Analytic index theory

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1 Overview of the Field

The local index formula in noncommutative geometry, initially proved by Alain Connes and Henri Moscovici in 1995, [9], is an analytic formula for the index of Fredholm operators that can be shown to imply the Atiyah-Singer local index formula for Dirac type operators (as shown by Raphael Ponge, [13]) but is more general in that it provides a computable formula for noncommutative index problems. The formula is cohomological in the sense that it is seen to be connected to Connes (b, B) complex in cyclic theory.

The connection to the classical case is that for a compact manifold M, cyclic cohomology of $C^\infty(M)$ is just the de Rham homology of M with complex coefficients. In the noncommutative case the hope is that the cyclic theory, with its long exact sequences, can add computational and conceptual tools to the computation of index pairing problems.

However the original formulation requires working with the standard trace on the algebra of bounded operators on Hilbert space whereas long ago Alain Connes and Joachim Cuntz, [8], showed that any study of cyclic cohomology leads very naturally to traces on semifinite von Neumann algebras not just on the bounded operators on Hilbert space. In [4] and [1] examples of situations where noncommutative geometry methods may be used to compute a semifinite index formula were found.

Subsequently, inspired by a new proof of the local index formula by Higson, [11], the participants in this workshop discovered a new version of the local index formula that is proved in the context of unital semifinite spectral triples under weaker hypotheses than the original theorem [5, 6, 7]. It was proposed that this current workshop would investigate whether the theory had reached a sufficiently stable state so that writing a monograph on the topic would be of use to researchers interested in the problem.

2 Recent Developments and Open Problems

In the course of preparing for the workshop the relevance of a new viewpoint involving nonunital spectral triples (in the sense of [2]) became apparent. Nonunital spectral triples are relevant to the index theory of Dirac type operators on noncompact manifolds, and particularly to the study of pseudo-Riemannian manifolds which is a topic of considerable interest to physicists.

The notion of nonunital spectral triple has proved rather difficult to properly formulate, and efforts in this direction have actually required rather substantial advances in the theory of operator ideals in von Neumann algebras.

In current work, [3], a very general approach to the problem of proving the local index formula in semifinite noncommutative geometry for nonunital algebras is being explored. It relies on the proof of the index formula in [7], *all* other approaches being impossible *on quite fundamental grounds*.

In finding the minimal conditions and methods needed to push this proof through, this project helped constrain the possibilities for the definition of a nonunital spectral triple.

3 Scientific Progress Made

The main discussion point for the meeting was whether there were examples of nonunital spectral triples satisfying the tentative definition implied by the approach of [3]. Given that this latter paper was far from being in a final state there was considerable fluidity in the formulation and much interplay between putative examples and the formulation of general theory.

The key examples which should be covered by any useful index theory for nonunital spectral triples are:
• the generalisation of the Atiyah-Singer index formula to noncompact manifolds, as studied in [10],
• the L^2-index theorem for noncompact manifolds, previously unstudied,
• the Phillips-Raeburn theorem for actions of $\mathbb{R}$ on arbitrary C^*-algebras with unbounded trace.

A good understanding of all three problems was reached, and how they all fit into the one context provided by nonunital spectral triples.

One key unifying feature which we wish to point out is the discovery of a single analytic framework dealing with summability and smoothness of spectral triples. The summability and smoothness conditions are the main ingredients in the local index formula, and finding a single framework to discuss these two, seemingly disparate, features was a major conceptual breakthrough.

Less pleasingly, it was realised that *all* papers on the local index formula contain a hidden continuity assumption, which is nontrivial. It is known how to prove this continuity in the case of classical manifolds, though it requires the sophisticated machinary of complex powers of pseudodifferential operators. Higson presents a generalisation and simplification of this method in [11], and shows that it gives the required continuity for more exotic examples arising from manifolds. This continuity question is difficult to address in general, and a case by case approach is still required to check that it is satisfied.

4 Outcome of the Meeting

A consensus was reached on the correct definition of smooth finitely summable semifinite spectral triple. This was tested by extensive calculations undertaken both at BIRS and in the subsequent week spent by some of the participants at the University of Victoria. Three conjectures were made. The first was in the context of complete Riemannian manifolds M where a formula of Gromov-Lawson type [10] was proposed for Dirac type operators on vector bundles over M. The second was on a smooth version of the nonunital index formula of Phillips-Raeburn [12] for generalised Toeplitz operators. The third was on the extension of the L^2-index theorem to noncompact manifolds. In addition the text of [3] was re-worked to take into account the nature of these two conjectures.

The main outcome of the meeting is expected to be an extended account of the nonunital theory in a rather lengthy text that will greatly extend the current preliminary form of [3]. It is likely the Phillips-Raeburn theorem will be dealt with in a separate text.

References

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