Herz-Schur multipliers and spherical functions on groups and trees

Schur multipliers, X set

$\psi : X \times X \to \mathbb{C}$ is a Schur multiplier if

$\exists C \geq 0 :$ $\forall x_1, \dots, x_n \in X$ distinct elements $\forall a = (a_{ij})_{i,j=1}^{n} \in M_n(\mathbb{C})$

$$\left\| \left(\sum \psi(x_j, x_k)\, a_{jk} \right)_{j,k} \right\|_{M_n(\mathbb{C})} \leq C \, \|a\|_{M_n(\mathbb{C})}$$

The best constant C is denoted $\|\psi\|_{HS}$.

Herz-Schur multipliers

G locally compact group. $\varphi : G \to \mathbb{C}$ is called a Herz-Schur multiplier if φ is continuous and

$$\psi(x, y) = \varphi(y^{-1} x)$$

is a Schur multiplier on $X \times X$. In this case one put

$$\|\varphi\|_{HS} = \|\psi\|_S .$$

<u>Theorem</u> (Bożejko - Fendler 1984, De Canne + Haagerup 1984)

G locally compact group.

$\varphi : G \to \mathbb{C}$ continuous fct. The TFA:

1) φ is a Herz - Schur multiplier

2) φ is bounded and the map

$$\lambda(f) \to \lambda(\varphi \cdot f) \quad , \quad f \in L^1(G)$$

extends to a completely bounded map M_φ on $C_r^*(G)$

3) φ is bounded and the map

$$\lambda(x) \to \varphi(x) f(x), \quad x \in G$$

extends to a σ-weakly continuous map $\bar{M}_\varphi$ on $L(G)$.

4) φ is a complete bounded multiplier on the Fourier algebra $A(G) \simeq L(G)_*$

If the above equivalent conditions are satisfied then

$$\|\varphi\|_{HS} = \|M_\varphi\|_{cb} = \|\bar{M}_\varphi\|_{cb} = \|\varphi\|_{M_{cb}A(G)}.$$

Def (Cowling + H).
A locally compact group G is weakly amenable if $\exists \Lambda \geq 0$ and a net $(\varphi_\alpha)_{\alpha \in A}$ of Herz-Schur multipliers on G, such that

- $\|\varphi_\alpha\|_{HS} \leq \Lambda \quad \forall \alpha \in A$
- Each φ_α has compact support
- $\varphi_\alpha \to 1$ uniformly on compact sets.

Let $\Lambda(G)$ denote the smallest possible constant Λ when $\Lambda(G)$ is weakly amenable, $\Lambda(G) = +\infty$ otherwise.

Thm (Cowling + H), 1989

1) $H \subseteq G$ closed subgroup
Then $\Lambda(H) \leq \Lambda(G)$.

2) If G is unimodular and $H \leq G$ is a lattice (i.e. a closed, discrete subgroup of G), s.t. G/H has a finite G-invariant measure). Then
$$\Lambda(H) = \Lambda(G).$$

Theorem (H), unpublished

G discrete group. TFAE

(1) G is weakly amenable
(2) $C^*_r(G)$ has CBAP
(3) $L(G)$ has the w^*-CBAP.

Moreover $\Lambda(G) = \Lambda(C^*_r(G)) = \Lambda(L(G))$, where $\Lambda(C^*_r(G))$, $\Lambda(L(G))$ are the best upper bound for the approx property

Best constant CBAP

<u>Thm</u> (De Cannière + H, 1984, Cowling + H 1989)
All simple Lie groups of real rank one are weakly amenable. Specifically

(ℝ) $\Lambda(SO(n,1)) = 1$

(ℂ) $\Lambda(SU(n,1)) = 1$

(ℍ) $\Lambda(Sp(n,1)) = 2n-1$

(𝕆) $\Lambda(F_{4(-20)}) = 21$.
$n=2$

<u>Thm</u> (UH, unpublished)
If G is simple with real rank ≥ 2 (eg. $SL(n,\mathbb{R})$, $n \geq 3$) then G is ~~not~~ not weakly amenable.

<u>Thm</u> (V. Lafforgue & M. de la Salle (April 2010)
$SL(n,\mathbb{R})$, $n \geq 3$ fail to have the more general property AF (of (H + Kraus))
In particular for $n \geq 3$:

$C_r^*(SL(n,\mathbb{R}))$ has not OAP

$L(SL(n,\mathbb{R}))$ has not w^*-OAP

Moreover

$L^p(L(SL(n,\mathbb{R})))$ has not OAP for
when $p > 4$ or $1 < p < \frac{4}{3}$

Spherical functions

(G, K) Gelfand pair:

G l.c.-group $K \leq G$ compact subgroup s.t. $C_c(K\backslash G/K)$ is abelian with respect to convolution.

$\varphi : G \to \mathbb{C}$ continuous fct. is a spherical function if

$$\Lambda_\varphi(f) = \int_G f \cdot \varphi \, df \quad , \quad f \in C_c(K\backslash G/K)$$

is a character on $C_c(K\backslash G/K)$.

Simple Real rank one groups

G	$SO(n,1)$,	$SU(n,1)$,	$Sp(n,1)$	$F_{4(-20)}$
	$\cup$	$\cup$	$\cup$	$\cup$
K	$SO(n)$	$SU(n)$	$Sp(n)$	C

Spherical function index φ_s $s \in \mathbb{C}$, s.t.

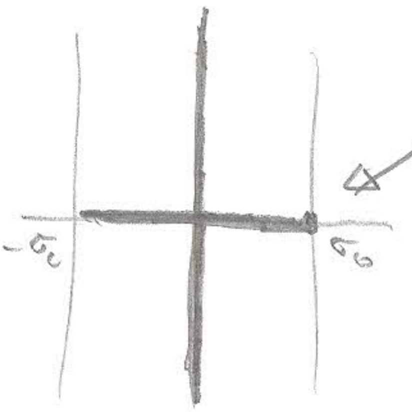
- $\varphi_s = -\varphi_{-s}$
- $\|\varphi_s\|_\infty < \infty \iff |\mathrm{Re}\, s| \leq \sigma_0$

$\varphi_s = \tilde{\varphi}_s \iff s \in \mathbb{R} \cup i\mathbb{R}$. $\quad (\tilde{\varphi}_s(x) = \overline{\varphi_s(x)}$

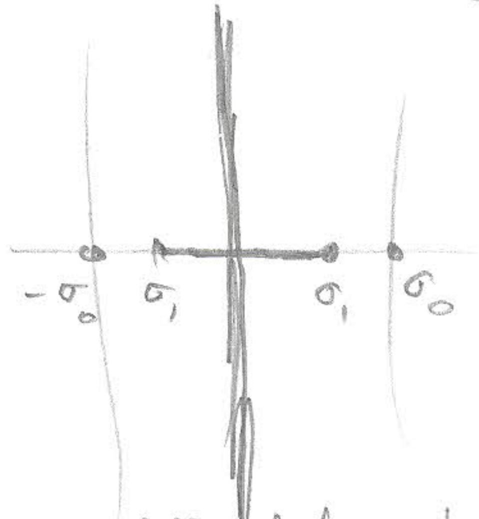
$\|\varphi_s\|_{\cancel{HS}} = 1 \iff \varphi_s$ positive def.

Property H (not T) | Property T (not H) 6

SO(n,1) SU(n,1) | Sp(n,1), $F_{4(-20)}$



pos. def. sph. fct.



pos. def sph. fct.

For these groups it turned out that

$$\Lambda(G) = \lim_{\sigma \to \sigma_0 -} \|\varphi_\sigma\|_{HS} = \begin{cases} 1 & SO(n,1),\ SU(n,1) \\ >1 & Sp(n,1),\ F_{4(-20)} \end{cases}$$

<u>Thm</u> (T. Steenstrup, 2009).

For SO(n,1), $s_0 = \frac{n-1}{2}$. Write $m = n-1$

Then for

$s = \sigma + it \qquad |\sigma| < m, \ t \in \mathbb{R}$

$$\|\varphi_s\|_{HS} = \frac{\Gamma(\frac{m}{2}+\sigma)\Gamma(\frac{m}{2}-\sigma)\Gamma(\frac{m}{2}+it)\Gamma(\frac{m}{2}-it)}{\Gamma(\frac{m}{2})\Gamma(\frac{m}{2})\left|\Gamma(\frac{m}{2}+s)\Gamma(\frac{m}{2}-s)\right|}$$

$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$

<u>Corollary</u> (case $n=2$)

For SO(2,1) $\quad \|\varphi_s\|_{HS} = (1 + \tan\pi\sigma \cdot \tanh\pi t)^{1/2}$

Groups acting on trees

a) Groups, who's Cayley graph is a (homogeneous) tree: X

$$\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{M} * \underbrace{(\mathbb{Z}_2 * \cdots * \mathbb{Z}_2)}_{N}$$

$\mathrm{Degree}(X) = 2M + N \qquad q = 2M + N - 1$

b) $PGL(2, \mathbb{Q}_q)$ acts on a Homogeneous tree V of degree $q+1$, Stabilizes group $PGL(2, \mathbb{Z}_q)$.

<u>Definition</u> (Figa-Talamanca, Picardello, Nebbia)

A spherical fct. on a ~~tree~~ homogenic tree ~~with~~ of degree $q+1$ (X, x_0) ~~function~~ is a fct

$$\varphi : X \to \mathbb{C}$$

~~radial~~ $\varphi(x)$ only depends on $d(x, x_0)$

$\cancel{L\varphi = s\varphi}$

$$L\varphi = s \cdot \varphi, \quad s \in \mathbb{C}.$$

where $L\varphi =$ "Laplace of φ"

$$L\varphi(x) = \frac{1}{q+1} \sum_{d(y,x)=1} \varphi(y).$$

$s \in \mathbb{C}$ is called the Eigenvalue of φ.

$\varphi(e) = 1, \quad \varphi(x) = s$, when $d(x,1) = s$.

$s \to \varphi_s$ is one to one

$$\boxed{\|\varphi_s\|_\infty < \infty \iff |\mathrm{Re}\, s|^2 + \left(\frac{q+1}{q-1}\right)^2 |\mathrm{Im}\, s|^2 \le 1}$$

~~Thm~~ (H + Steenstrup + Schwarz, 2009)

$$\|\varphi_s\|_{HS} < \infty \iff \begin{cases} |\mathrm{Re}\, s|^2 + \left[\frac{q+1}{q-1}\right]^2 |\mathrm{Im}\, s|^2 < 1 \\ \text{or } s = \pm 1 \end{cases}$$

In the first case

$$\|\varphi_s\|_{HS} = \frac{|1-s^2|}{1 - |\mathrm{Re}\, s|^2 - \left(\frac{q+1}{q-1}\right)^2 |\mathrm{Im}\, s|^2}$$

and in the second case,

~~and~~ $\|\varphi_1\|_{HS} = \|\varphi_{-1}\|_{HS} = 1$.

d) Corolly ~~Thm~~ (H + Steenstrup + Schwarz 2009)

The same formulas apply to the spherical functions on the Gelfand Pair

$$G = PGL(2, \mathbb{Q}_q)$$

$$K = PGL(2, \mathbb{Z}_q)$$

$\mathbb{Q}_q$ = q-adic numbers (q prime) ~~(...)~~

$\mathbb{Z}_q \subseteq \mathbb{Q}_q$ the set of p-adic integers.

$\mathbb{Q}_q$ = completion of $\mathbb{Q}$ in the metric

$d_p(x,y) = |x-y|_p$

where $|q|_p = q^{-k}$, where k is the power of q in the prime factorization of q

$\mathbb{Z}_q = \{q \in \mathbb{Q}_q : |q| \leq 1\}$.

The case $q = \infty$

The laplace does not make sense.

Use instead

$$\varphi_s(x) = s^{d(x, x_0)}$$

as "spherical functions". Natural limits for $q \to \infty$.

The $~~\|\varphi\|~~$

$$\|\varphi\|_{HS} = \frac{|1-s^2|}{1-|s|^2}$$

Corollary/ For $G = F_N$ $(2 \le N \le \infty)$. Then

$$\|M_{\varphi_s}\|_{cb} \le \frac{|1-s^2|}{1-|s|^2}, \qquad |s| < 1$$

Theorem (ess. due to Ozawa)

Let G be a finitely generated group and let $|x|$ denote the word length of $x \in G$ w.r.t. a given set of generators. If moreover

$$\varphi_s(x) = s^{|x|} \qquad |s| < 1$$

satisfies

$$\|\varphi_s\|_{HS} \le C \frac{|1-s|}{1-|s|} \qquad |s| < 1$$

for some constant $C \ge 1$, then G is weakly amenable with constant $\Lambda(G) \le C$.

<u>corollary</u> (originally due to DeCanniere–Haagerup).
$C^*_r(F_N)$ has CCAP.

<u>proof</u> $$\frac{|1-s^2|}{1-|s|^2} = \frac{|1+s|}{1+|s|} \cdot \frac{|1-s|}{1-|s|} \leq \frac{|1-s|}{1-|s|}$$

an appl. of corollary 1. and Ozawa's result with $p=1$.

<u>Thm</u> (J. Wysoczański, JFA 1995)
Let $G = \ast_{i \in I} G_i$ (free product)
$|x|$ = block length of $x \in G$ i.e. if

$$x = s_1 s_2 \dots s_n$$

where $x \in s_k \in G_{i_k} \setminus \{e\}$ $i_1 \neq i_2 \dots \neq i_n$
then $|x| = 1$.

Then with $\varphi_s(x) = s^{|x|}$ $(|s| < 1)$

$$\|\varphi_s\|_{HS} \leq \frac{|1-s|}{1-|s|}, \quad |s| < 1$$

and equality holds for $|I| = \infty$
and $|G_i| = \infty$ for all i

<u>corollary</u>
$G = G_1 * \dots * G_n$, $|G_i| < \infty$
Then $C^*_r(G)$ has CCAP.

Recent results

Thm (H + Möller, June 2010)

let

$$(A,\varphi) = \mathop{*}_{i\in I} (A_i, \varphi_i)$$

be an arbitrary free product of unital C^*-alg. with states φ_i (w. faithful GNS).

Then for each $s \in \mathbb{C}$, $|s|<1$

$\exists!$ $\Phi_s : A \to A$ completely bounded, st.

$$\Phi_s(1) = 1$$

$$(*) \quad \Phi_s(a_1 \dots a_n) = s^n a_1 \dots a_n$$

whenever $a_k \in A_{i_j}^0 = \ker(\varphi_{i_j})$, $i_1 \neq i_2 \neq \dots \neq i_k$

Moreover

$$\|\Phi_s\|_{cb} \leq \frac{|1-s|}{1-|s|}, \quad |s|<1$$

Corollary (Dykema, see also Ricard + Xu)

If $|I| < \infty$ and $A_1 \dots A_n$ are finite dimensional Then

$$(A,\varphi) = \mathop{*}_{i\in I} (A_i, \varphi_i)$$

has the CCAP.

<u>Refined result</u> (H + Müller, June 26, 2010)

What happens if s^n in (*) is exchanged by an arbitrary series of numbers. ~~The~~ $\varphi(n)$, $n \geq 0$:

$$M_\varphi : a_1 \ldots a_n \to \varphi(n)\, a_1 \ldots a_n$$

with $a_1 \ldots a_n$ as above and

$$M_\varphi(1) = \varphi(0) \cdot 1.$$

<u>Then</u>

$$\|M_\varphi\|_{cb} \leq |c| + \left\| \begin{pmatrix} \varphi(0)-\varphi(1) & 0 & \varphi(1)-\varphi(2) & \cdots \\ 0 & & 0 & \\ \varphi(1)-\varphi(2) & & \varphi(2)-\varphi(3) & \\ 0 & & & \\ \varphi(2)-\varphi(3) & & & \\ \vdots & & \vdots & \end{pmatrix} \right\|_{T(\ell^2(\mathbb{Z}))}$$

where $c = \lim_{n \to \infty} \varphi(n)$ (exists)

[Equality holds for $I = \infty$, and $\dim(A_i) = +\infty$. (follows from Wysoczanski's result.