Excluded minors for real-representable matroids

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7 August 2009

Excluded minors

Definition

Let F be a field. M is an excluded minor for F-representability if M is not F-representable, but deleting or contracting any element produces an F-representable matroid.

Excluded minors for GF(2)

Theorem (W. T. Tutte, 1958)

The only excluded minor for the class of GF(2)*-representable matroids is* $U_{2,4}$.

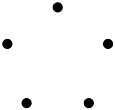
$U_{2,4}$

Excluded minors for GF(3)

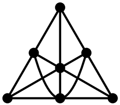
Theorem (R. Reid, R. Bixby, P. Seymour, 1971/1979)

The excluded minors for the class of GF(3)*-representable matroids are* $U_{2,5}$, $U_{3,5}$, F_7, *and* F_7^*.

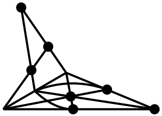
$U_{2,5}$



$U_{3,5}$



F_7

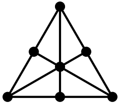


F_7^*

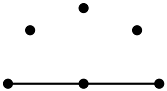
Excluded minors for GF(4)

Theorem (J. Geelen, B. Gerards, A. Kapoor, 1997)

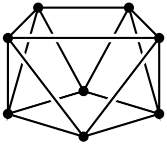
The excluded minors for the class of GF(4)*-representable matroids are* $U_{2,6}$, $U_{4,6}$, F_7^-, $(F_7^-)^*$, P_6, P_8, *and* P_8''.



F_7^-



P_6



P_8

Rota's conjecture

Conjecture (G. C. Rota, 1971)

If $\mathbb{F}$ is a finite field, then there are only finitely many excluded minors for $\mathbb{F}$-representability.

The excluded minors for F-representability are only known in the case that F is GF(2), GF(3), or GF(4).

Lazarson's Theorem

In contrast to Rota's conjecture, we have:

Theorem (T. Lazarson, 1958)

There are infinitely many excluded minors for real-representability.

Proof.

If $p > 2$ is a prime, then the matroid represented over GF(p) by the matrix

$$\left[\begin{array}{c|ccccc} & 0 & 1 & 1 & \cdots & 1 \\ & 1 & 0 & 1 & \cdots & 1 \\ I_{p+1} & 1 & 1 & 0 & \cdots & 1 \\ & \vdots & \vdots & \vdots & \ddots & \vdots \\ & 1 & 1 & 1 & \cdots & 0 \end{array}\right]$$

is an excluded minor for real-representability. □

Geelen's conjecture

Conjecture (J. Geelen, 2008)

If M is a real-representable matroid, then there is an excluded minor, N, for real-representability, such that M is a minor of N.

A proof of Geelen's conjecture

We have proved Geelen's conjecture.

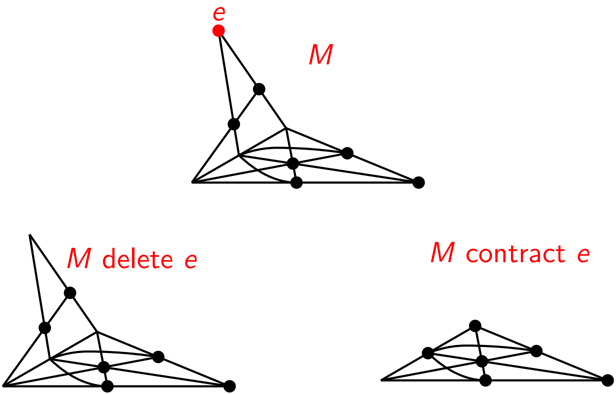
Theorem (D. Mayhew, M. Newman, G. Whittle, 2008)

Let $\mathbb{K}$ be any infinite field, and let M be a $\mathbb{K}$-representable matroid. There is an excluded minor, N, for $\mathbb{K}$-representability, such that M is a minor of N.

Equivalently, the excluded minors for $\mathbb{K}$-representability form a maximal antichain in the minor order.

Geometric representations of minors

When we delete a point, we remove it from the diagram. When we contract, we project onto a hyperplane (maximal non-spanning set).



The proof

The proof uses a lot of geometrical reasoning.

We frequently exploit the following phenomenon:

Suppose M is a matroid with ground set E, and M is representable over $\mathbb{K}$, an infinite field. We can think of this representation as an embedding of E in a projective geometry P over the field $\mathbb{K}$.

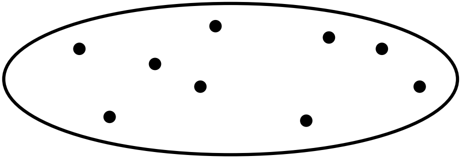
Let X be a subspace of P. Because $\mathbb{K}$ is infinite, there is a point $e \in X \backslash E$, such that if $Y \subseteq E$ spans e, then Y spans X.

Adding e to E is called adding e freely to X relative to E.

We can perform this operation and remain $\mathbb{K}$-representable.

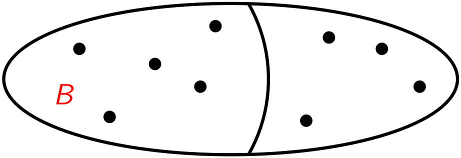
A partition into two bases

Henceforth M is a rank-r $\mathbb{K}$-representable matroid with ground set E.
A basis is a maximal independent set. We start by showing that we can assume M is partitioned into two bases.
We embed M in the projective space $P = \mathrm{PG}(r-1, \mathbb{K})$.



A partition into two bases

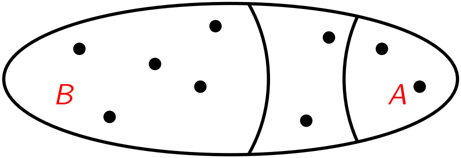
Let B be a basis of M.



A partition into two bases

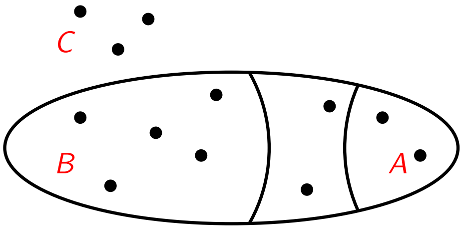
Let B be a basis of M.

Let A be a maximal independent set in $E - B$.



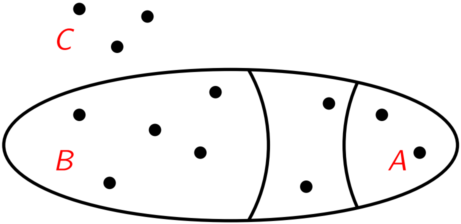
A partition into two bases

We add a set, C, of points freely to P, where $|C| = r - |A|$.



A partition into two bases

Next, we add an element in series to each element of $E - (A \cup B)$.



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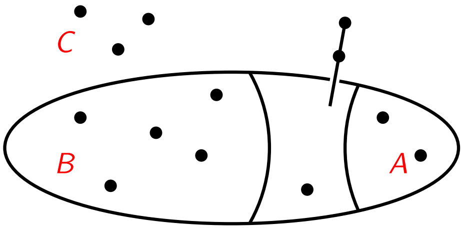

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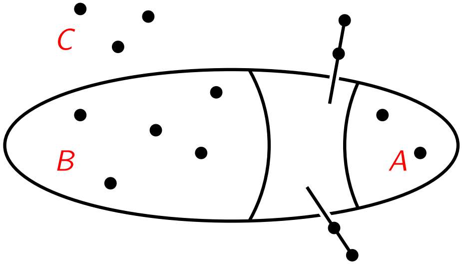
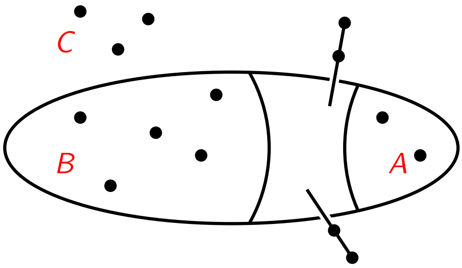


A partition into two bases

The resulting matroid is partitioned into two bases.



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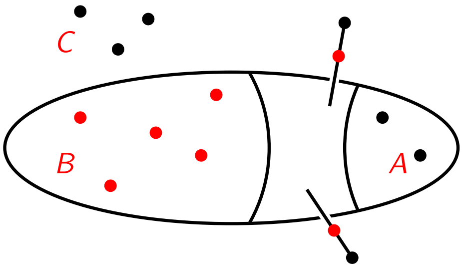

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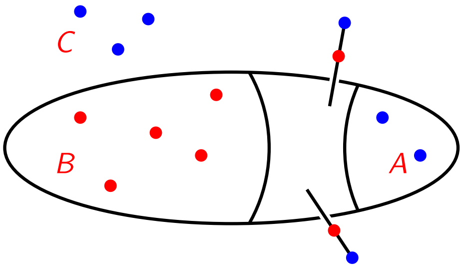
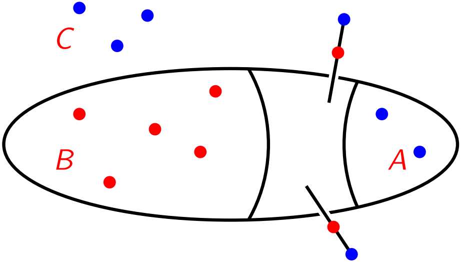


A partition into two bases

The resulting matroid is partitioned into two bases.
It certainly has M as a minor, so henceforth we assume M is partitioned into two bases.

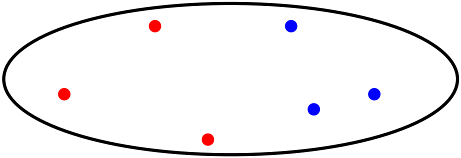


A partition into two independent hyperplanes

Next we claim that we can assume that M is partitioned into two independent hyperplanes.

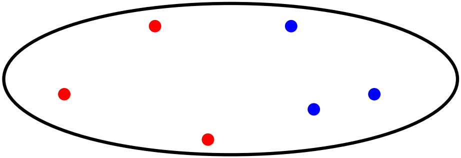
A partition into two independent hyperplanes

We embed M in $P = \mathrm{PG}(r+1, \mathbb{K})$, so $r(E) = r(P) - 2$. Let B_0 and B_1 be the bases that partition M.



A partition into two independent hyperplanes

Add points p and q freely to P.



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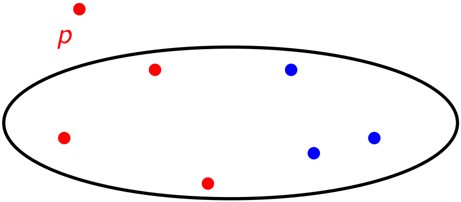

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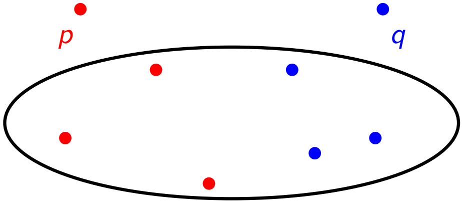
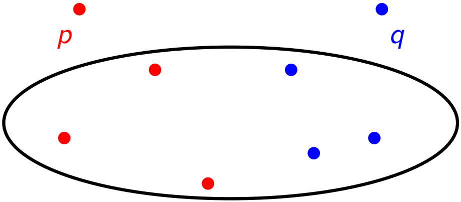


A partition into two independent hyperplanes

For each point $b \in B_0$ add a point freely to $\langle\{b, p\}\rangle$. For each point $b' \in B_1$ add a point freely to $\langle\{b', q\}\rangle$.



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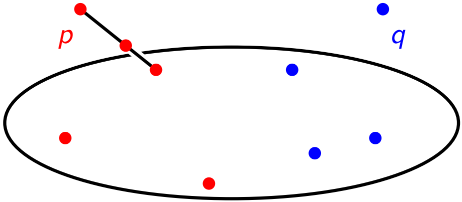

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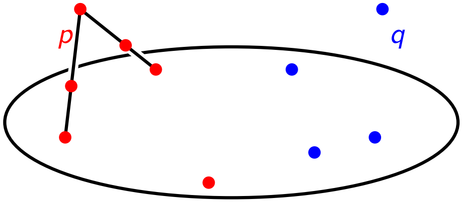

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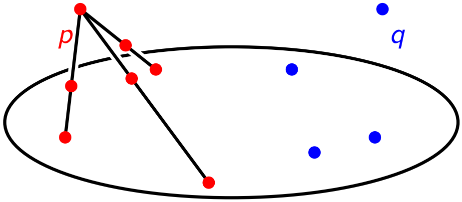

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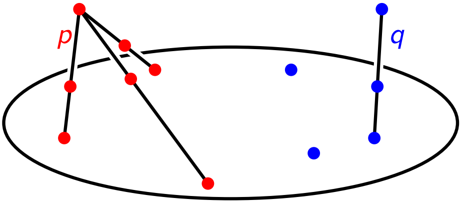

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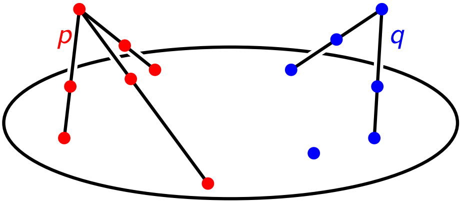

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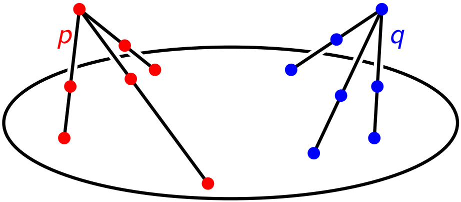
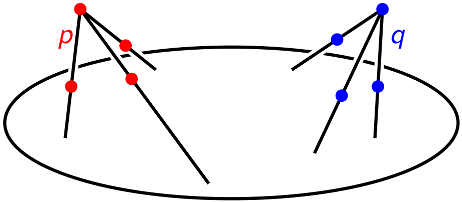


A partition into two independent hyperplanes

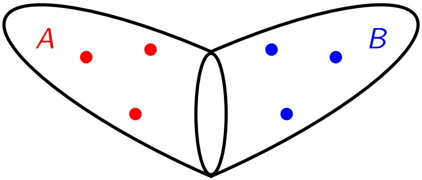
We delete the original points of M. The resulting matroid has a partition into two independent hyperplanes. It has M as a minor, so henceforth we assume M to be partitioned into two independent hyperplanes.



Invoking Ingleton's condition

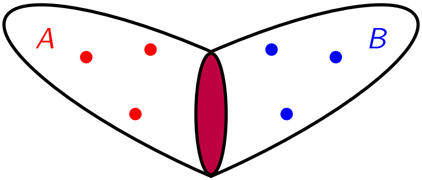
We embed M in $P = \mathrm{PG}(r, \mathbb{K})$, so that $r(E) = r(P) - 1$.

Let A and B be the two independent hyperplanes that partition M.



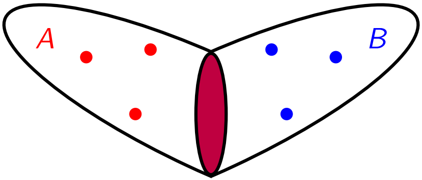
Invoking Ingleton's condition

Let V be the intersection of the spans of A and B.



Invoking Ingleton's condition

We add two points, p and q, freely to P.



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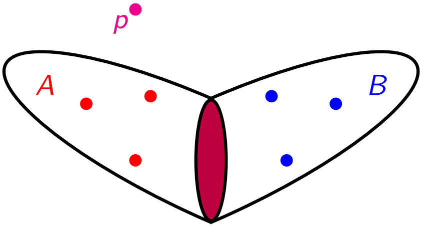

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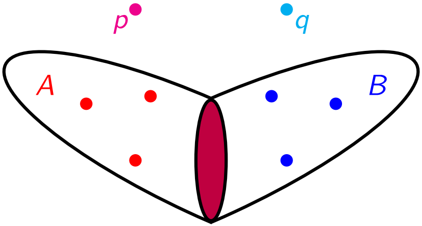
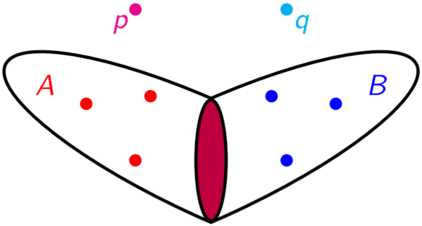


Invoking Ingleton's condition

Add a set, C, of points freely to $\langle V \cup \{p\} \rangle$, and a set, D, freely to $\langle V \cup \{q\} \rangle$, where $|C| + |D| = r + 1$.



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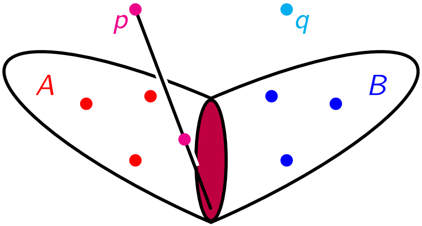

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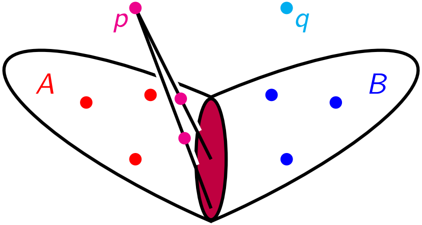

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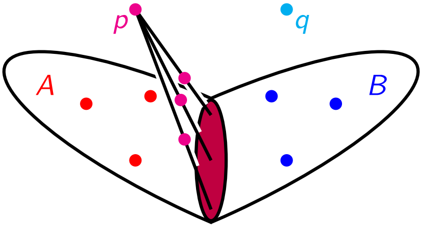

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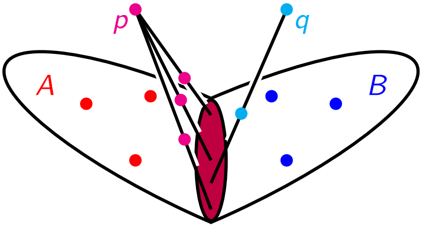

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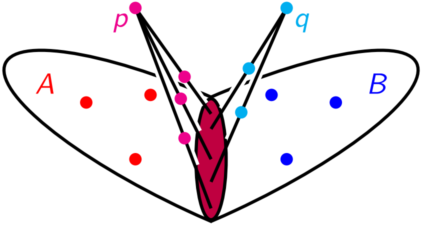
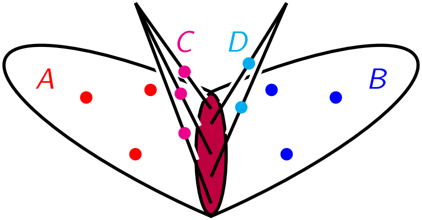


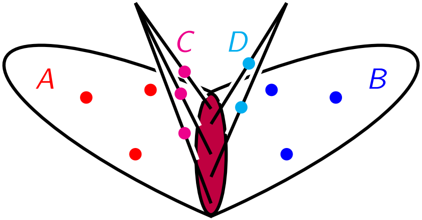
Invoking Ingleton's condition

Let N' be the matroid represented over $\mathbb{K}$ by the set of points $A \cup B \cup C \cup D$.



Invoking Ingleton's condition

A circuit is a minimal non-independent set. $C \cup D$ is a circuit-hyperplane of N'. Therefore, we can declare $C \cup D$ to be a basis. The resulting matroid is N.

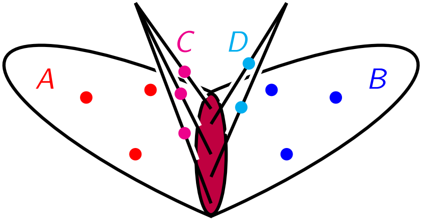


Invoking Ingleton's condition

Ingleton (1969) proved that if a matroid is representable over a field, then

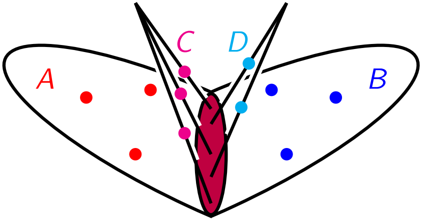
$$r(A) + r(B) + r(A \cup B \cup C) + r(A \cup B \cup D) + r(C \cup D) \leq \\ r(A \cup B) + r(A \cup C) + r(A \cup D) + r(B \cup C) + r(B \cup D)$$

for any subsets, A, B, C, and D.



Invoking Ingleton's condition

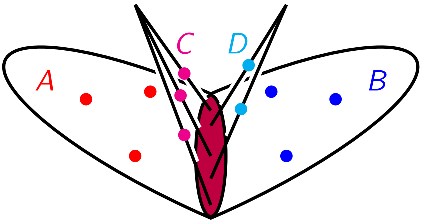
However $r(X \cup Y) = r$ in N, for any distinct $X,\ Y \in \{A,\ B,\ C,\ D\}$, as long as $\{X,\ Y\} \neq \{C,\ D\}$.



Invoking Ingleton's condition

Moreover,

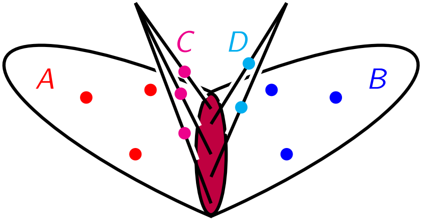
$r(A) = r(B) = r-1$, and $r(A \cup B \cup C) = r(A \cup B \cup D) = r(C \cup D) = r+1$.



Invoking Ingleton's condition

Therefore

$$r(A) + r(B) + r(A \cup B \cup C) + r(A \cup B \cup D) + r(C \cup D) = 5r + 1 >$$
$$5r = r(A \cup B) + r(A \cup C) + r(A \cup D) + r(B \cup C) + r(B \cup D)$$



Invoking Ingleton's condition

We conclude that N is not representable over any field.

It is fairly easy to see that deleting or contracting any element from N produces a $\mathbb{K}$-representable matroid. Hence N is an excluded minor for $\mathbb{K}$-representability.

N has an M-minor, so the proof is complete.