Models for Spatial Extremes

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Work supported in part by NSF-DMS 0905315

Outline

1. Introduction
   - Statistics for Extremes
   - Climate and Weather
2. Modeling Climate Spatial Effects for Extremes
   - Historical Methods
   - Bayesian Hierarchical Models
3. Models for Weather Spatial Effects for Extremes
   - Max-stable Processes
   - Fitting Max-stable Processes: Composite Likelihoods
   - Other approaches
4. Modeling Both Climate and Weather Effects
   - Possible Approaches
   - Employing a Max Stable Process Model
   - Appropriate Bayesian Inference for Composite L'hoods
   - Implementation in a BHM and Results

Extreme Value Analyses

General Idea: Distributions suggested by the asymptotic theory are fit using only data considered to be extreme.

Two approaches:

1. *Block maximum data*
   - GEV(μ, σ, ξ)
2. *Threshold exceedance data*
   - GPD($\tilde{\sigma}_u$, ξ, ζ)
   - PPE(μ, σ, ξ)

- ξ determines the type of tail and is difficult to estimate.
- always data poor in extreme value analyses.
- risk often summarized in terms of return levels.

Extreme Values in more than one dimension

Multivariate Extremes

- Theory well developed.
- Some existing parametric models for moderate dimension.
- Block maximum and threshold exceedance approaches have both been used.

Extremal Processes

- Theory well developed.
- Some existing models for max-stable processes.
- Only block maximum approaches to this point.
- Although process models ($d = \infty$), only the bivariate (trivariate?) joint distributions known in closed form.

In practice, marginal effects and dependence structure are handled separately, often unit-Frechet ($P(Z \leq z) = \exp(-z^{-1})$) are chosen.

This is vaguely "copula like"...

...but don't say "copula" at an EV meeting!

"Recipe for Disaster: The Formula That Killed Wall Street"
By Felix Salmon, *Wired Magazine* 02.23.09

$$\Pr[T_A < 1, T_B < 1] = \phi_2(\phi^{-1}(F_A(1)), \phi^{-1}(F_B(1)), \gamma)$$

Special issue of *Extremes* devoted to copulas:
Extremes 9:1, March 2006.

To be an EV (max-stable) copula:

$$C(u_1^t, \ldots, u_d^t) = C^t(u_1, \ldots, u_d)$$

Spatial Extremes, Climate, and Weather

There are two spatial effects at work in the data: climate and weather effects.

Q: What's the difference?

Climate vs. Weather

"Climate is what you expect, weather is what you get"

...but this doesn't really apply when doing an EV analysis.

Instead: Climate is the *distribution* from which weather is drawn (not just the expected value).

Climate and Weather Spatial Effects

Spatial dependence in data from two sources:

- local dependence due to events that effect more than one location (weather).
- regional dependence due to similar characteristics between locations (climate).

In terms of a statistical model:

climate effects: how the marginal distribution changes with location.

weather effects: the joint behavior of multiple locations.

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Goal of Modeling Climate Extremes

Precipitation Atlases:
NOAA's provides "official" estimates for extreme precipitation for US locations. `http://www.nws.noaa.gov/ohd/hdsc/noaaatlas2.htm`

Colorado 40°N 105°W (Boulder)

Map	Prcp (inches)	Prcp Intensity (in/hr)
2-year 6-hour	1.49	0.25
2-year 24-hour	2.13	0.09
100-year 6-hour	3.68	0.61
100-year 24-hour	5.06	0.21

Estimates based on NOAA Atlas 2, 1973.

- Precip. atlases used for designing bridges, drainage, etc.
- Other uses for climatological extremes:
  - Wind: structural design of buildings, power.
  - Sea level: storm surge needed for sea walls.

Historical Approach for Modeling Climate Extremes

Regional Frequency Analysis

- Traces roots to Dalrymple (1960).
- Thorough treatment of modern practice: Hosking and Wallis (1997)
- Basic idea is "trading space for time"
- Basic steps:
  1. Define and test homogeneous regions
  2. Normalize (block maximum) data using "index flood" – often a mean of annual maxima.
  3. Combine data and estimate parameters (L-moments)
  4. Un-normalize data by index flood to get estimates at each location
- NOAA using a variant of RFA to update its maps.

Recent Approach: Spatial Hierarchical Models

Basic idea: Assume there is a latent spatial process that characterizes the behavior of the data over the study region.

- much of early work done in epidemiology
- Diggle et al. (1998)
- Banerjee et al. (2004)
- Climate/weather applications: Cressie, Wikle, Berliner, others

Why bother? Latent process too complex to capture with fixed effects; covariates not rich enough.

Spatial HM General Framework

Bayesian formulation, three levels.

$$\pi(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 | \boldsymbol{y}) \propto \underbrace{\pi(\boldsymbol{y}|\boldsymbol{\theta}_1)}_{data} \underbrace{\pi(\boldsymbol{\theta}_1|\boldsymbol{\theta}_2)}_{process} \underbrace{\pi(\boldsymbol{\theta}_2)}_{prior},$$

Data level: Likelihood characterizing the observed data *given the parameters at the process level*.
Process level: Latent process captured by spatial model for the data level parameters.
Prior level: Prior distributions put on the parameters in the process level.

Typically an assumption of *conditional independence* made at the data level and joint likelihood is the product of the individual likelihoods at each location.

Overview of a Case Study: Regional Climate Model Precipitation Extremes

Cooley and Sain (2008)

Data: Output from a RCM for the western US, $\sim$ 2500 locations. Both control and future runs modeled simultaneously.

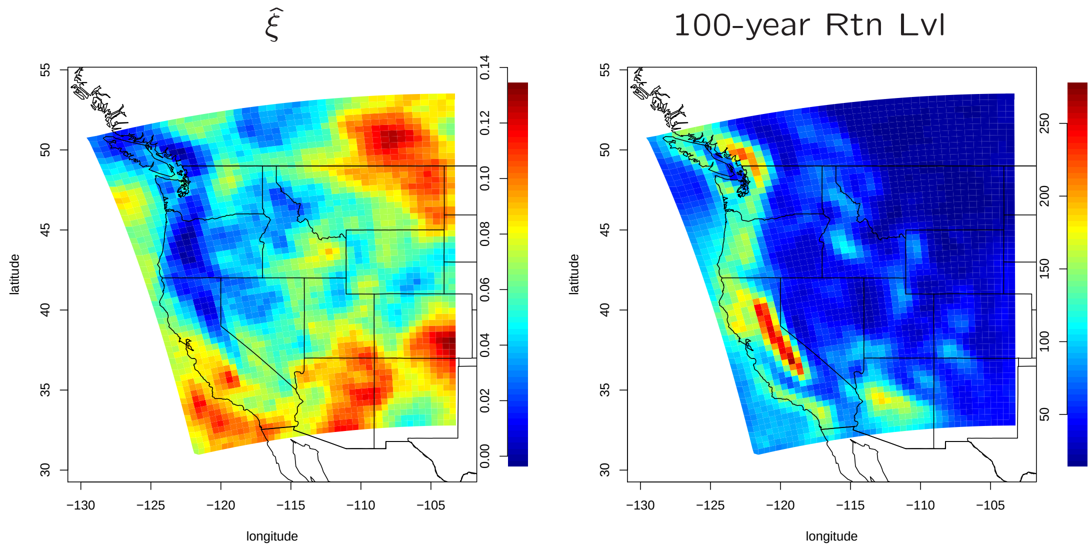
$$\pi(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 | \boldsymbol{y}) \propto \underbrace{\pi(\boldsymbol{y}|\boldsymbol{\theta}_1)}_{data} \underbrace{\pi(\boldsymbol{\theta}_1|\boldsymbol{\theta}_2)}_{process} \underbrace{\pi(\boldsymbol{\theta}_2)}_{prior},$$

Data Level: Point process model for threshold exceedances. Conditional independence assumed.

Process Level: Multivariate IAR model for $(\boldsymbol{\mu}, \boldsymbol{\sigma}, \boldsymbol{\xi})$.

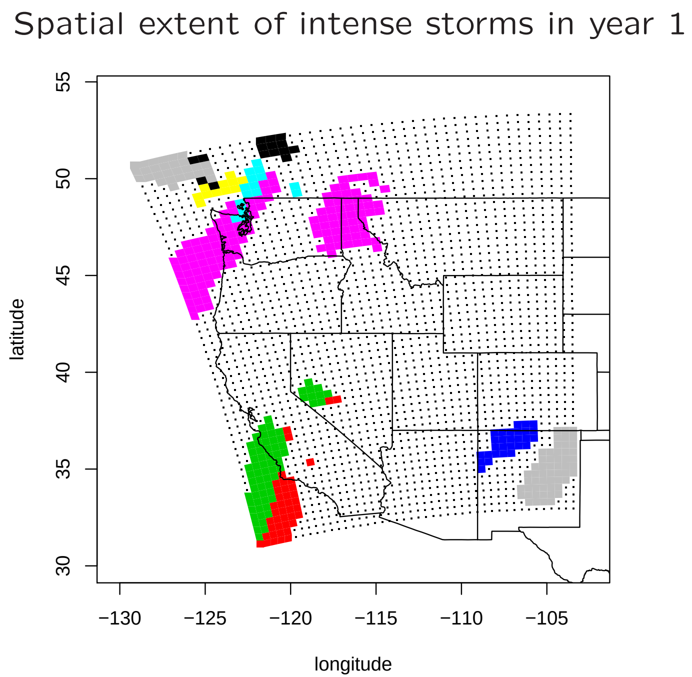
- Q has dimension 14784 x 14784.
- 29598 (non-indep) parameters, effective number $\sim$ 4250
- Inference via Gibbs Sampler

Model Results



- seems to capture *climate effects*
- latent spatial model needed
  - available covariates inadequate to capture effects
  - borrows strength across locations for estimates
- uncertainty easily obtained from MCMC runs

What about the weather effects?



Clearly, conditional independence assumption does not hold.

Q1: How to model extreme weather events' spatial effects?
Q2: If one is *only* interested in climate extremes questions, does one need to worry about ignoring the weather effects?

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Max-stable Processes

Let $Y_m(\boldsymbol{x}), \boldsymbol{x} \in \mathcal{D}, m = 1, \ldots, n$ be independent copies of $Y(\boldsymbol{x})$, and let $M_n(\boldsymbol{x}) = \max_m Y_m(\boldsymbol{x})$. $Y(\boldsymbol{x})$ is termed max-stable if there exist $a_n(\boldsymbol{x})$ and $b_n(\boldsymbol{x})$ such that

$$\mathbb{P}\left(\frac{M_n(\boldsymbol{x}) - b_n(\boldsymbol{x})}{a_n(\boldsymbol{x})} \leq y(\boldsymbol{x})\right) = \mathbb{P}(Y(\boldsymbol{x}) \leq y(\boldsymbol{x})).$$

- Max-stability is foundation of EVT.
- Limit process for site-wise maxima.
- Use is justified by extreme value theory.
- Suitable for modeling fields of block (annual) maximum data.
- From an EV perspective, max-stable processes are the correct answer to Q1.
- A few models have been proposed.

Point process formulation of max-stable processes

(de Haan and Ferreira, 2006, Corollary 9.4.5)

$Z(\boldsymbol{x}), \boldsymbol{x} \in \mathcal{D}$ is max-stable with unit Fréchet marginals

iff

There exist iid positive stochastic processes $V_1(\boldsymbol{x}), V_2(\boldsymbol{x}), \ldots$ with $E[V_i(\boldsymbol{x})] = 1$ for all $\boldsymbol{x} \in \mathcal{D}$ and $E[\sup_{\boldsymbol{x} \in \mathcal{D}} V(\boldsymbol{x})] \leq \infty$ and an independent point process $\{\eta_i\}_0^\infty$ on $(0, \infty]$ with intensity measure $r^{-2}dr$ such that $Z(\boldsymbol{x}) \stackrel{d}{=} \max_{i=1,2,\ldots} \eta_i V_i(\boldsymbol{x})$.

Models for Max-stable Processes

- Smith (1990)
- Schlather (2002)
- de Haan and Pereira (2006)
- Kabluchko et al. (2009)

Smith Model for a Max-stable Process

$$\Pr[Z(x_i) \leq z_1 \quad , Z(x_j) \leq z_2] =$$

$$\exp\left\{ -\frac{1}{z_1}\Phi\left(\frac{a}{2} + \frac{1}{a}\log\frac{z_2}{z_1}\right) - \frac{1}{z_2}\Phi\left(\frac{a}{2} + \frac{1}{a}\log\frac{z_1}{z_2}\right) \right\}$$

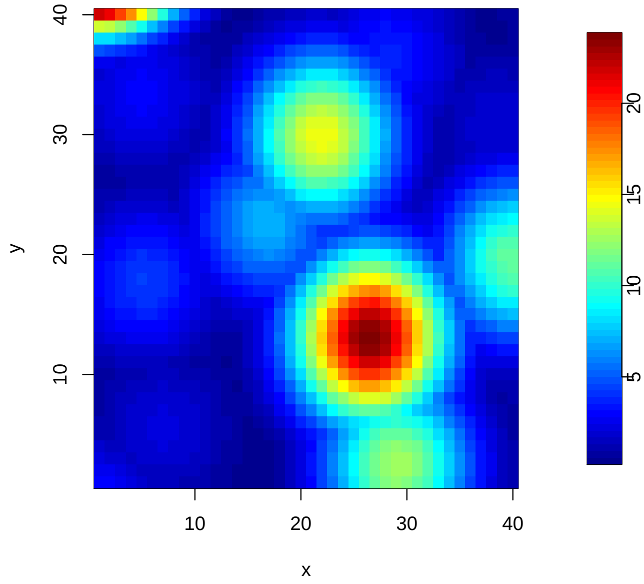
where $a^2 = (x_i - x_j)^T \Sigma^{-1} (x_i - x_j)$.

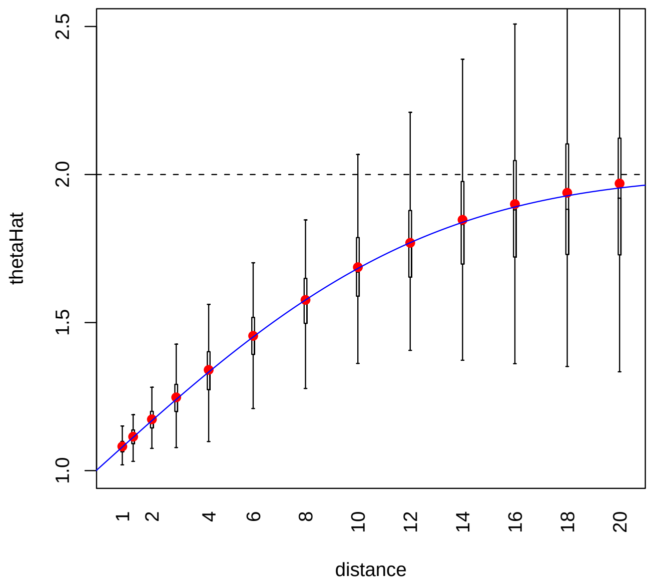
- Characterized by the parameter $\Sigma = \begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{1,2} & \sigma_{2,2} \end{bmatrix}$
- Assumes the marginals are unit Fréchet: $\mathbb{P}(Z(x_i) \leq z) = \exp(-z^{-1})$.
- If marginals not Fréchet, then a marginal transformation can be performed.
- Has a "storm" interpretation.
- We use this in the data level of our hierarchical model described later.

Smith Model for a Max-stable Process

$$\Pr[Z(x_i) \le z_1 \quad, Z(x_j) \le z_2] = \exp\left\{-\frac{1}{z_1}\Phi\left(\frac{a}{2}+\frac{1}{a}\log\frac{z_2}{z_1}\right)-\frac{1}{z_2}\Phi\left(\frac{a}{2}+\frac{1}{a}\log\frac{z_1}{z_2}\right)\right\}$$

where $a^2 = (x_i - x_j)^T \Sigma^{-1}(x_i - x_j)$.

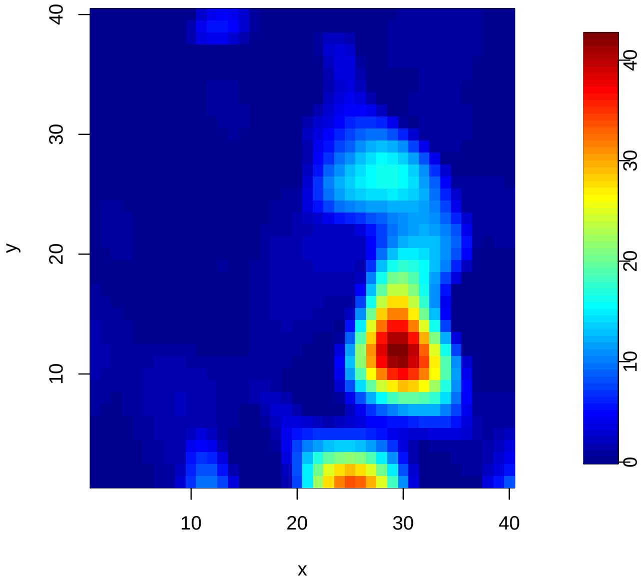


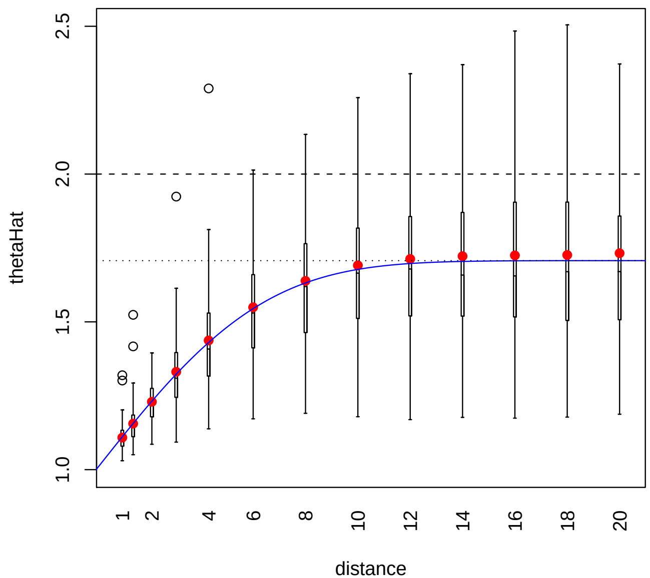


Schlather (2002) Model for a Max-stable Process

$$\Pr[Z(x_i) \le z_1, Z(x_j) \le z_2] =$$
$$\exp\left\{-\frac{1}{2}\left(\frac{1}{z_1}+\frac{1}{z_2}\right)\left(1+\sqrt{1-2(\rho(z_1-z_2)+1)\frac{z_1 z_2}{(z_1+z_2)^2}}\right)\right\}$$

where ρ is the correlation function of the standard normal Gaussian process.





Brown-Resnick Model for a Max-stable Process

Kabluchko et al. (2009)

$$\text{Pr}[Z(x_i) \leq z_1, Z(x_j) \leq z_2] =$$
$$\exp\left\{ - \left[e^{-z_1} \Phi \left(\frac{\sqrt{\rho(x_i - x_j)}}{2} + \frac{z_2 - z_1}{\sqrt{\rho(x_i - x_j)}} \right) + e^{-z_2} \Phi \left(\frac{\sqrt{\rho(x_i - x_j)}}{2} + \frac{z_1 - z_2}{\sqrt{\rho(x_i - x_j)}} \right) \right] \right\}$$

- Gives realistic-looking realizations.
- As distance increases, observations become independent.
- Interesting construction from Brownian motion with drift–the stationary process is constructed as a maximum of non-stationary processes.

Fitting Max-stable Processes

For the max-stable process models, only the *bivariate* distributions are known (trivariate for Smith model?). How does one fit a model to K observations?

A: Composite Likelihoods

Composite likelihoods are used to obtain estimating equations when the true likelihood is too difficult or impossible to obtain (Lindsay, 1988).

Pairwise Likelihoods

Since we have the bivariate distributions we will use the *pair-wise* likelihood.

Assume independent observations $\boldsymbol{y}_m = (y_m^{(1)}, y_m^{(2)}, \ldots, y_m^{(k)})^T$, $m = 1, \ldots, m$ arise from a probability model with density $f(\boldsymbol{y}; \theta)$ which has bivariate marginals $f(y_m^{(i)}, y_m^{(j)}; \theta)$. Then the pairwise log-likelihood is given by

$$\ell_p(\theta; \mathbf{y}) = \sum_{m=1}^{n} \sum_{i=1}^{K-1} \sum_{j=i+1}^{K} \log f(y_m^{(i)}, y_m^{(j)}; \theta).$$

Things to keep in mind:

- Not a true likelihood.
- Over-uses the data – each observation appears $K-1$ times.

Frequentist Methods for Composite Likelihoods

Point estimation achieved by maximizing the composite likelihood; MCLE denoted $\widehat{\theta}_c$.

1. Estimation is unbiased.
2. Uncertainty estimates achieved via the information sandwich approach.

$$\sqrt{n}\{H(\theta_0)J(\theta_0)^{-1}H(\theta_0)\}^{1/2}(\widehat{\theta}_c - \theta_0) \longrightarrow N(0, \mathrm{Id}_p),$$

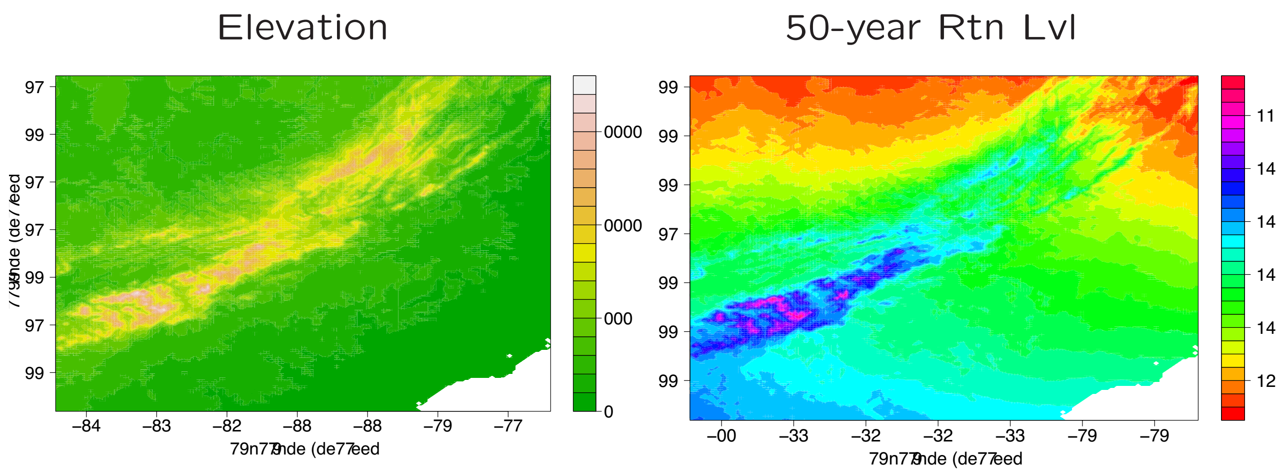
where $H(\theta_0) = \mathbb{E}[\nabla^2 \ell_c(\theta_0; Y)]$ and $J(\theta_0) = \mathrm{Var}[\nabla \ell_c(\theta_0; Y)]$.

Pairwise Likelihoods for Max-stable Processes

Padoan et al. (2009)

Use pairwise likelihood approach for annual maximum precipitation data.

Marginals: trend surface of longitude, latitude, and elevation.



Findings:

1. Improved modeling of joint occurrence.
2. Some reason to question the fit of marginal distributions.

Are MS Processes *Really* the Right Answer?

From a theoretical EV perspective, yes.
But, there are some limitations.

+ Justified by EV theory.
+ Able to describe asymptotic dependence.
– Describe everything that is asymptotically independent as exactly independent.
– Suitable for observations that are annual maxima.
  - Do we lose information about storm dependence?
  - How can we deal with threshold exceedances?

Despite the theoretical justification, there may be reason to use models other than max-stable process models.

At least it is a debate worth having.

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Modeling *Both* Climate and Weather Effects

SHM Revisited

$$\pi(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 | \boldsymbol{y}) \propto \underbrace{\pi(\boldsymbol{y}|\boldsymbol{\theta}_1)}_{data} \underbrace{\pi(\boldsymbol{\theta}_1|\boldsymbol{\theta}_2)}_{process} \underbrace{\pi(\boldsymbol{\theta}_2)}_{prior},$$

- To capture the weather effects, one needs to capture the dependence among observations due to weather events.
- Would have spatial model in *both* data and process levels.

Possible Approaches for Likelihoods

- Multivariate Extreme Value Models
- Copula Approaches
- Max-stable Process Models

Possible Approaches for Likelihoods

- Multivariate Extreme Value Models
  - + Models can be applied to both block maxima or threshold exceedance data.
  - – Existing models (e.g., Tawn (1990); Cooley et al. (2010)) applicable only to data of relatively low dimension ($d \sim 5$).
- Copula Approaches
- Max-stable Process Models

Possible Approaches for Likelihoods

- Multivariate Extreme Value Models
- Copula Approaches
  - Gaussian copula (Sang and Gelfand, 2010).
    - \+ Very natural, intuitive approach.
    - \+ Computationally feasible.
    - ◦ Leads to an asymptotically independent model.

$$\lim_{u \to \infty} \mathbb{P}(X_i > u | X_j > u) = 0$$

  - Dirichlet process (Fuentes et al., 2009)
    - \+ Both asymptotic dependence and asymptotic independence possible.
    - ◦ Unclear ties to EVT.
- Max-stable Process Models

Possible Approaches for Likelihoods

- Multivariate Extreme Value Models
- Copula Approaches
- Max-stable Process Models

Capturing the Weather Effects with a Max-stable Process

$$\pi(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 | \boldsymbol{y}) \propto \underbrace{\pi(\boldsymbol{y}|\boldsymbol{\theta}_1)}_{data} \underbrace{\pi(\boldsymbol{\theta}_1|\boldsymbol{\theta}_2)}_{process} \underbrace{\pi(\boldsymbol{\theta}_2)}_{prior},$$

We aim to employ a likelihood from a max-stable process at the data level.

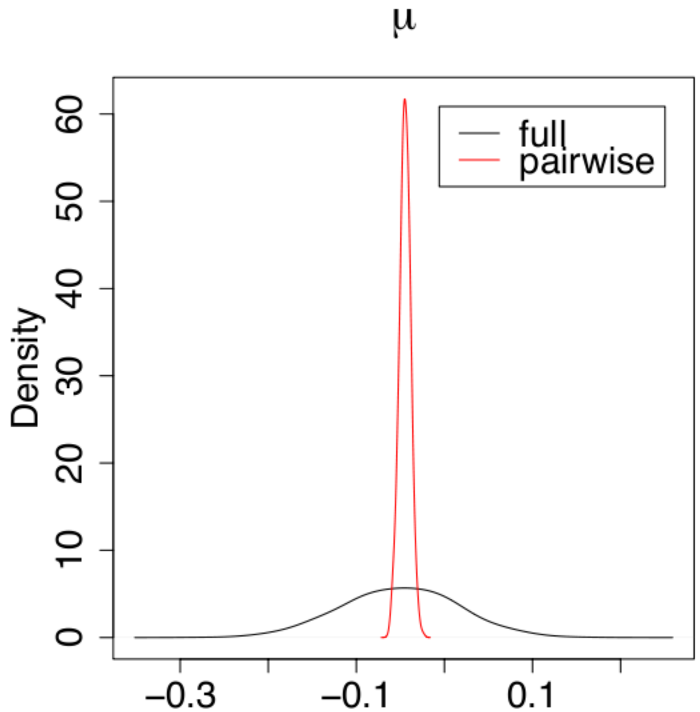
Challenge:

- Since only the bivariate distributions are known, we cannot use the correct likelihood $\rightarrow$ composite likelihood.
- Frequentist methodology exisits for composite likelihoods, can ideas be extended to obtain appropriate inference in a Bayesian setting?

Bayesian Methods for Composite Likelihoods?

posterior $\propto$ likelihood * prior

If true likelihood is replaced by pairwise likelihood, resulting posterior is too narrow.



50 replicates of a mean-zero Gaussian process with exponential covariance structure, observed at 20 locations

Appropriate Bayesian Inference

Ribatet, Cooley, and Davison (2009) investigate appropriate Bayesian inference for composite likelihoods by adjusting the MH acceptance ratio.

Two approaches suggested:

1. Magnitude adjustment
2. Curvature adjustment

As the curvature adjustment appears to outperform the we focus on it here.

Curvature Adjustment

In the context of performing likelihood ratio tests (frequentist), Chandler and Bate (2007) suggest replacing the composite likelihood with an adjusted inference function:

$$\ell_A(\boldsymbol{\theta}; y) = \ell_c(\boldsymbol{\theta}^*; y),$$
$$\boldsymbol{\theta}^* = \widehat{\boldsymbol{\theta}}_c + M^{-1} M_A(\boldsymbol{\theta} - \widehat{\boldsymbol{\theta}}_c),$$
$$\text{where } M^T M = -H(\boldsymbol{\theta}_0),\ M_A^T M_A = H(\boldsymbol{\theta}_0) J(\boldsymbol{\theta}_0)^{-1} H(\boldsymbol{\theta}_0).$$

- $\ell_A(\widehat{\boldsymbol{\theta}}_c) = \ell_c(\widehat{\boldsymbol{\theta}}_c)$.
- the Hessian of $\ell_A(\hat{\boldsymbol{\theta}}_{\boldsymbol{c}}; y)$ is $H_A(\boldsymbol{\theta}_0) = H(\boldsymbol{\theta}_0) J(\boldsymbol{\theta}_0)^{-1} H(\boldsymbol{\theta}_0)$.
- Note: to calculate $\ell_A(\boldsymbol{\theta})$, $\widehat{\boldsymbol{\theta}}_c$ must be known.

Behavior of Approximation

Unlike frequentist work, need to approximate the likelihood at *all* values of θ rather than only the "true" value θ_0.

True Likelihood:

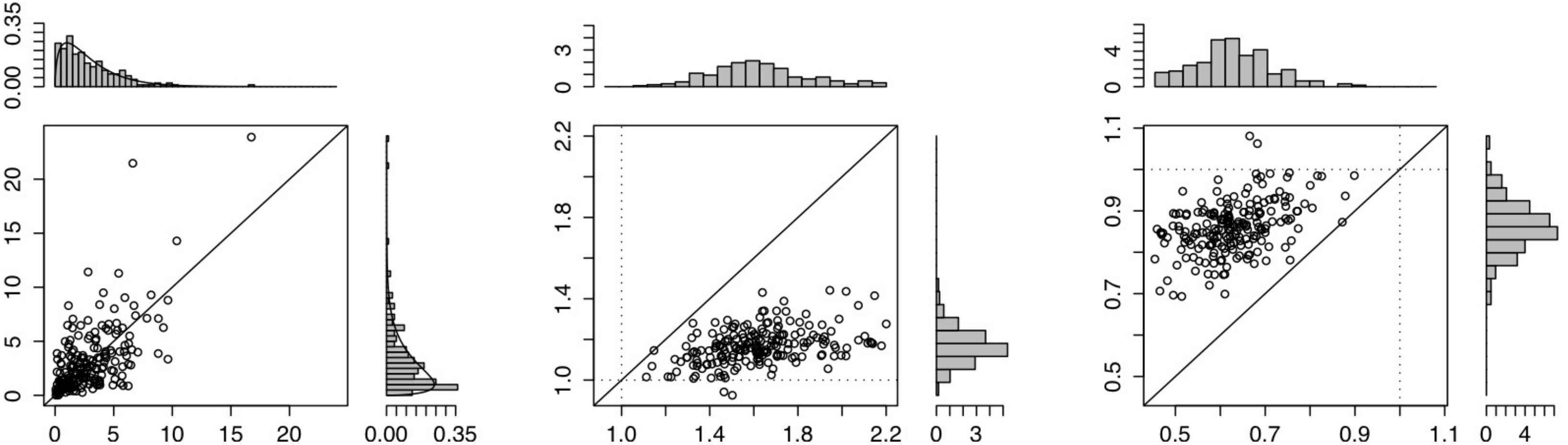
$$2\{\ell(\widehat{\theta}) - \ell(\theta_1)\} \xrightarrow{d} Y^T Y, \text{ where}$$

$$Y \sim N(I(\theta_0)^{1/2}(\theta_1 - \theta_0), \mathrm{Id}_p)$$

Adjusted Pairwise Likelihood:

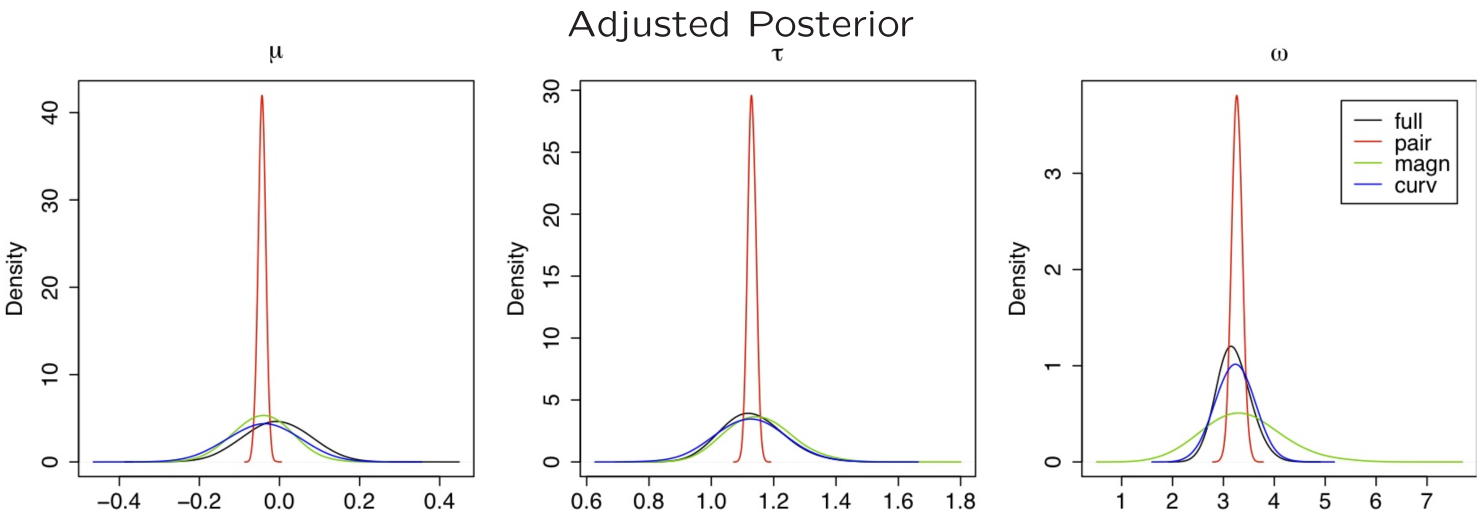
$$2\{\ell_{\text{curv}}(\widehat{\theta}_c) - \ell_{\text{curv}}(\theta_1)\} \xrightarrow{d} X^T X, \text{ where}$$

$$X \sim N(\{H(\theta_0)J^{-1}(\theta_0)H(\theta_0)\}^{1/2}(\theta_1 - \theta_0), \mathrm{Id}_p)$$



Simulation results

$\ell_A(\boldsymbol{\theta})$ used in MH ratio, can show detailed balance condition is met.



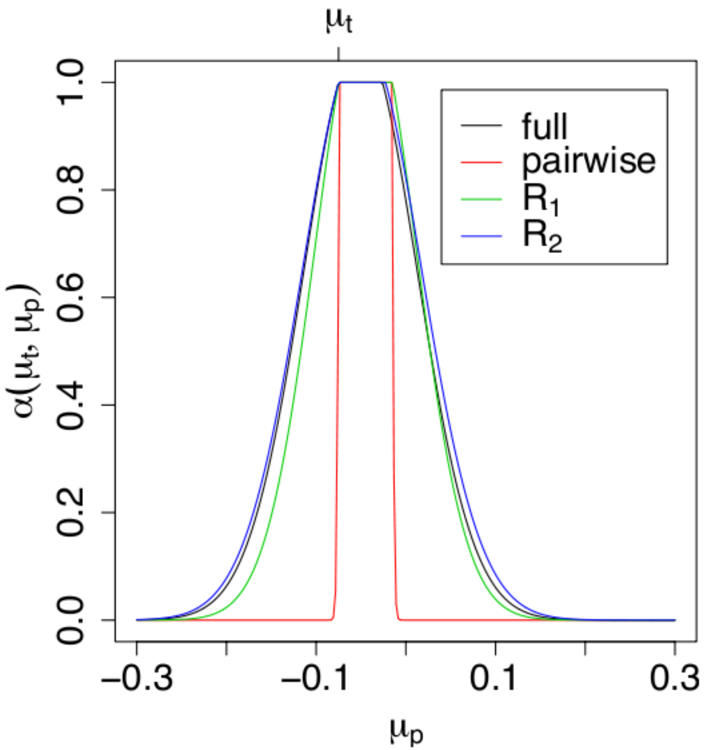
Coverage rates for 95% Credible Intervals

	Full			Magnitude			Curvature			Unadjusted		
	μ	τ	ω	μ	τ	ω	μ	τ	ω	μ	τ	ω
$\omega = 3$	96	94	94	89	92	100	94	93	94	16	21	37
$\omega = 1.5$	94	95	96	85	93	100	94	94	93	19	22	53

Composite Likelihoods and Metropolis-Hastings

Acceptance Ratio

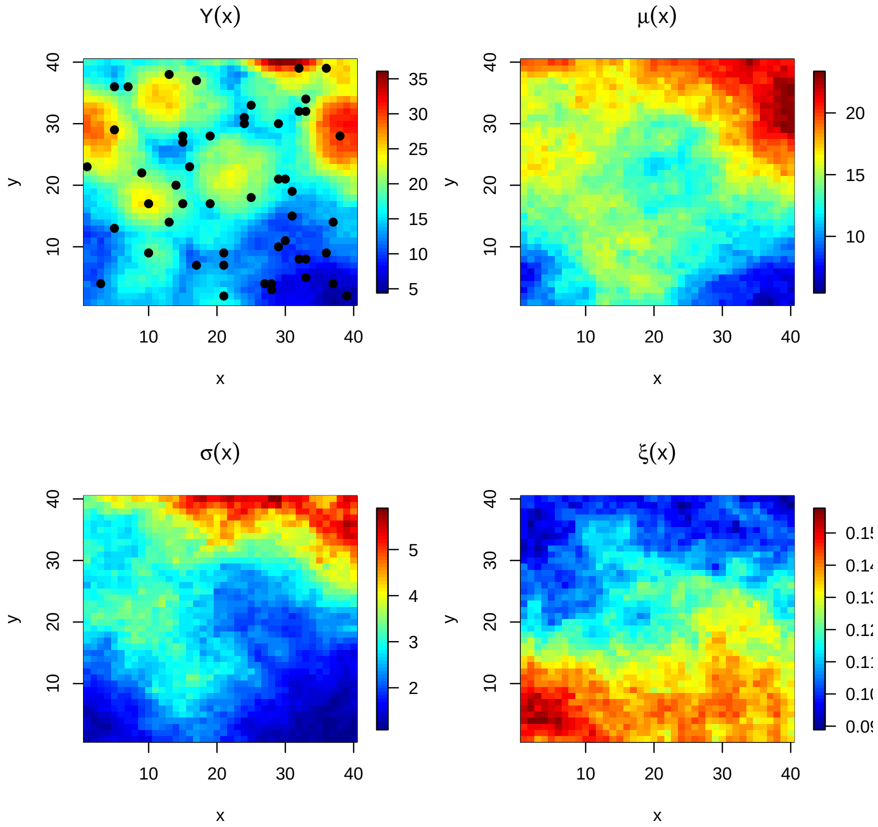
$$\alpha(\theta_t, \theta_p) = \min\left\{1, \frac{L(\theta_p; y)\pi(\theta_p)q(\theta_t|\theta_p)}{L(\theta_t; y)\pi(\theta_t)q(\theta_p|\theta_t)}\right\}$$



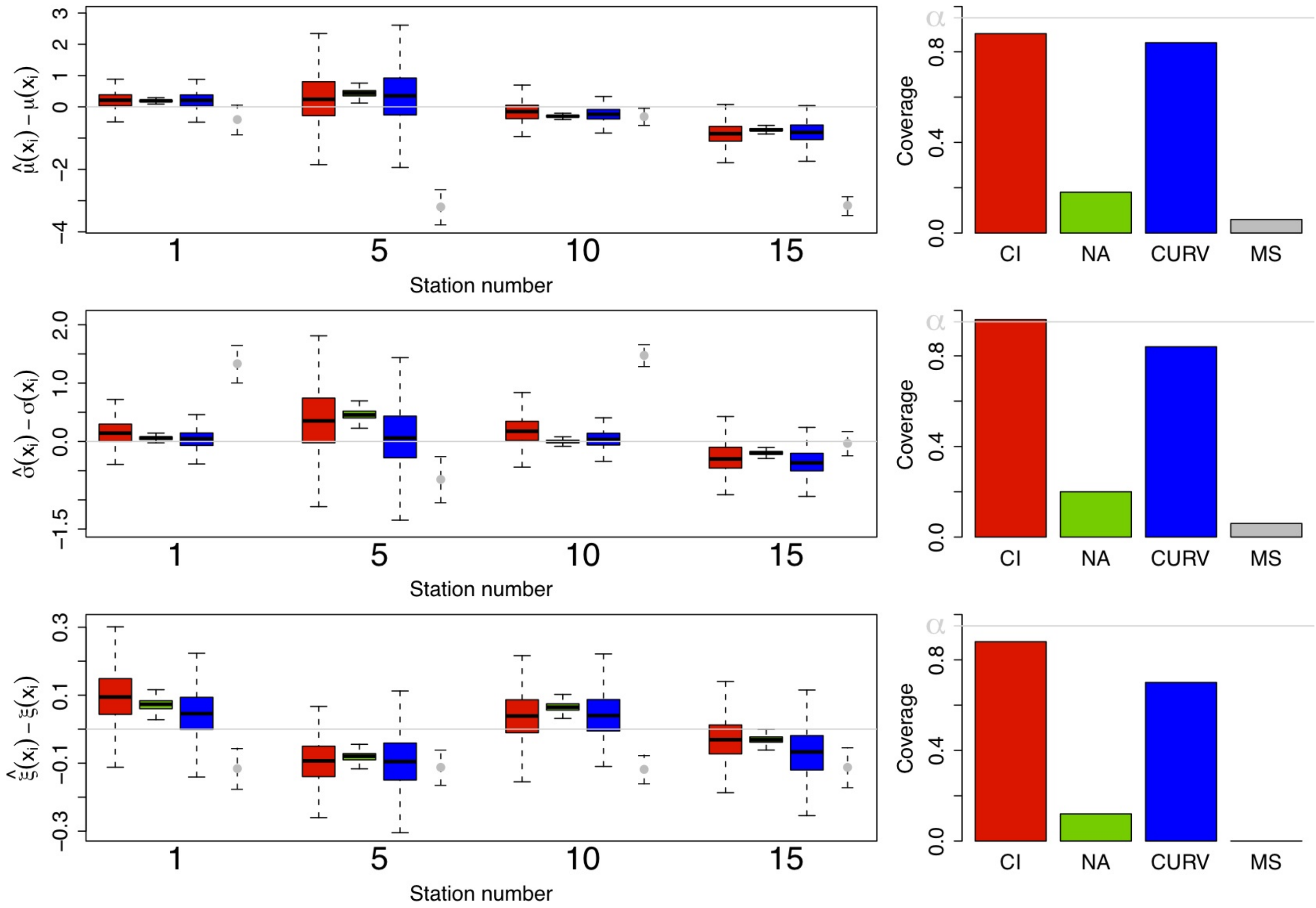
Simulation Study

$Z_j(x); j = 1, \ldots 50$ realizations of Smith's ms process. (weather)

Marginals: $Y(x) \sim GEV(\mu(x), \sigma(x), \xi(x))$;
$\mu(x), \sigma(x), \xi(x)$ are Gaussian process realizations. (climate)



Simulation Results: Estimating Marginals



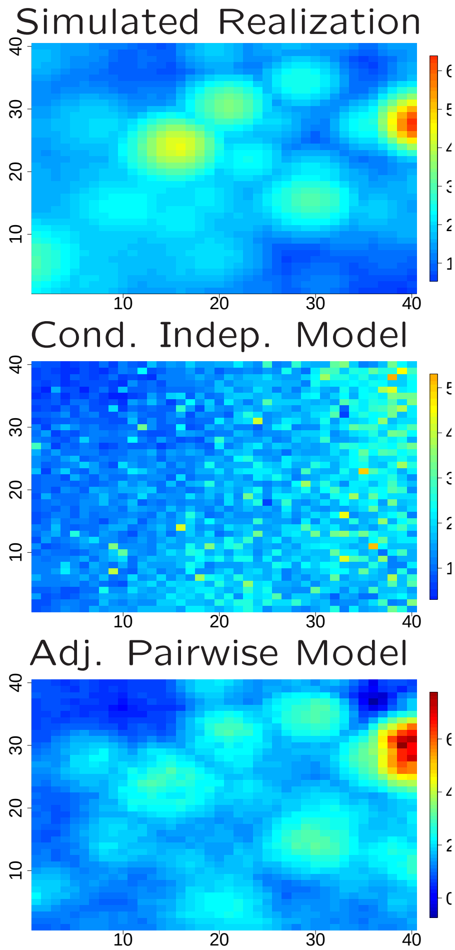
Simulation Results: Weather Dependence

Given by:

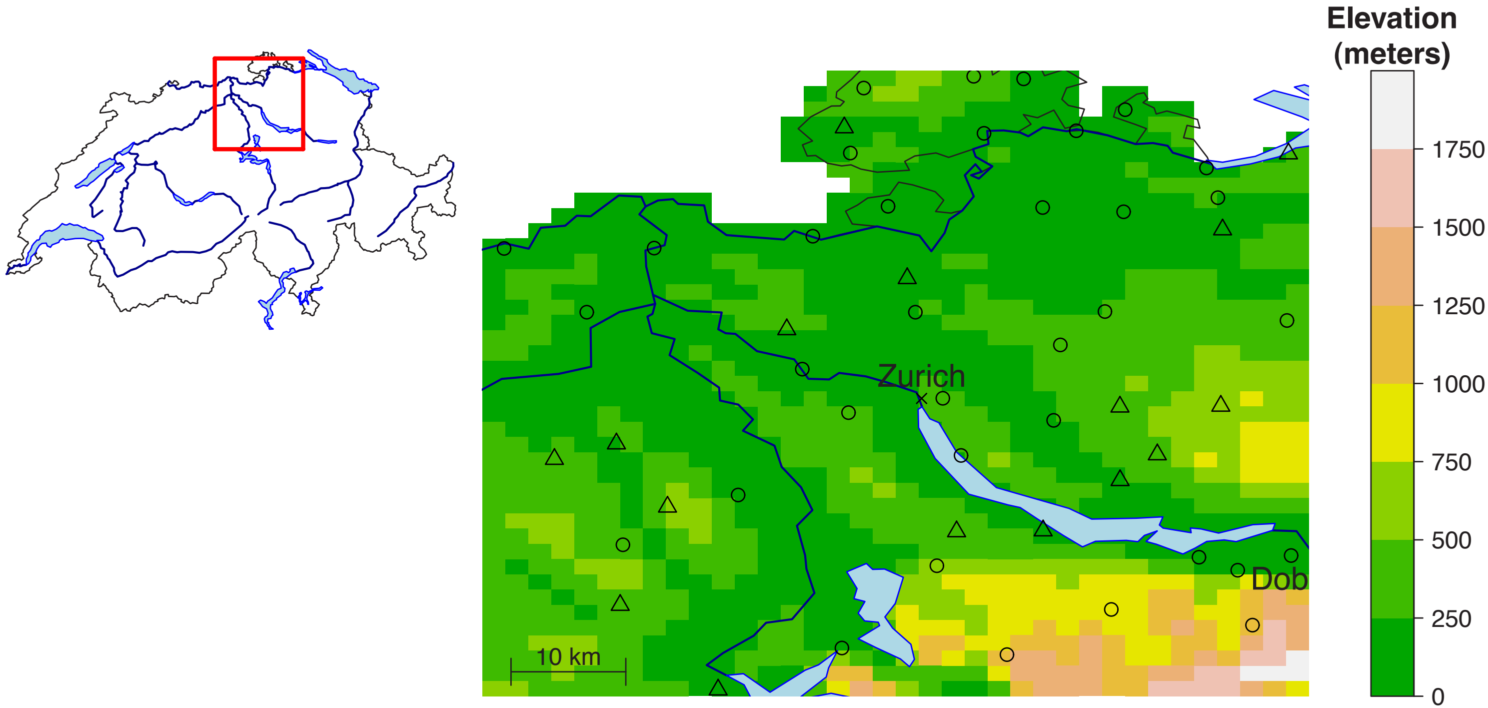
$$\Sigma = \begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{1,2} & \sigma_{2,2} \end{bmatrix}$$

Estimates

	Simulated	95% Cred. Int.
$\sigma_{1,1}$	6	(5.39, 8.76)
$\sigma_{1,2}$	0	(-1.28, 0.67)
$\sigma_{2,2}$	6	(5.58, 8.37)



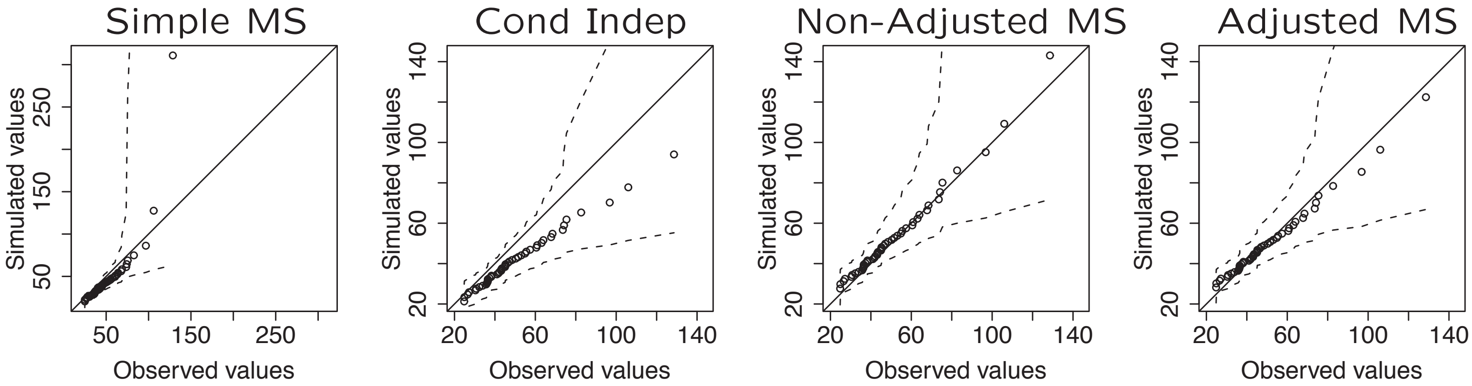
Application: Switzerland Precipitation



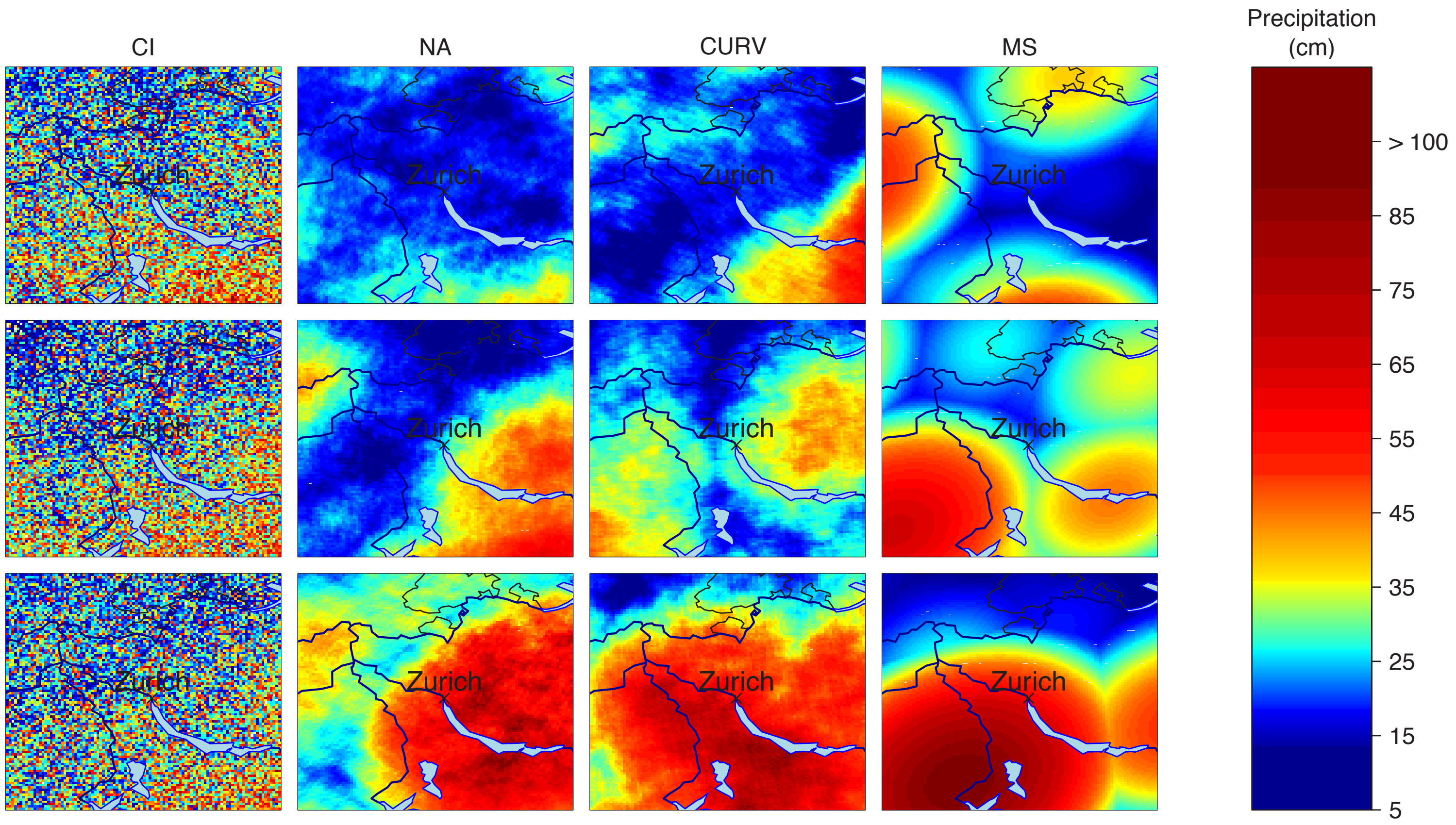
- 50 years of annual maximum data
- 35 stations used for model fitting
- 16 used for model validation

Results: Estimating Marginals

QQ plots of the annual max data at a validation station verses the model's estimated marginal.



Results: Describing Joint Behavior



Summary of this Study

By applying a max-stable model in the data level of a hierarchical model, we can account for both weather and climate spatial effects in extreme observations.

- Needed to employ a pairwise likelihood approach to fit the max-stable model.
- Needed to make adjustments to obtain appropriate inference.

+ Model at data level is grounded in extreme value theory.
+ Hierarchical approach allows for flexible modeling of climatological effects.

– As currently implemented, method is computationally feasible only for relatively small spatial problems.
  - ◦ Many avenues for speeding up computation.

– Currently, max-stable models have only been fit to annual maximum data.

Some Present Challenges for Spatial Extremes

- How does one handle spatial threshold exceedances?
  - What is the definition of a threshold exceedance?
  - Do you decluster?
  - Would the dependence estimates change with true knowledge of individual storms?
- How can one characterize the effects of climate change on extremes?
  - Sang and Gelfand (2009); Cooley and Sain (2008), Smith, unpublished.
- *Evaluation* of climate models with regards to extremes?
- Downscaling of extremes, Shamseldin et al. (2010)
- Space-time analysis of extremes.

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