

\( c_2 < c_1 \)
4. Properties of the hydrodynamic model
By time rescaling

\[
\partial_t \rho + \nabla_x \cdot (\rho \Omega) = 0 \\
\rho \left( \partial_t \Omega + c(\Omega \cdot \nabla)\Omega \right) + d \left( \text{Id} - \Omega \otimes \Omega \right) \nabla_x \rho = 0 \\
|\Omega| = 1
\]

where \( c = c_2/c_1 < 1 \), \( d = D/c_1 \)
Hydrodynamic Vicsek model

- By time rescaling

\[
\frac{\partial_t \rho}{\rho} + \nabla_x \cdot (\rho \Omega) = 0 \\
\rho \left( \frac{\partial_t \Omega}{\rho} + c (\Omega \cdot \nabla) \Omega \right) + d \left( \text{Id} - \Omega \otimes \Omega \right) \nabla_x \rho = 0 \\
|\Omega| = 1
\]

where \( c = \frac{c_2}{c_1} < 1 \), \( d = \frac{D}{c_1} \)

- Hyperbolic model with constraint
  - Non-conservative terms arise from the constraint
Hydrodynamic Vicsek model

By time rescaling

\[
\partial_t \rho + \nabla_x \cdot (\rho \Omega) = 0 \\
\rho \left( \partial_t \Omega + c (\Omega \cdot \nabla) \Omega \right) + d (\text{Id} - \Omega \otimes \Omega) \nabla_x \rho = 0 \\
|\Omega| = 1
\]

where \( c = c_2/c_1 < 1 \), \( d = D/c_1 \)

Hyperbolic model with constraint

- Non-conservative terms arise from the constraint

Velocity waves are slower than density waves

- Similar situation to traffic
Function $g/D$ as a function of $\omega \cdot \Omega$ for small values of $D$
Function $g/D$ as a function of $\omega \cdot \Omega$ for small values of $D$

c and $d$ as a function of noise level $D$
\( c \) as a function of noise level \( D \) for various apertures of vision cone (2D case)

The more forward individuals look, the more backwards velocity waves propagate
Mills are stationary solutions

\[ \text{Mills: } \rho = \rho(r), \quad \Omega = x^\perp / r \]

are solutions of macro CVA model iff:

\[ \rho(r) = \rho_0 \left( \frac{r}{r_0} \right)^{\frac{c}{d}} \]
Mills are stationary solutions

- Mills: \( \rho = \rho(r), \Omega = x^\perp / r \)
  - are solutions of macro CVA model iff:

\[
\rho(r) = \rho_0 \left( \frac{r}{r_0} \right)^{\frac{c}{d}}
\]

- Shape depends on noise level
  - Small noise: \( \rho \) convex function of \( r \): sharp edged mills
  - Large noise: \( \rho \) concave function of \( r \): fuzzy edges
Density at $t = 5$
Flux orientation at $t = 5$
Order parameter (after Vicsek)

Coeff. $c_1$ measures the order / disorder

$$c_1 = |\dot{j}_{M\Omega}|$$

- $c_1 \sim 1$: particle directions are aligned
- $c_1 \sim 0$: particle directions are random
Order parameter (after Vicsek)

- Coeff. $c_1$ measures the order / disorder

$$c_1 = |\dot{M}_\Omega|$$

- $c_1 \sim 1$: particle directions are aligned
- $c_1 \sim 0$: particle directions are random

- In our model: order parameter remains uniform
- $c_1$ fixed by the value of $D$
Order parameter (after Vicsek)

- Coeff. $c_1$ measures the order / disorder
  \[ c_1 = |\dot{j}_{M\Omega}| \]
  - $c_1 \sim 1$: particle directions are aligned
  - $c_1 \sim 0$: particle directions are random

- In our model: order parameter remains uniform
  - $c_1$ fixed by the value of $D$

- ≠ simulations: higher order at higher density
  - Possible cure: make $D(\rho)$.
  - Justification: Fluctuations in the mean-field limit
Simulation of Vicsek particle model

Left: Point position of the particles
Right: Density (black) and order parameter (red) profiles transverse to a band

After Chate et al, arXiv:0712.206.2V1
Phase transition as noise level varies

Left: Order parameter as a function of noise level $D$ (after Vicsek)
Right: Order parameter as a function of noise level $D$ (after hydro model)
Phase transition as density varies

Order parameter as a function of density (after Vicsek)

In hydro model, order parameter does not depend on density
Phase transition

- Hydro model unable to reproduce phase transition of Vicsek particle model
  - Unique equilibria (no bi-stability)
  - Hyperbolicity (no instability)
  - Smooth variation of the coefficients wrt noise level $D$
Phase transition

- Hydro model unable to reproduce phase transition of Vicsek particle model
  - Unique equilibria (no bi-stability)
  - Hyperbolicity (no instability)
  - Smooth variation of the coefficients wrt noise level \( D \)

- Possible explanation:
  - Vicsek particle simulations are not in hydro regime
  - Interaction radius \( R_{Vicsek} = O(1) \) \( \mid \) \( R_{Hydro} = O(\varepsilon) \)
  - \( \varepsilon_{Vicsek} \sim 0.03 \) not very small
  - requires a non-local collision operator with account of fluctuations of particle number
4. Conclusion
Hydrodynamics of Vicsek model derived under specific scaling hypotheses
Hydrodynamics of Vicsek model derived under specific scaling hypotheses

Non-standard features have been outlined
  Lack of collision invariants
Hydrodynamics of Vicsek model derived under specific scaling hypotheses

Non-standard features have been outlined
  - Lack of collision invariants

A new concept has been proposed
  - Generalized collision invariant
Hydrodynamics of Vicsek model derived under specific scaling hypotheses

- Non-standard features have been outlined
  - Lack of collision invariants

- A new concept has been proposed
  - Generalized collision invariant

- Leads to the first derivation of a non-conservative model from kinetic theory
  - Published in [D. Motsch, M3AS, Vol. 18, (2008)]
Comparison of Vicsek and hydrodynamics

- Shows some deficiencies of hydro model
  - Constant order parameter
  - Lack of phase transition, ...
Comparison of Vicsek and hydrodynamics

- Shows some deficiencies of hydro model
  - Constant order parameter
  - Lack of phase transition, ...

- Possible cures are proposed
  - Non-local collision operator
  - Account of fluctuations
  - Diffusive corrections (Chapman-Enskog), ...
Future goals

Understanding

- Describe is not explain
- Start from 'first principles' principles
- Link with experiment
Future goals

- Understanding
  - Describe is not explain
  - Start from 'first principles' principles
  - Link with experiment

- Prediction
Future goals

➡️ **Understanding**
   ➡️ Describe is not explain
   ➡️ Start from 'first principles' principles
   ➡️ Link with experiment

➡️ **Prediction**

➡️ **Optimal design and control**