A Proximal Average Suitable for Nonconvex Functions

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Abstract

Recently, researchers have examine the question of how to smoothly transform one function into another. This is, given functions f_0 and f_1, how can we build a "well-behaved" parameterized function $F(x, p)$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$? For convex functions the idea of a "proximal average" has been shown to be highly effective. We explore the proximal average, provide some previous results regarding convex functions, and develop a method to extend these results to non-convex functions. In doing so we develop a new version of the proximal average, which is more complicated but provides stronger stability results.

Outline

1. Introduction, Convex Proximal Average
2. Proximal Envelopes, NC Proximal Average
3. Parametric Prox-Regularity
4. Conclusions

Introduction

Our question

Given f_0 and f_1 how can we create

$$F(x, \lambda)$$

such that

$$F(x,0) = f_0(x) \quad F(x,1) = f_1(x)$$

and F is well behaved in λ?

One Reason and Approach

One reason is multi-objective optimization

$$\min_x \{ f_0(x) \quad \text{and} \quad f_1(x) \}$$

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The common method in this case to set

$$F(x,\lambda) = (1-\lambda)f_0(x) + \lambda f_1(x)$$

One Reason and Approach

One reason is multi-objective optimization

$$\min_x \{ f_0(x) \quad \text{and} \quad f_1(x) \}$$

The common method in this case to set

$$F(x, \lambda) = (1 - \lambda) f_0(x) + \lambda f_1(x)$$

But if

$$\operatorname{dom} f_0 \cap \operatorname{dom} f_1 \neq \operatorname{dom} f_i$$

this does not “behave well”

An alternate approach

In 2008, Bauschke, Lucet, & Trienis proposed the Proximal Average

$$\widehat{\mathcal{PA}}(x,\lambda) = \left((1-\lambda)\left(f_0 + \frac{1}{2}q\right)^* + \lambda\left(f_1 + \frac{1}{2}q\right)^*\right)^*(x) - \frac{1}{2}q(x)$$

where

$$f^*(y) := \sup_x \{\langle x, y\rangle - f(x)\}$$

and

$$q(x) = ||x||^2$$

$\widehat{\mathcal{PA}}$ Behaves Well

Theorem:
Let f_0 and f_1 be proper **convex** lsc.
Then for $\lambda \in [0, 1]$,

1. $\widehat{\mathcal{PA}}$ is a well-defined and convex
2. $\widehat{\mathcal{PA}}(x, 0) = f_0(x)$ and $\widehat{\mathcal{PA}}(x, 1) = f_1(x)$
3. $\widehat{\mathcal{PA}}$ is epi-continuous in λ

F is Epi-cont in λ if
$\mathrm{epi} F(\cdot, \lambda_k) = \{(x, \alpha) : \alpha \geq F(x, \lambda_k)\}$
converges setwise to $\mathrm{epi} F(\cdot, \lambda)$ as $\lambda_k \to \lambda$.

Proximal Average for nonconvex functions

Can a Proximal Average work if f_i are nonconvex?

Proximal Envelopes and the NC-Proximal Average

Relation to Proximal Envelopes

The Proximal Envelope* and Proximal Point Mapping

$$e_r f(x) := \inf_y \left\{ f(y) + \frac{r}{2}|y-x|^2 \right\}$$

$$\mathcal{P}_r f(x) := \mathrm{argmin}_y \left\{ f(y) + \frac{r}{2}|y-x|^2 \right\}$$

We call r the **prox-parameter** and x the **prox-center**

f is **prox-bounded** if $e_r f$ is well-defined for some $r > 0$

threshold of prox-boundedness $=$ greatest lower bound on such r

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* Moreau Envelope, Yosida Regularization, Tikhonov Regularization, etc...

Relation to Proximal Envelopes

Proximal Envelopes are related to conjugate functions via

$$\left(f + \frac{r}{2}q\right)^* (rx) = \left(-e_r f + \frac{r}{2}q\right)(x)$$

Using this, we see

$$\begin{aligned}
\widehat{\mathcal{PA}}(x,\lambda) &= \left((1-\lambda)\left(f_0 + \tfrac{1}{2}q\right)^* + \lambda\left(f_1 + \tfrac{1}{2}q\right)^*\right)^*(x) - \tfrac{1}{2}q(x) \\
&= -e_1(-(1-\lambda)e_1 f_0 - \lambda e_1 f_1)(x)
\end{aligned}$$

So

$$\widehat{\mathcal{PA}}(x,0) = -e_1(-e_1 f_0)(x) \text{ and } \widehat{\mathcal{PA}}(x,1) = -e_1(-e_1 f_1)(x)$$

Insight

$$\widehat{\mathcal{PA}}(x,0) = -e_1(-e_1 f_0)(x) \quad \text{and} \quad \widehat{\mathcal{PA}}(x,1) = -e_1(-e_1 f_1)(x)$$

Why/when does $f_i = -e_1(-e_1 f_i)$?

Insight

$$\widehat{\mathcal{PA}}(x,0) = -e_1(-e_1 f_0)(x) \text{ and } \widehat{\mathcal{PA}}(x,1) = -e_1(-e_1 f_1)(x)$$

Why/when does $f_i = -e_1(-e_1 f_i)$?

[Rockafellar & Wets, '98]

$$-e_r(-e_r f) = f \quad \Leftrightarrow \quad f + \frac{r}{2}q \text{ is convex and lsc}$$

Proximal-Average

This suggests a new, broader, form for the proximal average

$$\mathcal{PA}(x,\lambda) = -e_r \ \ (-(1-\lambda)e_r f_0 - \lambda e_r f_1)(x)$$

Another interesting fact

[Rockafellar & Wets, '98]

Let f be proper, lsc, and prox-bounded with threshold r
If $r_2 > r_1 > r$, then

$$-e_{r_2}(-e_{r_1}f) \in \mathcal{C}^{1+}$$

NC Proximal-Average

These two insights suggest a new NC-Proximal Average:

$$\mathcal{PA}(x,\lambda) = -e_{r+\lambda(1-\lambda)}(-(1-\lambda)e_r f_0 - \lambda e_r f_1)(x)$$

Basic Results

$$\mathcal{PA}(x,\lambda) = -e_{r+\lambda(1-\lambda)}(-(1-\lambda)e_r f_0 - \lambda e_r f_1)(x)$$

Implies

$$\mathcal{PA}(x,i) = -e_r(-e_r f_i) \quad \text{for } i = 0,1$$

so

$$\mathcal{PA}(x,i) = f_i \quad \text{if } f_i + \frac{r}{2}q \text{ is convex and lsc}$$

Basic Results

$$\mathcal{PA}(x,\lambda) = -e_{r+\lambda(1-\lambda)}(-(1-\lambda)e_r f_0 - \lambda e_r f_1)(x)$$

has

$$\text{“}r_2 > r_1\text{”} \quad \text{when} \quad \lambda \in (0,1)$$

so
if f_i is proper, lsc, and prox-bounded, then for fixed $\lambda \in (0,1)$

$$\mathcal{PA}(x,\lambda) \in \mathcal{C}^{1+} \text{ as a function of } x$$

Basic Results

$$\mathcal{PA}(x,\lambda) = -e_{r+\lambda(1-\lambda)}(-(1-\lambda)e_r f_0 - \lambda e_r f_1)(x)$$

has

$$r + \lambda(1-\lambda) > r \quad \text{when} \quad \lambda \in (0,1)$$

so
if f_i is proper, lsc, and prox-bounded, then for fixed $\lambda \in (0,1)$

$$\mathcal{PA}(x,\lambda) \in \mathcal{C}^{1+} \text{ as a function of } x$$

Lower-$\mathcal{C}^2$

f is **lower-$\mathcal{C}^2$** on $\mathcal{O}$ if

for all $x \in \mathcal{O}$ there exists $\rho > 0$ such that $f + \frac{\rho}{2} q$ is convex on $\mathcal{O}$

Basic Results Revisited

Theorem:
f_0, f_1 lsc, proper, prox-bounded
r be greater than the threshold of prox-boundedness for f_0 and f_1
Then

1. for all $\lambda \in [0,1]$ $\mathcal{PA}$ is proper (in x)
2. for all $\lambda \in (0,1)$ $\mathcal{PA}$ is lower-$\mathcal{C}^2$ and $\mathcal{C}^{1+}$ (in x)
3. if $f_i + \frac{r}{2}q$ is convex, then $\mathcal{PA}_r(x,i) = f_i(x)$

Parametric Prox-regularity

Sub-differentials

In this talk we will assume regularity, so

$$\partial f(\bar{x}) := \{w : f(x) \geq f(\bar{x}) + \langle w, x - \bar{x}\rangle + o(|x - \bar{x}|)$$

Prox-regular

f is **prox-regular** (**PR**) at $\bar{x}$ for $\bar{v} \in \partial f(\bar{x})$ if
f is locally lsc at $\bar{x}$ and there exist $\rho > 0$ such that

$$f(x') \geq f(x) + \langle v, x' - x \rangle - \frac{\rho}{2}|x' - x|^2$$

whenever x, x' near $\bar{x}$, $f(x)$ near $f(\bar{x})$, $v \in \partial f(x)$ near $\bar{v}$

Convex $\Rightarrow$ lower-$\mathcal{C}^2$ $\Rightarrow$ prox-regular

Parametric Prox-regular

$f(x,\lambda)$ is parametrically prox-regular (para-**PR**) at $\bar{x}$
with **compatible parameterization in** λ at $\bar{\lambda} \in \mathrm{dom} f(\bar{x},\cdot)$
for $\bar{v} \in \partial_x f(\bar{x},\bar{\lambda})$
if there exits $\rho > 0$ such that

$$f(x',\lambda) \geq f(x,\lambda) + \langle v, x'-x \rangle - \frac{\rho}{2}|x'-x|^2$$

whenever x, x' near $\bar{x}$, λ near $\bar{\lambda}$, $f(x,\lambda)$ near $f(\bar{x},\bar{\lambda})$, $v \in \partial f(x)$ near $\bar{v}$

Example of Parametric PR

Lemma:

f_0, f_1 lower-$\mathcal{C}^2$ near $\bar{x}$

Define

$$F(x, \lambda) := (1 - \lambda) f_0(x) + \lambda f_1(x)$$

Then F is para-**PR** at $\bar{x}$ with compatible parameterization in λ at any $\bar{\lambda} \in [0, 1]$

Example of Parametric PR

Lemma:

f_0, f_1 lower-$\mathcal{C}^2$ near $\bar{x}$

Define

$$F(x,\lambda) := (1-\lambda)f_0(x) + \lambda f_1(x)$$

Then F is para-**PR** at $\bar{x}$ with compatible parameterization in λ at any $\bar{\lambda} \in [0,1]$

Question: True for f, g **PR**?

Corollary

Lemma:

f_0, f_1 lsc, proper, prox-bounded

r be greater than the threshold of prox-boundedness

Then

$$F(x, \lambda) := -(1-\lambda)e_r f_0(x) - \lambda e_r f_1(x)$$

is para-**PR** at any $\bar{x}$ with compatible parameterization in λ at any $\bar{\lambda} \in [0, 1]$

Tool

Theorem [H. & Poliquin, '07]
$F(x,\lambda)$ prox-bounded and para-**PR** at $\bar{x}$ with compatible parameterization in λ at $\bar{\lambda}$
Suppose:

1. $(0,y) \in \partial^\infty F\left(\bar{x},\bar{\lambda}\right) \Rightarrow y = 0$,
2. $(0,\lambda') \in D^*(\partial_x F)\left(\bar{x},\bar{\lambda}|\bar{v}\right)(0) \Rightarrow \lambda' = 0$,
3. there exists $\rho > 0$ such that
   $(x',\lambda') \in D^*(\partial_x F)\left(\bar{x},\bar{\lambda}|\bar{v}\right)(v'),\ \ v' \neq 0 \Rightarrow \langle x', v'\rangle > -\rho|v'|^2$,
4. $\partial_x F(\bar{x},\cdot)$ has a continuous selection near $\bar{\lambda}$,

Then for $\bar{r}$ and K sufficiently large
$\mathcal{P}_r F_\lambda(x)$ is single-valued, with

$$\left|\mathcal{P}_r F_\lambda(x) - \mathcal{P}_{r'} F_{\lambda'}(x')\right| \leq K\left|(r(x-\bar{x}) - r'(x'-\bar{x}), \lambda - \lambda', r - r')\right|,$$

near $(\bar{x} + (1/r)\bar{v}, \bar{\lambda}, \bar{r})$
$(F_\lambda(x) = F(x,\lambda))$

Stability of NC-Proximal Average

f_0, f_1 lsc, proper, prox-bounded
Suppose $\mathcal{P}_r f_0$ and $\mathcal{P}_r f_1$ are Lipschitz with

$$lip\{r(\lambda \mathcal{P}_r f_0 + (1-\lambda)\mathcal{P}_r f_1 - I)\} \leq r,$$

then, for r sufficiently large and $\lambda \in (0,1)$ we have

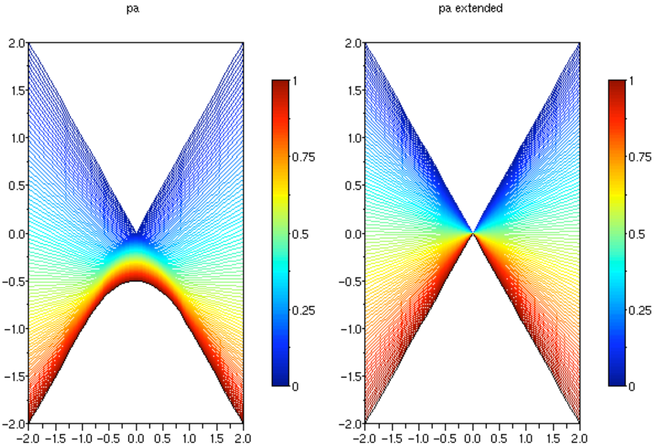
i. $\mathcal{PA}_r$ is $\mathcal{C}^{1+}$ in x,
ii. $\mathcal{PA}_r$ is locally Lipschitz continuous in λ, and
iii. $\nabla_x \mathcal{PA}_r$ is locally Lipschitz continuous in λ.

If for either $i = 0$ or $i = 1$ one has that $f_i + \frac{r}{2}q$ is convex, then $\mathcal{PA}_r(x, i) = f_i(x)$ for all x

Cororllary: If $f_0 + \frac{r}{2}q$ and $f_1 + \frac{r}{2}q$ are convex then all of the above holds

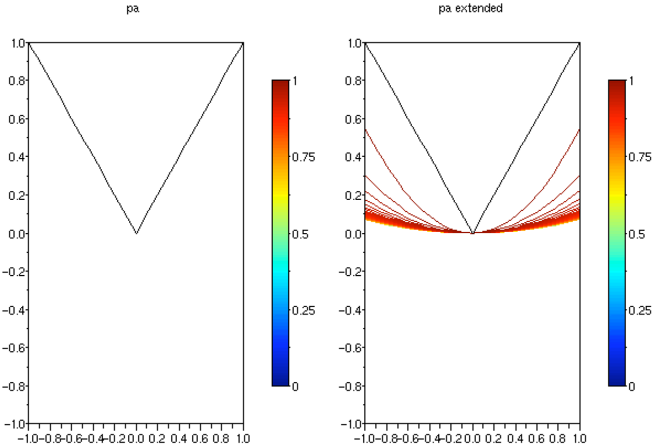
Pretty Pictures

$\widehat{\mathcal{PA}}$ and $\mathcal{PA}$ of $f_0(x) = |x|$, $f_1(x) = -|x|$



* Figures thanks to Yves Lucet

$\widehat{\mathcal{PA}}$ and $\mathcal{PA}$ of $f_0(x) = |x|, f_1(x) = |x|$



* Figures thanks to Yves Lucet

Conclusions and Future Directions

Conclusions

- Smoothly transforming one function into another is more challenging than it looks.
- One method is the Proximal Average

$$\widehat{\mathcal{PA}} = -e_1(-(1-\lambda)e_1 f_0 - \lambda e_1 f_1)(x)$$

- Another method is the NC-Proximal Average

$$\mathcal{PA}(x,\lambda) = -e_{r+\lambda(1-\lambda)}(-(1-\lambda)e_r f_0 - \lambda e_r f_1)(x)$$

- This NC-Proximal Average enjoys stronger stability, but may have other drawbacks

One research direction

What is

$$\min \mathcal{P}\mathcal{A}_r? \qquad \mathrm{argmin}\mathcal{P}\mathcal{A}?$$

One research direction

What is

$$\min \mathcal{PA}_r? \quad \operatorname{argmin} \mathcal{PA}?$$

Example:
Let $f_0 = i_A$ and $f_1 = i_B$, where A and B are convex sets. Then

$$\min_y \mathcal{PA}_r = \min_y \left\{ \frac{r}{2} \left((1-\lambda)\mathrm{dist}^2(y, A) + \lambda \mathrm{dist}^2(y, B) \right) \right\}$$

Results

Results

CMS: Windsor Dec 6th, 2009

Thank You

References

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