Beta distributions on matrices and applications

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Hermitian matrices

We take care simultaneously of the real case ($d = 1$), the complex case ($d = 2$) and the quaternionic case ($d = 4$). It is convenient to sometimes denote by F_1, F_2 and F_4 respectively the real numbers, the complex numbers and the quaternions. We fix a positive integer r. We denote by M_r the *real* linear space of (r, r) matrices $X = (x_{ij})_{1\leq i,j\leq r}$ with elements in F_d. The adjoint $X^* = (y_{ij})_{1\leq i,j\leq r}$ of $X = (x_{ij})_{1\leq i,j\leq r}$ is defined by $y_{ij} = \overline{x_{ji}}$. The group K_r is the group of $u \in M_r$ such that $uu^* = u^*u = 1$. Thus K_r is the orthogonal group for $d = 1$, the unitary group for $d = 2$ and the symplectic group for $d = 4$. Furthermore $X \in M_r$ is said to be Hermitian if $X = X^*$.

Determinants of quaternionic Hermitian matrices and their eigenvalues need some care to be properly defined.

You
people of random matrices,
you who call ensemble*
what others call
probability law on matrices
remember that what you call β
is the Peirce constant d
of Jordan algebras.
The number $f = d/2$ is
the half Peirce constant.

*after Boltzmann

We denote by V_r the real linear space of Hermitian matrices (therefore real symmetric for $d = 1$ and quaternionic Hermitian for $d = 4$.) If $X \in V_r$ then X is said to be positive definite if for any $z \in F_d^r$ written as a column then the number z^*Xz is a real number and is positive. We denote by Ω the cone of positive definite Hermitian matrices of order r. We denote by I_r the identity matrix. We denote by Ω the cone of positive definite Hermitian matrices of order r .

Wishart

If $\sigma \in \Omega$ and if

$$p \in \Lambda = \{f, 2f, \ldots, (r-1)f\} \cup ((r-1)f, \infty)$$

the Wishart distribution $\gamma_{p,\sigma}(dx)$ is the distribution on $\overline{\Omega}$ whose Laplace transform is

$$\int_{\overline{\Omega}} e^{-\mathrm{tr}\,(\theta x)} \gamma_{p,\sigma}(dx) = \det(I_r + \sigma^{1/2}\theta\sigma^{1/2})^{-p}$$

Of course, for $d = 1, 2$ the Laplace transform can be given the simpler form

$$\int_{\overline{\Omega}} e^{-\mathrm{tr}\,(\theta x)} \gamma_{p,\sigma}(dx) = \det(I_r + \theta\sigma)^{-p}.$$

When $p > (r-1)f$ then $\gamma_{p,\sigma}(dx)$ is :

$$\frac{1}{\Gamma_\Omega(p)} e^{-\mathrm{tr}\,(\sigma^{-1}x)} (\det x)^{p-1-(r-1)f} (\det \sigma)^{-p} \mathbf{1}_\Omega(x) dx,$$

where

$$\Gamma_\Omega(p) = C \prod_{j=1}^{r} \Gamma(p - (j-1)f). \qquad (1)$$

The numerical constant C does not depend on p but only on d and r.

If $p = kf$ with $k = 1, 2 \ldots, r - 1$ then $\gamma_{p,\sigma}$ is concentrated on the elements of $\overline{\Omega}$ with rank k and is a singular distribution. Λ is called the Gyndikin set, since Gyndikin has shown in 1975 that the above Laplace transform is not the Laplace transform of a positive measure if $p \notin \Lambda$ but a Schwartz distribution.

Beta

Proposition 1. If U and V are independent rv with respective distributions $\gamma_{p,\sigma}$ and $\gamma_{q,\sigma}$ where p and q are in Λ and are such that $p+q > (r-1)f$ then

1. $S = U + V$ is invertible.
2. $Z = S^{-1/2}US^{-1/2}$ is independent of S
3. The distribution of Z does not depend on σ.
4. The distribution of Z is invariant by the action of K_r defined by $z \mapsto uzu^*$.
5. If $p > (r-1)f$ and $q > (r-1)f$ then Z has a density concentrated on $\Omega \cap (I_r - \Omega)$:

   $C(\det z)^{p-1-(r-1)f}(\det(I_r - z))^{q-1-(r-1)f}$

   with $C = \frac{\Gamma_\Omega(p+q)}{\Gamma_\Omega(p)}$

   The distribution of such a Z is called the beta distribution $B_{p,q}$ with parameters p, q.

Some linear algebra : We equip V_r with the inner product $\langle a, b\rangle = \mathrm{tr}\,(ab)$. Thus V_r becomes Euclidean. Since V_r is Euclidean, speaking of symmetric linear operators acting on V_r makes sense. By definition such a symmetric operator $\varphi : V_r \mapsto V_r$ must satisfy for all $a, b \in V_r$

$$\mathrm{tr}\,(a\varphi(b)) = \mathrm{tr}\,(\varphi(a)b).$$

Denote by $\mathcal{S}(V_r)$ the linear space of these symmetric operators φ on V_r. Here are two important examples of elements of $\mathcal{S}(V_r)$

Example 1 : the operator $\mathbb{P}(z)$. If $z \in V_r$ and $a \in V_r$ denote $\mathbb{P}(z)(a) = zaz$. Thus for fixed $z \in V_r$ the map $\mathbb{P}(z)$ defined by $a \mapsto zaz$ is linear. Furthermore it is symmetric since $\mathrm{tr}\,(a\mathbb{P}(z)(b)) = \mathrm{tr}\,(\mathbb{P}(z)(a)b)$ or $\mathrm{tr}\,(azbz) = \mathrm{tr}\,(zazb)$ by commutativity of traces.

Example 2 : the operator $z \otimes z$. If $z \in V_r$ and $a \in V_r$ define $(z \otimes z)(a) = z\,\mathrm{tr}\,(za)$. Thus $a \mapsto (z \otimes z)(a)$ is linear from V_r to V_r and it defines a symmetric operator on V_r since

$$\mathrm{tr}\,((z\otimes z)(a)b) = \mathrm{tr}\,(za)\,\mathrm{tr}\,(zb) = \mathrm{tr}\,(a(z\otimes z)(b)).$$

The magic operator Ψ. We want to avoid calculations (actually hidden in the proof of next proposition). The magic operator Ψ is a special linear map of $\mathcal{S}(V_r)$ into itself which has the property that $\Psi(z \otimes z) = \mathbb{P}(z)$ for all $z \in V_r$. Since this is for all $z \in V_r$ and since the elements of rank one $z \otimes z$ in $\mathcal{S}(V_r)$ are actually numerous enough to generate $\mathcal{S}(V_r)$ itself this is not surprising that $\Psi(z \otimes z) = \mathbb{P}(z)$ defines at most one Ψ.

Proposition 2 There exists one and only one endomorphism Ψ of $\mathcal{S}(V_r)$ such that for all z

$$\Psi(z \otimes z) = \mathbb{P}(z)$$

Furthermore (recall that $f = d/2$) :

$$\Psi(\mathbb{P}(z)) = f z \otimes z + (1 - f)\mathbb{P}(z).$$

It is not advisable to represent the operator Ψ by a matrix : $\dim \mathcal{S}(V_r) = (r+d)(r+d+1)/2$. Suppose that $d = r = 2$: then $\dim \mathcal{S}(V_2) = 10$ and after having chosen a basis of $\mathcal{S}(V_2)$ you have still to find the representative matrix of Ψ corresponding to this basis. A colleague became convinced of the usefulness of Ψ after he had completely written the hundred entries of such a matrix..

The Olkin and Rubin Lemma. For simplicity, for u in K_r denote $k_u(z) = uzu^*$ for all $z \in V_r$. Since V_r is Euclidean, when k is a linear operator on V_r, it makes sense to define the adjoint operator k^* by $\mathrm{tr}\,(ak(b)) = \mathrm{tr}\,(k^*(a)b)$ for all a and b in V_r. In particular $(k_u)^* = k_{u^*}$. We denote by K the image of K_r by $u \mapsto k_u$.

The lemma looks for all the elements f of $\mathcal{S}(V_r)$ such that for all $k \in K$ one has $f = kfk^*$. An example of such an f is $I_r \otimes I_r$: If $k = k_u$ let us compute $f(z)$ and $kfk^*(z)$ for all $z \in V_r$. Thus $f(z) = I_r \,\mathrm{tr}\,(z)$ and

$$\begin{aligned} kfk^*(z) &= kf(u^*zu) = k(I_r \,\mathrm{tr}\,(u^*zu)) = k(I_r \,\mathrm{tr}\,(z)) \\ &= \mathrm{tr}\,(z)k(I_r) = \mathrm{tr}\,(z)uI_ru^* = \mathrm{tr}\,(z)I_r. \end{aligned}$$

An other example of f such that $f = kfk^*$ is simply the identity on V_r : if $f = \mathrm{id}_{V_r}$ then $f(z) = z$ and $kfk^*(z) = uu^*zuu^* = z$ since $uu^* = I_r$. The lemma says that these two examples are essentially the only ones. More specifically

Lemma . Let $f \in \mathcal{S}(V_r)$. Then $f = kfk^*$ for all $k \in K$ if and only if there exists two real numbers λ and μ such that

$$f = \lambda \mathrm{id}_{V_r} + \mu I_r \otimes I_r.$$

First and second moments of Wishart

Proposition 3. Let $U \sim \gamma_{p,\sigma}$ where $p \in \Lambda$ and $\sigma \in \Omega$. Then $\mathbb{E}(U) = p\sigma$ and

$$\begin{aligned} \mathbb{E}(U \otimes U) &= p^2 \sigma \otimes \sigma + p\mathbb{P}(\sigma) \\ \mathbb{E}(\mathbb{P}(U)) &= pf\sigma \otimes \sigma + (p^2 + p(1-f))\mathbb{P}(\sigma) \end{aligned}$$

which means

$$\begin{aligned} \mathbb{E}(\operatorname{tr}(Ua)\operatorname{tr}(Ub)) &= p^2 \operatorname{tr}(\sigma a)\operatorname{tr}(\sigma b) + p\operatorname{tr}(a\sigma b\sigma) \\ \mathbb{E}(\operatorname{tr}(aUbU)) &= pf\operatorname{tr}(\sigma a)\operatorname{tr}(\sigma b) \\ &\quad + (p^2 + p(1-f))\operatorname{tr}(a\sigma b\sigma) \end{aligned}$$

Proof of Proposition 3. The computations of $\mathbb{E}(U)$ and of $\mathbb{E}(U \otimes U)$ are directly obtained from the Laplace transform. The computation of $\mathbb{E}(\mathbb{P}(U))$ is an application of Proposition 2 : apply Ψ-the-Magic to both sides of (9). We get by linearity

$$\begin{aligned} \mathbb{E}(\mathbb{P}(U)) = \mathbb{E}(\Psi(U \otimes U)) &= p^2\Psi(\sigma \otimes \sigma) + p\Psi(\mathbb{P}(\sigma)) \\ &= p^2\mathbb{P}(\sigma) + pf\sigma \otimes \sigma \\ &\quad + p(1-f)\mathbb{P}(\sigma) \end{aligned}$$

First and second moments of beta

Proposition 4. Let $Z \sim B_{p,q}$ where $p, q \in \Lambda$ and $p+q > (r-1)f$. Then $\mathbb{E}(Z) = \frac{p}{p+q} I_r$ and

$$\mathbb{E}(Z \otimes Z) = \lambda_1 \mathrm{id}_{V_r} + \mu_1 I_r \otimes I_r \qquad (2)$$
$$\mathbb{E}(\mathbb{P}(Z)) = \lambda_2 \mathrm{id}_{V_r} + \mu_2 I_r \otimes I_r \qquad (3)$$

where

$$\lambda_1 = \frac{p}{p+q} \times \frac{q}{(p+q)^2 + (p+q)(1-f) - f}$$
$$\mu_1 = \frac{p}{p+q} \times \frac{p(p+q+1-f) - f}{(p+q)^2 + (p+q)(1-f) - f}$$

and $\lambda_2 = (1-f)\lambda_1 + \mu_1$ and $\mu_2 = f\lambda_1$.

(Note the simplicity of formulas in the complex case $f = 1$).

Proof. We use Proposition 1 and write $Z = S^{-1/2}US^{-1/2}$ with $S = U + V$ and $U \sim \gamma_{p,\sigma}$ $V \sim \gamma_{q,\sigma}$ independent. Since the distribution of Z is invariant by K thus $m = \mathbb{E}(Z)$ is invariant by K, that means $umu^* = m$ for all $u \in K_r$: this implies that there exists a real number λ such that $m = \lambda I_r$ (just diagonalize m to verify this fact). For computing λ we write $\lambda I_r = \mathbb{E}(Z|S)$ since Z and S are independent. This implies by applying $\mathbb{P}(S^{1/2})$ to both sides

$$\lambda S = \mathbb{P}(S^{1/2})(\mathbb{E}(Z|S)) = \mathbb{E}(\mathbb{P}(S^{1/2})(Z)|S) = \mathbb{E}(U|S)$$

Now we take the expectation of both sides : $\lambda\mathbb{E}(S) = \mathbb{E}(U)$ which leads from Proposition 3 to $\lambda(p+q)\sigma = p\sigma$ (recall that $S \sim \gamma_{p+q,\sigma}$ from the Laplace transform). Finally $\mathbb{E}(Z) = \frac{p}{p+q}I_r$ as desired.

For the second moments, we use the Olkin and Rubin lemma above. Since the distribution of Z is invariant by K, then $f = \mathbb{E}(Z \otimes Z)$ must satisfy $f = kfk^*$ for all $k \in K$ and there exists two numbers λ and μ such that

$$\mathbb{E}(Z \otimes Z) = \lambda \mathrm{id}_{V_r} + \mu I_r \otimes I_r.$$

We translate this into

$$\mathbb{E}(\mathrm{tr}\,(aZ)\,\mathrm{tr}\,(bZ)) = \lambda\,\mathrm{tr}\,(ab) + \mu\,\mathrm{tr}\,(a)\,\mathrm{tr}\,(b).$$

To determine λ and μ we use the representation $Z = S^{-1/2}US^{-1/2}$ as usual with Z and S independent. We take a and b as functions of S : $a = \mathbb{P}(S^{1/2})(A)$ and $b = \mathbb{P}(S^{1/2})(B)$ where A and B are constant elements of V_r. Thus

$$\begin{array}{ll} & \lambda\, \mathrm{tr}\,(ASBS) + \mu\, \mathrm{tr}\,(AS)\, \mathrm{tr}\,(BS) \\ = & \mathbb{E}(\,\mathrm{tr}\,(\mathbb{P}(S^{1/2})(A)Z)\, \mathrm{tr}\,(\mathbb{P}(S^{1/2})(B)Z)|S) \\ = & \mathbb{E}(\,\mathrm{tr}\,(AU)\, \mathrm{tr}\,(BU))|S) \end{array}$$

Taking the expectations of both sides we get

$$\lambda\mathbb{E}(\,\mathrm{tr}\,(ASBS)) + \mu\mathbb{E}(\,\mathrm{tr}\,(AS)\, \mathrm{tr}\,(BS)) = \mathbb{E}(\,\mathrm{tr}\,(AU)\, \mathrm{tr}\,(BU$$

that one may rather write as

$$\lambda\mathbb{E}(\mathbb{P}(S)) + \mu\mathbb{E}(S \otimes S) = \mathbb{E}(U \otimes U)$$

Now we use Proposition 3 to rewrite this as

$$\begin{array}{l} \lambda[(p+q)f\sigma \otimes \sigma + ((p+q)^2 + (p+q)(1-f))\mathbb{P}(\sigma)] \\ +\mu[(p+q)^2\sigma \otimes \sigma + (p+q)\mathbb{P}(\sigma)] \\ = p^2\sigma \otimes \sigma + p\mathbb{P}(\sigma) \end{array}$$

Now we identify (since $r \geq 2$) the coefficients of $\sigma \otimes \sigma$ and $\mathbb{P}(\sigma)$ on both sides :

$$\begin{aligned} \lambda(p+q)f + \mu(p+q)^2 &= p^2 \\ \lambda[(p+q)^2 + (p+q)(1-f)] + \mu(p+q) &= p \end{aligned}$$

Solving this linear system we get $\lambda = \lambda_1$ and $\mu = \mu_1$ and (2) is proven. To prove (3) we apply Ψ to (2) and we get since $\mathrm{id}_{V_r} = \mathbb{P}(I_r)$ and by using Proposition 2

$$\begin{aligned} \mathbb{E}(\mathbb{P}(Z)) &= \mathbb{E}(\Psi(Z \otimes Z)) = \lambda_1 \Psi(\mathbb{P}(I_r)) + \mu_1 \Psi(I_r \otimes I_r) \\ &= \lambda_1 (f I_r \otimes I_r + (1-f)\mathrm{id}_{V_r}) + \mu_1 \mathrm{id}_{V r} \\ &= (\lambda_1(1-f) + \mu_1)\mathrm{id}_{V_r} + \lambda_1 f I_r \otimes I_r \\ &= \lambda_2 \mathrm{id}_{V_r} + \mu_2 I_r \otimes I_r \end{aligned}$$

Invariant moments of the beta distributions.

Let Z be $B_{p,q}$ distributed, for $d = 1, 2$ or 4. We explain how to compute the *invariant moments* of Z namely $\mathbb{E}(Q(Z))$ where $Q(z)$ is a real polynomial with respect to the $n = r + fr(r-1)$ real entries of the matrix $z \in E$ with the following invariance property : for each $u \in K_r$ we have

$$Q(uzu^*) = Q(z).$$

Such a polynomial Q is called an *invariant polynomial*. Examples are $Q(z) = \mathrm{tr}\,(z^k)$ for all non negative integers k. Actually Q is invariant if and only if $Q(z)$ depends only on the eigenvalues $(\lambda_1, \dots, \lambda_r)$ of z, more specifically being a symmetric polynomial in $(\lambda_1, \dots, \lambda_r)$. For instance $\mathrm{tr}\,(z^k) = \lambda_1^k + \cdots + \lambda_r^k$.

The set of invariant polynomials is obviously an algebra (we mean that it is closed by linear combination and multiplication). The algebra of invariant polynomials is linearly generated by the family of *spherical polynomials* $\Phi = \{\Phi_{\mathbf{m}} \ ; \ \mathbf{m} \in M\}$ that we are going to describe. Furthermore Φ is a *basis*. By this we mean that for any invariant polynomial Q there exists a unique set $(\lambda_{\mathbf{m}}(Q))_{\mathbf{m} \in M}$ of real numbers such that

$$Q = \sum_{\mathbf{m} \in M} \lambda_{\mathbf{m}}(Q) \Phi_{\mathbf{m}}.$$

Of course, in this compact expression only a finite number of $\lambda_{\mathbf{m}}(Q)$ are not zero.

The basis Φ of spherical polynomials is not the only remarkable basis of the set of spherical polynomials and the book by Macdonald (1999) describes several of them. However Φ is important for our purposes here since we are able in Proposition 6 below to compute $\mathbb{E}[\Phi_{\mathbf{m}}(Z)]$ explicitly, as well as $\mathbb{E}[\Phi_{\mathbf{m}}(Z^{-1})]$ when it exists. From this proposition we can to compute the expectation of $Q(Z)$ or $Q(Z^{-1})$ for any invariant polynomial Q if we are able to compute $(\lambda_{\mathbf{m}}(Q))_{\mathbf{m}\in M}$.

Consider a Hermitian matrix $x = (x_{ij})_{1\leq i,j\leq r}$ of order r. For $1 \leq k \leq r$ we denote $\Delta_k(x) = \det(x_{ij})_{1\leq i,j\leq k}$. Consider a sequence of integers $\mathbf{m} = (m_1, \ldots, m_r)$ such that $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$. We denote $|\mathbf{m}| = m_1 + m_2 + \cdots + m_r$. Let

$$\begin{aligned}\Delta_{\mathbf{m}}(x) &= (\Delta_1(x))^{m_1-m_2}(\Delta_2(x))^{m_2-m_3}\ldots\\ &\quad (\Delta_{r-1}(x))^{m_{r-1}-m_r}(\Delta_r(x))^{m_r}.\end{aligned}$$

The spherical polynomial $\Phi_{\mathbf{m}}(x)$ of parameter $\mathbf{m}$ is defined by the following integral on the group K_r endowed with the Haar measure du (normalized in order to have total mass one) :

$$\Phi_{\mathbf{m}}(x) = \int_{K_r} \Delta_{\mathbf{m}}(u^{-1}xu)du.$$

We have obviously $\Phi_{\mathbf{m}}(I_r) = 1$. The spherical polynomial $\Phi_{\mathbf{m}}(x)$ is a homogeneous polynomial of degree $|\mathbf{m}|$ with respect to the entries x_{ij} of the Hermitian matrix x. An other important remark is that for any v in K_r we have

$$\Phi_{\mathbf{m}}(x) = \Phi_{\mathbf{m}}(vxv^*). \tag{4}$$

A consequence of (4) is that actually, $\Phi_{\mathbf{m}}(x)$ depends only on the eigenvalues of x. They are sometimes the perfect analogue of powers in one dimension, as shown by the following proposition :

Proposition 5. Let A and X be independent random variables valued in M_r and V_r respectively such that $X \sim uUu^*$ for all $u \in K_r$. Then for all $m \in M$ we have

$$\mathbb{E}(\Phi_{\mathbf{m}}(AXA^*)) = \mathbb{E}(\Phi_{\mathbf{m}}(AA^*))\mathbb{E}(\Phi_{\mathbf{m}}(X))$$

Proof. It relies on the following important formula, proved for instance in Faraut and Koranyi (1994), Corollary XI.3.2 : for $x \in V_r$ and $a \in M_r$

$$\int_{K_r} \Phi_{\mathbf{m}}(auxu^*a^*)du = \Phi_{\mathbf{m}}(aa^*)\Phi_{\mathbf{m}}(x) \quad (5)$$

Since $uXu^* \sim X$ we are allowed to write

$$\begin{aligned}
\mathbb{E}(\Phi_{\mathbf{m}}(AXA^*)) &= \int_{K_r} \mathbb{E}(\Phi_{\mathbf{m}}(AX^*A^*))du \\
&= \int_{K_r} \mathbb{E}(\Phi_{\mathbf{m}}(AuXu^*A^*))du \\
&= \mathbb{E}\left(\int_{K_r} \Phi_{\mathbf{m}}(AuXu^*A^*)du\right) \\
&= \mathbb{E}(\Phi_{\mathbf{m}}(AA^*)\Phi_{\mathbf{m}}(X)) \\
&= \mathbb{E}(\Phi_{\mathbf{m}}(AA^*))\mathbb{E}(\Phi_{\mathbf{m}}(X)).
\end{aligned}$$

We introduce the symbols

$$(p)_{\mathbf{m}} = \frac{\prod_{j=1}^{r} \Gamma(m_j + jf + p)}{\prod_{j=1}^{r} \Gamma(jf + p)} \tag{6}$$

$$(q^*)_{\mathbf{m}} = \frac{\prod_{j=1}^{r} \Gamma(-m_j + jf + q)}{\prod_{j=1}^{r} \Gamma(jf + q)} \tag{7}$$

Proposition 6. Let Z be $B_{p,q}$ distributed, for $d = 1, 2$ or 4. Then for $\mathbf{m} \in M$ we have

$$\mathbb{E}[\Phi_{\mathbf{m}}(Z)] = \frac{(p)_{\mathbf{m}}}{(p+q)_{\mathbf{m}}}.$$

Furthermore if $p > (r-1)f$ we have

$$\mathbb{E}[\Phi_{\mathbf{m}}(Z^{-1})] = \frac{((p-rf)^*)_{\mathbf{m}}}{((p+q-rf)^*)_{\mathbf{m}}}.$$

Proof. As usual we write $Z = S^{-1/2}US^{-1/2}$ with $S = U + V$ where U and V are independent rv such that $U \sim \gamma_{p,I_r}$ and $V \sim \gamma_{q,I_r}$ with $p + q > (r-1)f$. Now we apply Proposition 5 to $X = Z$ and to $A = S^{1/2}$. Since Z and S are independent we have

$$\begin{aligned}\mathbb{E}(\Phi_{\mathbf{m}}(U)) &= \mathbb{E}(\Phi_{\mathbf{m}}(S^{1/2}ZS^{1/2})) \\ &= \mathbb{E}(\Phi_{\mathbf{m}}(S))\mathbb{E}(\Phi_{\mathbf{m}}(Z)).\end{aligned}$$

We then use the fact that

$$\mathbb{E}(\Phi_{\mathbf{m}}(U)) = (p)_{\mathbf{m}}\Phi_{\mathbf{m}}(\sigma)$$

and that $\mathbb{E}(\Phi_{\mathbf{m}}(S)) = (p+q)_{\mathbf{m}}\Phi_{\mathbf{m}}(\sigma)$ to get the result. The second part has a similar proof based on the fact that

$$\mathbb{E}(\Phi_{\mathbf{m}}(U^{-1})) = ((p-rf)^*)_{\mathbf{m}}\Phi_{\mathbf{m}}(\sigma^{-1}).$$

In the complex case (d=2) there is a third and very complete way to investigate the moments of the beta distributions : see the very complete paper by Mireille Capitaine and Muriel Casalis, *Indiana University Mathematics Journal*, **53** (2004) 397 -431

Asymptotic Freeness by Generalized moments for Gaussian and Wishart Matrices. Application to Beta Random Matrices

A first application : the Thomae formula for hypergeometric functions on Hermitian matrices. We now use Proposition 6 for extending the following Thomae formula

$$ {}_3F_2(a,b,c;d,e;1)\frac{\Gamma(d+e-b-c)\Gamma(d+e-a-c)\Gamma(c)}{\Gamma(d)\Gamma(e)\Gamma(d+e-a-b-c)} = $$

$$ {}_3F_2(d-c,e-c,d+e-a-b-c;d+e-a-c,d+e-b-c;1) $$

to hypergeometric functions defined on V_r. Some definitions are in order. Let us introduce first the *zonal polynomials* $(C_{\mathbf{m}}(x))_{\mathbf{m}\in M}$. They are mere multiples of the spherical polynomials :

$$ C_{\mathbf{m}}(x) = C_{\mathbf{m}}\Phi_{\mathbf{m}}(x) $$

where the complicated constant $C_{\mathbf{m}}$ is

$$ C_{\mathbf{m}} = \frac{|\mathbf{m}|!}{(1-f+rf)_{\mathbf{m}}}d_{\mathbf{m}} $$

where $d_{\mathbf{m}}$ is the dimension of the linear space $\mathcal{P}_{\mathbf{m}}$ generated by the set of polynomials in $x \in V_r$:

$$ \{\Delta_{\mathbf{m}}(axa^*)\ ;\ a \in M_r\}. $$

We said before that the spherical polynomials $\Phi_{\mathbf{m}}(x)$ are sometimes the perfect analogue of the powers x^m in one dimension. This is not always true : the zonal polynomials $C_{\mathbf{m}}(x)$ can be better analogues since they replace powers in the extension to V_r of many one variable classical formulas. One of them is

$$\det(I_k - x)^{-p} = \sum_{\mathbf{m} \in M} \frac{(p)_{\mathbf{m}}}{|\mathbf{m}|!} C_{\mathbf{m}}(x) \quad (8)$$

We now define for the integer $q \geq 0$

$$\begin{aligned} & {}_{q+1}F_q(a_0, a_1, \ldots, a_q; b_1, \ldots, b_q; x) \\ &= \sum_{\mathbf{m} \in M} \frac{(a_0)_{\mathbf{m}}(a_1)_{\mathbf{m}} \ldots (a_q)_{\mathbf{m}}}{|\mathbf{m}|!(b_1)_{\mathbf{m}} \ldots (b_q)_{\mathbf{m}}} C_{\mathbf{m}}(x). \end{aligned}$$

Before stating the Thomae formulae for these hypergeometric functions in Proposition 8, let us use them for giving the distribution of $U^{1/2}VU^{1/2}$ when U and V are independent Beta matrix variables. In the proof we shall use the notation $n = r + r(r-1)f$ for the dimension of V_r.

Proposition 7. If $U \sim B_{a,b}$ and $V \sim B_{c,d}$ are independent then the distribution of $X = I_r - U^{1/2}VU^{1/2}$ is

$$\frac{\Gamma_\Omega(a+b)\Gamma_\Omega(c+d)}{\Gamma_\Omega(a)\Gamma_\Omega(b+c+d)}\,{}_2F_1(b, c+d-a; b+d; x)\beta_{b+d,c}(dx)$$

(We skip the long but standard proof : the major ingredient is the Gauss formula for the Jordan algebra V_r).

Here is the Thomae Formula :

Proposition 8.

$$
\begin{aligned}
&{}_3F_2(a,b,c;d,e;I_r) \\
&\times \frac{\Gamma_\Omega(d+e-b-c)\Gamma_\Omega(d+e-a-c)\Gamma_\Omega(c)}{\Gamma_\Omega(d)\Gamma_\Omega(e)\Gamma_\Omega(d+e-a-b-c)} =
\end{aligned}
$$

$$
{}_3F_2(d-c,e-c,d+e-a-b-c;d+e-a-c,d+e-b-c;I_r)
$$

Proof. The trick is to compute in two ways $\mathbb{E}((\det X)^{-p})$ when X is defined in Proposition 7. The first way uses the distribution of X as computed in Proposition 7.

$\mathbb{E}((\det X)^p)$ is

$$\int_{(I_r-\Omega)\cap\Omega} (\det x)^p \frac{\Gamma_\Omega(a+b)\Gamma_\Omega(c+d)}{\Gamma_\Omega(a)\Gamma_\Omega(b+c+d)}$$

$$ {}_2F_1(b, c+d-a; b+d; x)\beta_{b+d,c}(dx)$$

$$= \frac{\Gamma_\Omega(a+b)\Gamma_\Omega(c+d)}{\Gamma_\Omega(a)\Gamma_\Omega(b+c+d)}$$

$$\sum_{\mathbf{m}\in M} \frac{(b)_{\mathbf{m}}(c+d-a)_{\mathbf{m}}}{|\mathbf{m}|!(b+d)_{\mathbf{m}}} \int C_{\mathbf{m}}(x)(\det x)^p \beta_{b+d,c}(dx)$$

$$= \frac{\Gamma_\Omega(a+b)\Gamma_\Omega(c+d)\Gamma_\Omega(b+d+p)}{\Gamma_\Omega(a)\Gamma_\Omega(b+d)\Gamma_\Omega(b+c+d+p)}$$

$$\sum_{\mathbf{m}\in M} \frac{(b)_{\mathbf{m}}(c+d-a)_{\mathbf{m}}}{|\mathbf{m}|!(b+d)_{\mathbf{m}}} \int C_{\mathbf{m}}(x)\beta_{b+d+p,c}(dx)$$

Since from Proposition 6 we have

$$\int C_{\mathbf{m}}(x)\beta_{b+d+p,c}(dx) = \frac{(b+d+p)_{\mathbf{m}}}{(b+c+d+p)_{\mathbf{m}}} C_{\mathbf{m}}$$

we can claim that $\mathbb{E}((\det X)^p)$ is

$$\frac{\Gamma_\Omega(a+b)\Gamma_\Omega(c+d)\Gamma_\Omega(b+d+p)}{\Gamma_\Omega(a)\Gamma_\Omega(b+d)\Gamma_\Omega(b+c+d+p)}$$
$$\times\ {}_3F_2(b, c+d-a, b+d+p\,; b+c+d+p, b+d; I_r)$$

The second way is simpler and uses first Propositions 5 and 6 for writing

$$\begin{aligned}\mathbb{E}(C_{\mathbf{m}}(U^{1/2}VU^{1/2})) &= C_{\mathbf{m}}\mathbb{E}(\Phi_{\mathbf{m}}(U))\mathbb{E}(\Phi_{\mathbf{m}}(V)) \\ &= \frac{(a)_{\mathbf{m}}(c)_{\mathbf{m}}}{(a+b)_{\mathbf{m}}(c+d)_{\mathbf{m}}} C_{\mathbf{m}}\end{aligned}$$

We then use (8) applied to $x = U^{1/2}VU^{1/2}$ for writing

$$\begin{aligned}\mathbb{E}((\det X)^p) &= \mathbb{E}((\det(I_r - U^{1/2}VU^{1/2}))^p) \\ &= \sum_{\mathbf{m}\in M} \frac{(-p)_{\mathbf{m}}}{|\mathbf{m}|!}\mathbb{E}(C_{\mathbf{m}}(U^{1/2}VU^{1/2})) \\ &= \sum_{\mathbf{m}\in M} \frac{(-p)_{\mathbf{m}}(a)_{\mathbf{m}}(c)_{\mathbf{m}}}{|\mathbf{m}|!(a+b)_{\mathbf{m}}(c+d)_{\mathbf{m}}} C_{\mathbf{m}} \\ &= {}_3F_2(-p, a, c\,; a+b, c+d; I_r)\end{aligned}$$

A second application : expectation of

$$(X+Y)^{-1}X^2(X+Y)^{-1}$$

when X, Y are independent Wishart. We need first the result

Proposition 9. If $p > (r-1)f+1$ and if $X \sim \gamma_{p,\sigma}$ then

$$\begin{aligned} & \mathbb{E}[\operatorname{tr}(aX)\operatorname{tr}(bX^{-1})] \\ = \ & \frac{p}{p-1-(r-1)f}\operatorname{tr}(a\sigma)\operatorname{tr}(b\sigma^{-1}) \\ & -\frac{1}{p-1-(r-1)f}\operatorname{tr}(ab) \end{aligned}$$

In particular

$$\begin{aligned} & \mathbb{E}[X\operatorname{tr}(bX^{-1})] \\ = \ & \frac{p}{p-1-(r-1)f}\sigma\operatorname{tr}(b\sigma^{-1}) - \frac{1}{p-1-(r-1)f}b \\ & \mathbb{E}[X^{-1}\operatorname{tr}(aX)] \\ = \ & \frac{p}{p-1-(r-1)f}\sigma^{-1}\operatorname{tr}(a\sigma) - \frac{1}{p-1-(r-1)f}a \end{aligned}$$

Proof. Stokes' formula says that if f vanishes on the boundary of Ω and vanishes sufficiently fast at infinity then $\int_{\Omega} f'(x)(h)dx = 0$. We apply this to $f(x) = \frac{1}{\Gamma_{\Omega}(p)}(\det x)^q e^{-\operatorname{tr}(\theta x)}$ for $q = p - 1 - (r-1)f$ and $\theta \in \Omega$. In this case

$$f'(x)(h) = f(x)(q \operatorname{tr}(x^{-1}h) - \operatorname{tr}(\theta h)).$$

Thus

$$\int_{\Omega} \operatorname{tr}(x^{-1}h)f(x)dx = \frac{1}{q}\operatorname{tr}(\theta h)\int_{\Omega} f(x)dx.$$

Now we differentiate this last equality with respect to θ :

$$\begin{aligned} & -\int_{\Omega} \operatorname{tr}(x^{-1}h)\operatorname{tr}(xk)f(x)dx \\ = \ & \frac{1}{q}\operatorname{tr}(kh)\int_{\Omega} f(x)dx - \frac{1}{q}\operatorname{tr}(\theta h)\int_{\Omega} \operatorname{tr}(xk)f(x)dx. \end{aligned}$$

Now we do $h = -b, k = a$ and $\theta = \sigma^{-1}$ and we use $\mathbb{E}(X) = p\sigma$ -and therefore $\mathbb{E}(\operatorname{tr}(aX)) = p\operatorname{tr}(a\sigma)$ and we get the desired result.

Let X and Y be independent random variables such that $X \sim \gamma_{p,\sigma}$ and $Y \sim \gamma_{q,\sigma}$. We assume that $p + q > 1 + (r-1)f$. We denote $S = X + Y$ and $U = S^{-1}X^2S^{-1}$ and we are willing to compute $a = \mathbb{E}(U)$. For this we introduce $Z = S^{-1/2}XS^{-1/2}$ which is independent of S and beta distributed and we write $U = S^{-1/2}ZSZS^{-1/2}$. This implies

$$\mathbb{E}(U) = \mathbb{E}(\mathbb{E}(U|S)) = \mathbb{E}[S^{-1/2}\mathbb{E}(ZSZ|S)S^{-1/2}].$$

Now we use Proposition 4 : by applying $\mathbb{E}(\mathbb{P}(Z))$ to S we are able to compute $\mathbb{E}(ZSZ|S) = \lambda_2 S + \mu_2 I_r \operatorname{tr} S$. Thus

$$\mathbb{E}(U) = \lambda_2 I_r + \mu_2 \mathbb{E}(S^{-1} \operatorname{tr} S).$$

Now we apply Proposition 9 to $S \sim \gamma_{p+q,\sigma}$ and to $a = I_r$ for finally getting $\mathbb{E}(U) = \alpha I_2 + \beta\sigma^{-1}\,\mathrm{tr}\,(\sigma)$ where the coefficients α and β are given by the following formula :

Proposition 10 Let $X \sim \gamma_{p,\sigma}$ and $Y \sim \gamma_{p,\sigma}$ be two independent Wishart random variables and let $S = X + Y$ and $U = S^{-1}X^2S^{-1}$. Then

$$\begin{aligned}\mathbb{E}(U) &= \lambda_2 I_r + \mu_2[\frac{p+q}{p+q-1-(r-1)f}\sigma^{-1}\,\mathrm{tr}\,(\sigma) \\ &\quad -\frac{1}{p+q-1-(r-1)f}I_r]\end{aligned}$$

We now use the preceeding calculations and tools for comments on the Bryc-Bożejko problem.

If X, Y are iid random matrices of Ω and if $S = X + Y$ and $Z = S^{-1/2} X S^{-1/2}$ are independent then X is Wishart distributed : this has been proved by Olkin and Rubin in 1962 for $d = 1$ under the hypothesis of invariance by the orthogonal group $Z \sim uZu^*$ for all $u \in \mathbb{O}(n)$ and under the only assumption of a C^2 density by Bobecka and Wesołowski in 2002. Bryc and Bożejko have raised the following question : what is the distribution of X when rather S and $U = S^{-1} X^2 S^{-1}$ are independent ? As we are going to see, X cannot be Wishart except in the trivial case $r = 1$. There are trivial solutions : suppose that $n_1 + \cdots + n_k = n$, that X is diagonal, that $X = \mathrm{diag}(X_1 I_{n_1}, \ldots, X_k I_{n_k})$ and that $X_1, \ldots, X_r$ are real independent with $X_j \sim \gamma_{\lambda_j, a_j}$. Considering even $X' = uXu^*$ with $u \in K_r$ provides an artificial generality.

The basic differential system

Our first step is to translate the problem of finding all possible distributions of X in a system of differential equations.

Proposition 11. Let X and Y be iid random variables valued in the cone V_d^+ of positive definite elements of V_r. One assumes that $\mathbb{E}(e^{\langle\theta,x\rangle}) = e^{k(\theta)}$ exists in a non empty open convex set Θ. Let $S = X + Y$ and $U = S^{-1}X^2S^{-1}$ and $a = \mathbb{E}(U)$. If S and U are independent then for all $\theta \in \Theta$ one has

$$\Psi(k''(\theta))(a - \frac{1}{2}I_r) + k'(\theta)(2a - \frac{1}{2}I_r)k'(\theta) = 0 \tag{9}$$

Proof. We observe first that $\mathbb{E}(Xe^{\langle\theta,X\rangle}) = k'(\theta)e^{k(\theta)}$ and that

$$\mathbb{E}((X\otimes X)e^{\langle\theta,X\rangle}) = [k''(\theta)+k'(\theta)\otimes k'(\theta)]e^{k(\theta)}$$

Applying Ψ to the last equality gives the equality in the space $\mathcal{L}_s(V_r)$

$$\mathbb{E}(\mathbb{P}(X)e^{\langle\theta,X\rangle}) = [\Psi(k''(\theta))+\mathbb{P}(k'(\theta))]e^{k(\theta)}. \tag{10}$$

Now we compute $\mathbb{E}(X^2e^{\langle\theta,X\rangle})$ from the independence hypothesis : $a=\mathbb{E}(U)=\mathbb{E}(U|S)$ implies

$$SaS = S\mathbb{E}(U|S)S = \mathbb{E}(SUS|S) = \mathbb{E}(X^2|S)$$

$$SaSe^{\langle\theta,S\rangle} = \mathbb{E}(X^2|S)e^{\langle\theta,S\rangle} = \mathbb{E}(X^2e^{\langle\theta,S\rangle}|S)$$

We now take expectations :

$$\begin{aligned}\mathbb{E}(SaSe^{\langle\theta,S\rangle}) &= \mathbb{E}(XaXe^{\langle\theta,S\rangle}) + \mathbb{E}(YaYe^{\langle\theta,S\rangle}) \\ &\quad + \mathbb{E}(XaYe^{\langle\theta,S\rangle}) + \mathbb{E}(YaXe^{\langle\theta,S\rangle})\end{aligned}$$

$$= 2\mathbb{E}(XaXe^{\langle\theta,X\rangle})e^{k(\theta)} + 2k'(\theta)ak'(\theta)e^{2k(\theta)}.$$

Similarly

$$\mathbb{E}(\mathbb{E}(X^2|S)e^{\langle\theta,S\rangle}) = \mathbb{E}(X^2e^{\langle\theta,X\rangle})e^{k(\theta)}.$$

Equating we get

$$\mathbb{E}(\mathbb{P}(X)(a - \frac{1}{2}I_r)e^{\langle\theta,X\rangle}) + k'(\theta)ak'(\theta)e^{k(\theta)} = 0$$

Therefore if we apply 10 to $a - \frac{1}{2}I_r$ we get 9.

Let us mention an important corollary :

Corollary. If $a = \alpha I_r$ with $\alpha \neq \frac{1}{2}$ denote $c = \frac{2\alpha - 1/2}{-\alpha + 1/2}$ and $f(\theta) = e^{ck(\theta)}$. Then

$$\Psi(f''(\theta))(I_r) = 0.$$

Proof. From the proposition we have $\Psi(k'')(I_r) = ck'^2$ and

$$f' = -cfk', \quad f'' = c^2 fk' \otimes k' - cfk''.$$

Thus $\Psi(f'')(I_r) =$

$$c^2 f\mathbb{P}(k')(I_r) - cf\Psi(k'') = cf[ck'^2 - ck'^2] = 0.$$

Note that if $a = \alpha I_r$ and if $\alpha = \frac{1}{2}$ implies that X is Dirac by Proposition 1. Furthermore since

$$\mathbb{E}(S^{-1}(X - Y)^2 S^{-1}) = 4a - I_r$$

is semi positive definite- use

$$I_r = \mathbb{E}(S^{-1}(X + Y)^2 S^{-1}),$$

we have $\alpha \geq 1/4$. Thus c and $\frac{1}{2} - \alpha$ have the same sign.

Why Wishart distributions do not fit. When X is Wishart, Proposition 10 has shown that $a = \alpha I_r + \beta\sigma^{-1}\,\mathrm{tr}\,(\sigma)$ where the numbers α and β have been computed. Denote $\kappa(\theta) = -p\log\det(-\theta)$. Then if X is Wishart with shape parameter p we have $k(\theta) = \kappa(\theta+\theta_0) - \kappa(\theta_0)$. For simplification denote $\sigma = -(\theta+\theta_0)^{-1}$. Then standard calculation shows that

$$k'(\theta) = p\sigma, \quad k''(\theta) = p\mathbb{P}(\sigma).$$

We use

$$\Psi(\mathbb{P}(y)) = \frac{d}{2}y\otimes y + (1-\frac{d}{2})\mathbb{P}(y)$$

for claiming that if X is Wishart as above then

$$\Psi(k''(\theta)) = \frac{p}{2}((2-d)\mathbb{P}(\sigma) + d\sigma\otimes\sigma).$$

We carry this is in Proposition 11 and we get

$$\frac{p}{2}[\sigma(a-\frac{1}{2}I_r)\sigma+\sigma\,\mathrm{tr}\,(\sigma(a-\frac{1}{2}I_r))]+p^2\sigma(2a-\frac{1}{2}I_r)\sigma=0$$

which is clearly impossible if $n>1$ since it leads to an equality of type

$$A\sigma^2+B\sigma\,\mathrm{tr}\,(\sigma)=0$$

where the real coefficients A and B can be computed and satisfy $A+rB\neq 0$. Thus Wishart distributions cannot be a solution.

Solutions invariant by K_r

We are now concentrating on the search of distributions of X fulfilling independence of S and U such that the distribution of X is invariant by the transformations $X \mapsto uXu^*$ where $u \in K_r$. In this case $a = \mathbb{E}(U)$ has the form αI_r since the distribution of U is also invariant by K_r and finding the possible distributions of X is solving the 4 problems

1. Find all real analytic functions f defined on some open subset of the space V_r such that $\Psi(f'')(I_r) = 0$ which are furthermore invariant by $\theta \mapsto u\theta u^*$ for all orthogonal (correspondingly, unitary) matrices.
2. Among these f find the ones such that $\theta \mapsto f(\theta)^{-1/c}$ is the Laplace transform of some probability P on $\mathcal{S}_r$
3. Find the corresponding probabilities P.
4. Among them, decide which P's are such that X and Y iid with distribution P satisfy U independent of $X+Y$ with

$$U = (X+Y)^{-1}X^2(X+Y)^{-1}$$

Problem 1 involves the finding of functions f such that $\Psi(f'')(I_r) = 0$ and which are invariant by K_r, which means that $x \mapsto f(x)$ depends only on the eigenvalues of x. However, instead of writing $f(x) = g(\lambda_1, \dots, \lambda_r)$ computations will be simpler by introducing the elementary symmmetric functions $\sigma_1 = \lambda_1 + \cdots + \lambda_r$, $\sigma_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{r-1}\lambda_r$, $\sigma_r = \lambda_1 \cdots \lambda_r$ and by writing

$$f(x) = g(\sigma_1, \dots, \sigma_r).$$

A very long calculation now leads to the following partial differential equations (PDE) system. Note that it is

1. linear homogeneous
2. of the second order
3. with non constant coefficients which are polynomials of degree one with respect to the variables $\sigma_1, \dots, \sigma_r$.

For stating the PDE system we introduce n symmetric matrices $P_1, \ldots, P_r$ of order r. For $r = 2$ they are

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & -\sigma_2 \end{bmatrix}, \; P_2 = \begin{bmatrix} 0 & 1 \\ 1 & \sigma_1 \end{bmatrix},$$

For $r = 3$ they are

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\sigma_2 & -\sigma_3 \\ 0 & -\sigma_3 & 0 \end{bmatrix}, \; P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & \sigma_1 & 0 \\ 0 & 0 & -\sigma_3 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & \sigma_1 \\ 1 & \sigma_1 & \sigma_2 \end{bmatrix},$$

For $r = 4$ they are

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\sigma_2 & -\sigma_3 & -\sigma_4 \\ 0 & -\sigma_3 & -\sigma_4 & 0 \\ 0 & -\sigma_4 & 0 & 0 \end{bmatrix}, \; P_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & \sigma_1 & 0 & 0 \\ 0 & 0 & -\sigma_3 & -\sigma_4 \\ 0 & 0 & -\sigma_4 & 0 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & \sigma_1 & 0 \\ 1 & \sigma_1 & \sigma_2 & 0 \\ 0 & 0 & 0 & -\sigma_4 \end{bmatrix}, \; P_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \sigma_1 \\ 0 & 1 & \sigma_1 & \sigma_2 \\ 1 & \sigma_1 & \sigma_2 & \sigma_3 \end{bmatrix}$$

The general structure of P_j for general r is
$P_j(k, s) =$

$$\begin{cases} \sigma_{k+s-1-j} & \text{if } \max\{k, s\} \leq j \leq k + s - 1 \leq r; \\ -\sigma_{k+s-1-j} & \text{if } 1 \leq j < \min\{k, s\} \text{ and } \leq k + s - 1 \leq r; \\ 0 & \text{otherwise.} \end{cases}$$

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & -\sigma_2 & -\sigma_3 & -\sigma_4 & -\sigma_5 & & \dots & -\sigma_r \\ 0 & -\sigma_3 & -\sigma_4 & -\sigma_5 & \dots & & -\sigma_r & 0 \\ 0 & -\sigma_4 & -\sigma_5 & & & & 0 & 0 \\ 0 & -\sigma_5 & \vdots & & & & \vdots & \vdots \\ 0 & & & & & & & \\ \vdots & \vdots & -\sigma_r & 0 & \dots & & & \vdots \\ 0 & -\sigma_r & 0 & 0 & \dots & & \dots & 0 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & \sigma_1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & -\sigma_3 & -\sigma_4 & -\sigma_5 & \cdots & & -\sigma_r \\ 0 & 0 & -\sigma_4 & -\sigma_5 & & & & 0 \\ 0 & 0 & -\sigma_5 & & & & & \vdots \\ 0 & \vdots & \vdots & & & & & \\ \vdots & & & & & & & \vdots \\ 0 & 0 & -\sigma_r & 0 & \cdots & & \cdots & 0 \end{bmatrix}$$

$$P_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \sigma_1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \sigma_1 & \sigma_2 & 0 & \cdots & 0 \\ 0 & 1 & \sigma_1 & \sigma_2 & \sigma_3 & 0 & \cdots & 0 \\ 1 & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sigma_6 & & -\sigma_r \\ \vdots & \vdots & \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & -\sigma_r & \cdots & 0 \end{bmatrix}$$

$$
P_{r-1} = \begin{bmatrix}
0 & 0 & \cdots & & \cdots & 0 & 1 & 0 \\
0 & 0 & \cdots & \cdots & 0 & 1 & \sigma_1 & 0 \\
\vdots & \vdots & & & & \sigma_1 & \sigma_2 & 0 \\
 & \vdots & & & & \sigma_2 & \sigma_3 & 0 \\
\vdots & 0 & & & & & \vdots & \vdots \\
0 & 1 & \sigma_1 & \sigma_2 & & & \vdots & 0 \\
1 & \sigma_1 & \sigma_2 & \sigma_3 & \cdots & & \sigma_{r-2} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\sigma_r
\end{bmatrix}
$$

$$
P_r = \begin{bmatrix}
0 & 0 & \cdots & & \cdots & 0 & 0 & 1 \\
0 & 0 & \cdots & \cdots & 0 & 0 & 1 & \sigma_1 \\
\vdots & \vdots & & & & 1 & \sigma_1 & \sigma_2 \\
 & \vdots & & & & \sigma_1 & \sigma_2 & \sigma_3 \\
\vdots & 0 & & & & & & \vdots \\
0 & 0 & 1 & \sigma_1 & & & & \sigma_{r-3} \\
0 & 1 & \sigma_1 & \sigma_2 & & & \sigma_{r-3} & \sigma_{r-2} \\
1 & \sigma_1 & \sigma_2 & \sigma_3 & \cdots & \sigma_{r-3} & \sigma_{r-2} & \sigma_{r-1}
\end{bmatrix}
$$

Here is our main theorem with W. Bryc :

Theorem Let $f : V_r \to \mathbb{R}$ such that f invariant by K_r, and denote $f(x) = g(\sigma_1, \ldots, \sigma_r)$. We denote $g_i = \frac{\partial g}{\partial \sigma_i}$ and $g_{ij} = \frac{\partial^2 g}{\partial \sigma_i \partial \sigma_j}$. Then $\Psi(f'')(I_r) = 0$ if and only if g is the solution of the following PDE system :

$$-(r-j)fg_{j+1} + \sum_{m=1}^{r} \sum_{i=1}^{r} P_j(m,i)g_{mi} = 0$$

for $j = 1, 2, \ldots, r$.

We remark that this becomes

$$\sum_{m=1}^{n} \sum_{i=1}^{r} P_r(m,i)g_{mi} = 0$$

in the exceptional case $j = r$. Note also that the system can be written as

$$\mathrm{tr}\,(P_j g'') = (r-j)fg_{j+1}$$

for $j = 1, 2, \ldots, r$. The matrix P_j does not depend on d but the second member does.

For instance we have the following solutions
$g = \sigma_1$,

$$g = (\sigma_2 - \frac{\sigma_1^2}{4})^{(1-(r-1)d)/2}$$

for $(r, d) \neq (2, 1)$ and

$$g = \log(\sigma_2 - \frac{\sigma_1^2}{4})$$

for $(r, d) = (2, 1)$.

For $r = 2$ the general solution of the system can be made explicit :

$$g = C_1 + C_2\sigma_1 + C_3(\sigma_2 + \frac{d}{4}\sigma_1^2) + C_4(\sigma_2 - \frac{\sigma_1^2}{4})^{(1-d)/2}$$

for $d > 1$ and

$$g = C_1 + C_2\sigma_1 + C_3(\sigma_2 + \frac{1}{4}\sigma_1^2) + C_4 \log(\sigma_2 - \frac{\sigma_1^2}{4})$$

for $d = 1$.

For $r = 3$ Alban Quadrat has proved that the dimension of the space of solutions is 8 and one can conjecture that the dimension of the space of solutions for arbitrary r is 2^r. For $r = 3$ I know only of a space of dimension 5

$$\begin{aligned} g &= C_1 + C_2\sigma_1 + C_3(\sigma_2 + \frac{d}{2}\sigma_1^2) \\ &\quad + C_4(\sigma_3 + \frac{d}{4}\sigma_1\sigma_2 + \frac{d^2}{24}\sigma_1^3) \\ &\quad + C_5(\sigma_2 - \frac{\sigma_1^2}{4})^{(1-2d)/2} \end{aligned}$$

while the three other independent solutions whose existence has been shown by Alban Quadrat are not known explicitely. In general one observes that we have $r + 1$ independent polynomial solutions of weight $\leq r$, plus the universal one $(\sigma_2 - \frac{\sigma_1^2}{4})^{(1-(r-1)d)/2}$ and the $2^r - r + 2$ other ones unknown.

We get at least a lot of non trivial solutions for the Laplace transform of X :

$$\mathbb{E}(e^{\mathrm{tr}\,(\theta X)})$$
$$= \frac{1}{[1 + C_2\sigma_1 + C_3(\sigma_2 + \frac{d}{4}\sigma_1^2)]^p}$$
$$= \frac{1}{[1 + C_2\,\mathrm{tr}\,\theta + \frac{C_3}{2}((1 + \frac{d}{4})(\,\mathrm{tr}\,\theta)^2 - \,\mathrm{tr}\,(\theta^2)]^p}$$

The denominator involves a quadratic polynomial in θ and therefore the density can be made explicit. It exists under a proper choice of the constants C_2 and C_3. (See Letac and Wesołowski, TAMS 2008.) The distribution is concentrated on the cone

$$\{x \in V_r;\ (1 + f)(\,\mathrm{tr}\,x)^2 - \,\mathrm{tr}\,(x^2) > 0\}.$$