An inexact primal-dual deflected subgradient algorithm with augmented Lagrangians

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Outline

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The Primal Problem

X reflexive Banach space, H a Hilbert space

$$\text{minimize } \varphi(x) \quad \text{s.t.} \quad x \text{ in } X \tag{1}$$

$\varphi : X \to \mathbb{R}_{+\infty}$ proper, weakly-lsc
with weakly compact level sets

For constructing the dual, we need the following ingredients...

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Building up the Dual

Take a duality parameterization for (1), i.e.,

$$g : X \times H \to \mathbb{R}_{\pm\infty} \quad \text{such that } g(x,0) = \varphi(x) \;\; \forall x \in X.$$

and an augmenting function $\sigma : H \to \mathbb{R}$

proper, w-lsc, level-bounded, and:

$$\sigma(0) = 0, \quad \sigma(y) \geq \|y\| \, \forall y, \quad \text{and } \operatorname*{Argmin}_y \sigma(y) = \{0\}$$

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The “augmented” Dual

The augmented Lagrangian

$$L : X \times H \times \mathbb{R}_+ \to \mathbb{R}_{\pm\infty}$$

$$L(x, y, r) := \inf_{z \in H} \{g(x, z) - \langle z, y \rangle + r\sigma(z)\}$$

The dual function: $q : H \times \mathbb{R}_+ \to \mathbb{R}_{-\infty}$

$$q(y, r) = \inf_{x \in X} L(x, y, r)$$

with dual problem:

$$\textbf{maximize} \;\; q(y, r) \;\; \textbf{s.t.} \;\; (y, r) \;\; \textbf{in} \;\; H \times \mathbb{R}_+ \qquad (D)$$

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Augmented Lagrangians proposed by Rockafellar and Wets, 1997:

- Strong duality: dual optimal value = primal optimal value
- Saddle point properties: get primal solution using dual one
- Dual problem is convex: use known solution techniques

Also true in infinite dimensions [Burachik-Rubinov, 2007]

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Fix (y, r) a dual variable, and $\varepsilon \geq 0$, find:

$$(\tilde{x}, \tilde{z}) \in X_\varepsilon(y, r)$$

where

$$X_\varepsilon(y, r) := \{(x, z) \in X \times H : g(x, z) - \langle z, y \rangle + r\sigma(z) \leq q(y, r) + \varepsilon\}$$

Fact: $(-\tilde{z}, \sigma(\tilde{z})) \in \partial_\varepsilon q(y, r)$

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ϵ-solutions

Let $M_P := \inf_{x \in X} \varphi(x)$ $\longrightarrow$ optimal primal value

and $M_D := \sup_{(y,r) \in H \times \mathbb{R}_+} q(y, r)$ $\longrightarrow$ optimal dual value

Dual solutions$=D_*$

Fix $\epsilon_* \geq 0$:

$x_* \in X$ is ϵ_*-primal solution if $\varphi(x_*) \leq M_P + \epsilon_*$

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An ϵ-subgradient method

Update rule: Given current $w_k := (u_k, c_k)$, search along ϵ-subgradient direction $g_k \in \partial_\epsilon q(w_k)$:

$$w_{k+1} = w_k + s_k g_k$$

where step-size $s_k > 0$. An ϵ-subgradient of q at w_k is

$$g_k = (-z_k, \sigma(z_k))$$

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Deflected Subgradient Method

Let $\alpha > 0, \delta \in (0, 1), \epsilon_* > 0$

Step 0 Choose (u_0, c_0) with $c_0 \geq 0$ and choose $\epsilon_k \downarrow 0$.

Step k Given (u_k, c_k):

Step k.1 Find $(x_k, z_k) \in X_{\epsilon_k}(u_k, c_k)$. If $z_k = 0$ and $\epsilon_k \leq \epsilon_*$, STOP.

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Step k.3

$$\text{Set} \begin{cases} u_{k+1} := u_k - s_k z_k, \\ c_{k+1} := c_k + s_k(1 + \alpha_k)\sigma(z_k), \end{cases}$$

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(Gasimov 2002; Gasimov & Ismayilova 2004)

(Burachik & Gasimov & Ismayilova & Kaya, 2006, Burachik & Kaya 2007, Burachik & Kaya & Mammadov, 2009)

Basic Result

For every choice of s_k, α_k

If stops at iteration k, then

x_k is ϵ_*-primal optimal, and

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Bounded Stepsize

Take $\beta > \eta > 0$ and $\eta_k := \min\{\eta, \|z_k\|\}$, $\beta_k := \min\{\beta, \sigma(z_k)\}$

Choose $s_k \in [\eta_k, \beta_k]$

If $\epsilon_k \leq M_D - q(u_k, c_k) + R\sigma(z_k)$ and $D_* \neq \emptyset$, then

Call $q_k := q(u_k, c_k)$

- $\{q_k\}$ converges to M_D
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- If $0 < \alpha_k < \bar{\alpha}$, $\{x_k\}$ bded and accumulation points are primal solutions
- Dual sequence bded **iff** dual solutions exist

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Take $\beta > 0$ and a sequence $\{\theta_k\} \subset [0, \beta)$, $\sum \theta_k = \infty$

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