Representing Polyhedra by Few Polynomials

Martin Henk

Banff, February, 2010
Why?

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- Martin Grötschel Impact for hard combinatorial optimization problems?, Constructions?, Approximations by polynomial inequalities?
- Bröcker, Scheiderer, ’84, . . . , ’89. Every basic closed semi-algebraic set $S \subset \mathbb{R}^n$ can be represented by at most $n(n + 1)/2$ polynomial inequalities, i.e., there exist $p_1, \ldots, p_{n(n+1)/2} \in \mathbb{R}[x]$ such that

$$S = \{ x \in \mathbb{R}^n : p_1(x) \geq 0, \ldots, p_{n(n+1)/2}(x) \geq 0 \}.$$
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In the case of basic open semi-algebraic sets, $n$ polynomials suffice, and both bounds are best possible.
• **Open:** For instance, the positive orthant
\[ \{ x \in \mathbb{R}^n : x_i > 0, \ 1 \leq i \leq n \} \] cannot be described by less than \( n \) strict polynomial inequalities.
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• **Closed:** For instance, the family of stacked cubes cannot be described by less than \( n(n + 1)/2 \) polynomial inequalities.

• Can the bound be improved, e.g., for convex sets?
Consequences for polyhedra

- Every polyhedron

\[ P = \{ x \in \mathbb{R}^n : \langle a_i, x \rangle \leq b_i, \ 1 \leq i \leq m \}, \]

given as the intersection of finitely many linear inequalities, can be described by at most \( n(n + 1)/2 \) polynomial inequalities. The interior of a polyhedron can even be described by \( n \) polynomials.

Can the bound be improved? Yes!
Can we (really) construct these (few) polynomials? It depends...!
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(Trivial) Examples
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- The (regular) $n$-cube

$$C_n = \{x \in \mathbb{R}^n : -1 \leq x_i \leq 1, 1 \leq i \leq n\}$$
(Trivial) Examples

- The (regular) $n$-cube (or any other parallelepiped)

\[ C_n = \{ x \in \mathbb{R}^n : -1 \leq x_i \leq 1, \; 1 \leq i \leq n \} \]

\[ = \{ x \in \mathbb{R}^n : (x_i)^2 \leq 1, \; 1 \leq i \leq n \} \].
(Trivial) Examples

- The $n$-simplex

\[ T_n = \{ x \in \mathbb{R}^n : x_i \geq 0, \ x_1 + \cdots + x_n \leq 1 \} \]
**(Trivial) Examples**

- The $n$-simplex

\[
T_n = \{ x \in \mathbb{R}^n : x_i \geq 0, x_1 + \cdots + x_n \leq 1 \}
= \left\{ x \in \mathbb{R}^n : x_i \left( 1 - \sum_{k=i}^{n} x_k \right) \geq 0, 1 \leq i \leq n \right\}.
\]
The regular $n$-crosspolytope

\[ C_n^* = \left\{ x \in \mathbb{R}^n : \sum |x_i| \leq 1 \right\} \]
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• The regular $n$-crosspolytope

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• Bosse, 2003, $n = 3$:

\[ p_{1/2} = \text{product of 4 facet defining inequalities which do not have an edge in common.} \]

\[ p_0 = \text{circumsphere of } C_3^*. \]
Each facet defining linear polynomial $b_i - a_i$, $x^i$ is a factor of one of the polynomials in a polynomial representation. Hence, the sum of the degrees in any polynomial representation is at least the number of facets of the polyhedron.

For every $k$-face there exist at least $n - k$ polynomials in a polynomial representation vanishing on aff $F$. Hence, a polynomial representation of a polyhedron having a vertex consists of at least $n$ polynomials.

For prisms and pyramids a polynomial representation can be constructed from a polynomial representation of the basis plus one additional polynomial.

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- For bi-pyramids?
Dimension 2

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- **Bernig, 1998.** For each (bounded) polygon we can construct 2 polynomial inequalities representing the polygon.
Let $P = \{x \in \mathbb{R}^2 : \langle a_i, x \rangle \leq b_i, 1 \leq i \leq m \}$ be a polygon.
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$$p_1(x) = (b_1 - \langle a_1, x \rangle) \cdot (b_2 - \langle a_2, x \rangle) \cdot \ldots \cdot (b_m - \langle a_m, x \rangle)$$
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$$p_0(x) = \text{concave polynomial through the vertices}$$

\{p_1(x) \geq 0\}

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• \( p_0(x) \) is of the form

\[
p_0(x) = 1 - \sum_{i=1}^{m} \lambda_i \left[ \frac{\langle w_i, x \rangle - l_i}{u_i - l_i} \right]^{2k},
\]

where \( w_i \) are normal vectors of support hyperplanes of the vertices,

\[
 l_i = \min_{x \in P} \langle w_i, x \rangle, \quad u_i = \max_{x \in P} \langle w_i, x \rangle
\]

and \( \lambda_i > 0 \) and \( k \) are chosen such that \( p_0(x) \) vanishes on the vertices.
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and $\lambda_i > 0$ and $k$ are chosen such that $p_0(x)$ vanishes on the vertices.

• In particular, the degree depends on metric properties of the polygon.
The obvious generalization of that 2-dimensional approach to consider polynomials

\[ p_k(x) = \prod \text{support hyperplanes of } k\text{-faces}, \quad k = 1, \ldots, n - 1, \]

\[ p_0(x) = \text{concave polynomial through the vertices} \]

does not work for \( n \geq 3 \) (see, e.g., crosspolytope).
- Bosse & Grötschel & H., 2005. For every $n$-dimensional polyhedron we can construct $2n$ polynomial inequalities representing the polytope.
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• Consequence: Let

$$S = \left\{ x \in \mathbb{R}^n : f_1(x) \geq 0, \ldots, f_m(x) \geq 0 \right\}$$

with $\deg(f_i) \leq d$. Then we can find $2\binom{n+d}{n} - 2$ polynomials representing the set $S$.  

Simple polytopes seem to be simpler

- Averkov&H., 2008. For every $n$-dimensional simple polytope we can construct $n$ polynomial inequalities representing the polytope.
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- Rough idea:
  - Let $l_i(x) = b_i - \langle a_i, x \rangle$ and let

$$P = \{ x \in \mathbb{R}^n : l_i(x) \geq 0, 1 \leq i \leq m \}.$$
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- Rough idea:
  - Let \( l_i(x) = b_i - \langle a_i, x \rangle \) and let
    \[
    P = \{ x \in \mathbb{R}^n : l_i(x) \geq 0, \ 1 \leq i \leq m \}.
    \]
  - Let
    \[
    \sigma_j(x) = \sum_{J \subseteq \{1,...,m\}} \prod_{k \in J} l_k(x)
    \]
    be the \( j \)-th elementary symmetric polynomial of \( l_1(x), \ldots, l_m(x) \).
$P = \{ x \in \mathbb{R}^n : \sigma_i(x) \geq 0, \ 1 \leq i \leq m \}$
Let $x \in \mathbb{R}^n$ such that $\sigma_i(x) \geq 0$, $1 \leq i \leq m$. Let

$$f(t) = \prod_{i=1}^{m} (l_i(x) + t) = \sum_{i=0}^{m} \sigma_i(x) t^{m-i}.$$ 

All coefficients are non-negative and hence, the roots $-l_i(x)$, $1 \leq i \leq m$, are non-positive, i.e., $x \in P$. 
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$P = \{ x \in \mathbb{R}^n : \sigma_i(x) \geq 0, \ 1 \leq i \leq m \}.$

If $P$ is simple then there exists an $\epsilon > 0$ such that for $x \in P + \epsilon B_n$

$$\sigma_i(x) \geq 0, \ 1 \leq i \leq m - n.$$
Let \( x \in P \). Since \( P \) is simple, there exist at most \( n \) linear forms \( l_i(x) \) vanishing at \( x \).

- Hence at least \( m - n \) linear forms are positive at \( x \) and so

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\sigma_j(x) > 0, \quad j \leq m - n.
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Let $x \in P$. Since $P$ is simple, there exist at most $n$ linear forms $l_i(x)$ vanishing at $x$.

- Hence at least $m - n$ linear forms are positive at $x$ and so
  \[ \sigma_j(x) > 0, \quad j \leq m - n. \]

- Thus by continuity we can find an $\epsilon > 0$ such that for all $x \in P + \epsilon B_n$
  \[ \sigma_j(x) \geq 0, \quad j \leq m - n. \]
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- If $P$ is simple then there exists an $\epsilon > 0$ such that for $x \in P + \epsilon B_n$
  \[ \sigma_i(x) \geq 0, \ 1 \leq i \leq m - n. \]

- Thus

  \[ P = \{x \in \mathbb{R}^n : \sigma_{m-n+i+1}(x) \geq 0, \ 0 \leq i \leq n - 1, \ p_\epsilon(x) \geq 0\} , \]

where $\{x \in \mathbb{R}^n : p_\epsilon(x) \geq 0\}$ is a "good" approximation of $P$. 
A simple polytope $P = \{ x \in \mathbb{R}^n : l_i(x) \geq 0, 1 \leq i \leq m \}$ is described by the $n$ polynomial inequalities

$$p_i(x) := \sigma_{m-n+i+1}(x) \geq 0, \quad 1 \leq i \leq n-1, \quad p_0(x) \geq 0,$$

where $p_0(x)$ is a concave polynomial passing through the vertices of $P$ and which approximates $P$ $\epsilon$-well.
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where $p_0(x)$ is a concave polynomial passing through the vertices of $P$ and which approximates $P$ $\epsilon$-well.

In particular, $p_i(x)$ vanishes on the $i$-faces of $P$, $i = 0, \ldots, n - 1$. 
Example

• For a regular simplex $P \subseteq \mathbb{R}^3$ we can choose

$$l_1(x) = 1 + x_1 - x_2 + x_3, \quad l_2(x) = 1 - x_1 + x_2 + x_3$$

$$l_3(x) = 1 + x_1 + x_2 - x_3, \quad l_4(x) = 1 - x_1 - x_2 - x_3.$$
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•

$$p_2 = l_1 l_2 l_3 l_4$$
$$p_1 = l_1 l_2 l_3 + l_1 l_2 l_4 + l_1 l_3 l_4 + l_2 l_3 l_4$$
$$= 4 \left( 1 - x_1^2 - x_2^2 - x_3^2 - 2 x_1 x_2 x_3 \right)$$
$$p_0 = 3 - x_1^2 - x_2^2 - x_3^2.$$
For $J \subset \{0, 1, 2\}$ let $P_J = \{x \in \mathbb{R}^3 : p_j(x) \geq 0, j \in J\}$
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The general case

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For every 3-dimensional polyhedra we can construct 3 polynomials representing the polyhedra.
Averkov & Bröcker, 2010. Let

\[ S = \{ x \in \mathbb{R}^n : f_i(x) \geq 0, \ 1 \leq i \leq m \} \]

be a basic closed semi-algebraic set.

- If all \( f_i(x) \) are linear, i.e., \( S \) is a polyhedron, then \( S \) can be represented by \( n \) polynomials.

- The proofs are "semi-effective".

Separation theorems based on Stone-Weierstrass approximation.
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- Let \( d \) be the maximal number of polynomials vanishing at a point. Then there exist \( d + 1 \) polynomials \( p_0, \ldots, p_d \) representing \( S \).
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  - If there are only finitely many points where \( d \) polynomials \( f_i(x) \) vanish then \( d \) polynomials suffice.
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How many polynomials are needed if we fix the degree?
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- H.&Matzke, 2007. Let $P$ be a 2-polyhedron with $m$ edges and let $k \in \mathbb{N}$. Then one can construct $d$ polynomials $q_1, \ldots, q_d$ of degree at most $k$ and with

$$d \leq \left\lceil \frac{m}{k} \right\rceil + \lfloor \log_2(k - 1) \rfloor + 1$$

such that

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- Best possible for $k = O(m/\log_2 m)$. 

Averkov&Bey, 2010. $d \leq \max \left\lfloor \frac{m}{k} \right\rfloor, \lfloor \log_2(m) \rfloor$, and it is best possible for any $k$ among a certain family of polynomials.
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More Open Questions

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- \ldots
Thank you for your attention!