Examining Extremal Dependence in Continental USA Climate Data

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A presentation in
Extreme events in climate and weather — an interdisciplinary workshop
at the Banff Centre in Banff, Alberta, Canada
World example 1 of spatial extreme dependence

Satellite image of China precipitation

Figure 1: *China heavy rain real time image*
Does CO$_2$ cause the increase of temperature?

Figure 2: Temperature and CO$_2$ concentration in the atmosphere over the past 400,000 years.
World example 3 of conditional temporal and spatial extreme dependence

Temperatures rocketed and rainfall reached extremes

Figure 3: Russia’s Fires & Pakistan’s Floods: The Result of a Stagnant Jet Stream?
Illustration of bivariate tail (in)dependence

\[ N(0,1) \]

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In theory and methodology

- Introduce a class of **tail quotient correlation coefficients** (TQCC) which allows the underlying threshold values to be random and diverge to infinity almost surely.

- Test statistics for extremal independence are constructed and shown to have asymptotic **power one** under the alternative hypothesis of extremal dependence and M4 approximation.

- Introduce a class of **nonlinear quotient correlation coefficients** (NQCC) for studying nonlinear dependency between random variables.
Apply TQCC and NQCC to study spatial extremal dependency and nonlinear dependency of daily precipitation during 1950–1999 recorded at 5873 stations from NCDC Rain Gauge Data.

Our results suggest nonstationarity, asymmetry, spatial clusters, and extremal dependency in the data. They provide useful information for next generation climate models.
Extremal (in)dependence definition

Two identically distributed r.v. $X$ and $Y$ are called *extremely independent* if

$$
\lambda = \lim_{u \to x_F} P(Y > u \mid X > u)
$$

exists and equals 0, where $x_F = \sup\{x \in \mathbb{R} : P(X \leq x) < 1\}$. The quantity $\lambda$, if exists, is called the bivariate extremal dependence index. If $\lambda > 0$, then $(X, Y)$ is called *extremely dependent* and we say there are extreme co-movements between $X$ and $Y$.

Remark

The notion *extremal dependence*, also known as *tail dependence* or *asymptotic dependence*, between the components of a two-dimensional random vector, refers to the concurrence of extreme values in the components.
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Importance of Studying Extreme (In)dependence

In theory

Suppose \( \{(X_i, Y_i), i = 1, \ldots, n\} \) is a random sample of \((X, Y)\).

If \( \lambda = 0 \), then the limit joint bivariate extreme value distribution is the product of the univariate limit distributions, i.e.

\[
\lim_{n \to \infty} P\left\{ a_n(\max_{i} X_i - b_n) < x, \ c_n(\max_{i} Y_i - d_n) < y \right\} = \lim_{n \to \infty} P\left\{ a_n(\max_{i} X_i - b_n) < x \right\} \lim_{n \to \infty} P\left\{ c_n(\max_{i} Y_i - d_n) < y \right\}.
\]

When \( \lambda = 0 \), the limit theory of the joint maxima is simple and easy!

- Example: a bivariate normal random variable with correlation coefficient \( \rho \neq 1 \).

When \( \lambda > 0 \), the limit theory of the joint maxima does not show a unified parametric form!!

- Example: a bivariate \( t \) random variable with correlation coefficient \( \rho > 0 \).
Importance of Studying Extreme (In)dependence: In financial risk management.

Did Gaussian copula cause Wall Street crash?

![Figure 4: VaR comparison of portfolios of different combinations. Zhang and Huang (2006), Zhang and Shinki (2006), Zhang and Zhao (2009)](image)
Importance of Studying Extreme (In)dependence: In extremal climatic conditions.

Notes

- Global warming causes severe storms.
- Increased ocean temperatures cause increasingly intense hurricanes.
- Three major earthquakes struck within an hour and 10 minutes in the morning of October 8, 2009 near Vanuatu in the South Pacific, prompting a tsunami warning that was quickly lifted.

Reliability of climate models

- There are increasing concerns about the reliability of climate models.
- Climate models are used to predict climate changes, which draw the most attention and debate among politicians, environmentalists and even scientists.
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The specific question for forecasting future extreme weather events:

- How to account for historical records.

Two fundamental issues:

1. How to identify extremal dependency and nonlinear dependency between climatic variables.


Our present focus:

- The first issue.
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Hypotheses of extremal (in)dependence

\[ H_0 : X \text{ and } Y \text{ are extremely independent} \]
\[ \iff H_1 : X \text{ and } Y \text{ are extremely dependent}, \]

which can also be written as

\[ H_0 : \lambda = 0 \iff H_1 : \lambda > 0. \] (2)

In the remaining of the talk, we discuss how to test the null of (2) and how to estimate \( \lambda \) under the alternative hypothesis.

Remarks

- The null and alternative hypotheses in Ledford and Tawn (1996, 1997) are reversed in this talk, see also Peng (1999), Draisma et al. (2004), and others.
- Other significant tests include Falk and Michel (2006), Hüsler and Li (2009), Bacro, Bel, and Lantuéjoul (2010) etc.
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Motivations:

- In view of (1), the extremal dependence index $\lambda$ is mainly relying on a high threshold value $u$ and the dependence between tails of two random variables.

Examples of constructing extremal (in)dependence

- Let

$$\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$$

be a sequence of independent unit Fréchet random variables with distribution function $F(x) = e^{-1/x}, x > 0$. 

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$$\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$$

be a sequence of independent unit Fréchet random variables with distribution function $F(x) = e^{-1/x}, \ x > 0$. (3)
Example 2.1
Let \( \{(U_{ni}, Q_{ni}), i = 1, \ldots, n\} \) be a sample of independent random pairs, where \( U_{ni} \) and \( Q_{ni} \) are correlated and both are supported on \((0, u_n]\) for a positive high threshold value \( u_n \). Let

\[
X_i = \xi_i I_{\{\xi_i > u_n\}} + U_{ni} I_{\{\xi_i \leq u_n\}}, \quad Y_i = \eta_i I_{\{\eta_i > u_n\}} + Q_{ni} I_{\{\eta_i \leq u_n\}}
\]

for \( i = 1, \ldots, n \). Then it follows that tail values \( X_i I_{\{X_i > u_n\}} \) (\( = \xi_i I_{\{\xi_i > u_n\}} \)) and \( Y_i I_{\{Y_i > u_n\}} \) (\( = \eta_i I_{\{\eta_i > u_n\}} \)) are independent, but \( X_i I_{\{X_i \leq u_n\}} \) ((\( U_{ni} I_{\{\xi_i \leq u_n\}} \)) and \( Y_i I_{\{Y_i \leq u_n\}} \) (\( = Q_{ni} I_{\{\eta_i \leq u_n\}} \)) are dependent. Furthermore,

\[
(X_1, Y_1), \ldots, (X_n, Y_n)
\]

is a sample of independent and identically distributed random pairs with extremely independent margins.
Elementary facts

- **Ways to measure relative positions:**
  - the difference $X - Y$;
  - the quotient $X/Y$ for positive variables $X$ and $Y$.
    - $X/Y = 0$ ‘means’ no relation.
    - $X/Y = 1$ means they are identical.

- **In a ‘Normal’ world:** Pearson correlation coefficient is the sum of the products of the $Z$ scores,
  \[
  r_n = \frac{1}{n} \sum Z_{x_i} Z_{y_i}.
  \]

- **Quotient** correlation coefficients are based on the maxima of the quotients of the Fréchet scores,
  \[
  q_n = \frac{\max_{i \leq n} \{ Y_i / X_i \} + \max_{i \leq n} \{ X_i / Y_i \} - 2}{\max_{i \leq n} \{ Y_i / X_i \} \times \max_{i \leq n} \{ X_i / Y_i \} - 1} \tag{5}
  \]
Suppose now $X_i$ and $Y_i$ are two dependent unit Fréchet random variables. Define a sample based tail dependence measure by

$$q_{un} = \frac{\max_{1 \leq i \leq n} \left\{ \frac{\max(X_i,u_n)}{\max(Y_i,u_n)} \right\} + \max_{1 \leq i \leq n} \left\{ \frac{\max(Y_i,u_n)}{\max(X_i,u_n)} \right\} - 2}{\max_{1 \leq i \leq n} \left\{ \frac{\max(X_i,u_n)}{\max(Y_i,u_n)} \right\} \times \max_{1 \leq i \leq n} \left\{ \frac{\max(Y_i,u_n)}{\max(X_i,u_n)} \right\} - 1}. \quad (6)$$

In the particular case of $u_n = u$ (a constant), definition (6) coincides with the one defined in Zhang (2008). In this talk, $u_n$ is allowed to diverge to infinity.
Analytical properties of $q_{un}$

$$f(x, y) = \frac{x + y - 2}{xy - 1}, \text{ for } x \geq 1, \; y \geq 1, \; x + y > 2. \tag{7}$$

is a bounded and monotone function.

- $0 \leq f(x, y) \leq 1$, $f(1, y) = 1$, $f(x, 1) = 1$
- $f(x_1, y_1) \leq f(x_2, y_2)$, where $x_1 \geq x_2$ and/or $y_1 \geq y_2$.

Figure 5: Illustration of the function $f(x, y)$ in (7).
Example 2.2

Using (3), we define

\[ X_i^* = \max\{a\xi_i, (1 - a)\eta_i\}, \quad Y_i^* = \max\{(1 - b)\xi_i, b\eta_i\}, \]

where \(0 < a < 1\) and \(0 < b < 1\).

Suppose \(\{(\varepsilon_{1i}, \varepsilon_{2i}), \; i = 1, \ldots, n\}\) is a sample of independent random pairs from a bivariate standard normal random variables \((\varepsilon_1, \varepsilon_2)\) with correlation coefficient \(\rho\). For \(i = 1, \ldots, n\), define

\[ U_{ni} = -1/\log\{\Phi(\varepsilon_{1i})e^{-1/u_n}\}, \quad Q_{ni} = -1/\log\{\Phi(\varepsilon_{2i})e^{-1/u_n}\}, \]

where \(\Phi(\cdot)\) denotes \(N(0,1)\) distribution function, and define

\[ X_i = X_i^* I\{X_i^* > u_n\} + U_{ni} I\{X_i^* \leq u_n\}, \quad Y_i = Y_i^* I\{Y_i^* > u_n\} + Q_{ni} I\{Y_i^* \leq u_n\}. \tag{8} \]

Then for \(0 < a < 1\) and \(0 < b < 1\),

\[ q_{un} \xrightarrow{a.s.} \lim_{u \to \infty} P(X_i > u \mid Y_i > u) = \lim_{u \to \infty} P(X_i^* > u \mid Y_i^* > u) = \lambda^* > 0, \]
Illustration of extremal dependence

Figure 6: Scatterplot of \{ (\Phi^{-1}(\exp(-1/X_i)), \Phi^{-1}(\exp(-1/Y_i))) \}_{i=1}^{500} in Example 2.2 (8). Values at lower regions in three panels are drawn from a bivariate standard normal random variable with correlation coefficient $\rho$. 
How to determine extremal (in)dependence: Random thresholds at work

Suppose $X$ and $Y$ are unit Fréchet distributed, $u_n = W_{n,t}$, where $W_{n,t}$ is distributed as $e^{-n/w^t}$, for $w > 0$ and $t > 1$, and $W_{n,t}$ is independent of $(X, Y)$. Then

$$\lim_{u \to \infty} \frac{P(X > u, Y > u)}{P(X > u)} = \lim_{u \to \infty} \frac{P\{\max(X, W_{n,t}) > u, \max(Y, W_{n,t}) > u\}}{P\{\max(X, W_{n,t}) > u\}}.$$  

(9)

Furthermore, suppose $X$ and $Y$ are extremely independent satisfying

$$\frac{P(X > u, Y > u)}{P(X > u)} = O\{u^{-(t_0 - 1)}\}$$

for a fixed $t_0 > 1$.

Remarks

- The existence of $t_0$ is guaranteed.
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\lim_{u \to \infty} \frac{P(X > u, Y > u)}{P(X > u)} = \lim_{u \to \infty} \frac{P\{\max(X, W_{n,t}) > u, \max(Y, W_{n,t}) > u\}}{P\{\max(X, W_{n,t}) > u\}}.
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Remarks

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Random thresholds at work

Suppose $X'$ and $Y'$ are independent unit Fréchet random variables, and they are independent of $W_{n,t}$. Then

$$\frac{P(X > u, Y > u)}{P(X > u)} = O\left(\frac{P\{\max(X', W_{n,t_0}) > u, \max(Y', W_{n,t_0}) > u\}}{P\{\max(X', W_{n,t_0}) > u\}}\right)$$

(10)

and

$$\frac{P(X > u, Y > u)}{P(X > u)} = o\left(\frac{P\{\max(X', W_{n,t}) > u, \max(Y', W_{n,t}) > u\}}{P\{\max(X', W_{n,t}) > u\}}\right)$$

(11)

for $t \in (1, t_0)$. 
Random thresholds at work

- Equation (9) tells that under the null hypothesis of extremal independence, testing for \((X, Y)\) is equivalent to testing for \((\max(X, W_{n,t}), \max(Y, W_{n,t}))\) where \(W_{n,t}\) can be simulated values.

- Equation (9) implies

\[
P\{\max(X, W_{n,s}) > u, \max(Y, W_{n,s}) > u\} = O(P\{\max(X', W_{n,t}) > u, \max(Y', W_{n,t}) > u\})
\]

for all \(s > 1\) and \(t \in (1, t_0]\), which tells that the upper tail probability of \((\max(X, W_{n,s}), \max(Y, W_{n,s}))\) is ‘equivalent’ to that of \((\max(X', W_{n,t}), \max(Y', W_{n,t}))\).

- Equation (11) tells that if (10) holds, one can always theoretically choose a \(t\) such that \(1 < t < t_0\) and construct a test statistic based on a bivariate random sample from \((\max(X', W_{n,t}), \max(Y', W_{n,t}))\).

- On the other hand, if \(X\) and \(Y\) are extremely dependent, (10) can never be true, i.e. the above test statistic (procedure) is an asymptotic power one test under the alternative hypothesis of extremal dependence.
Limit distribution and extremal independence test

Suppose that random variables $X_i$, $Y_i$, $W_{n,t}$, $i = 1, \ldots, n$, are independent, where $X_i$ and $Y_i$ are unit Fréchet random variables, $W_{n,t}$ is distributed as $e^{-n/w^t}$, for $w > 0$ and $t > 1$. Define

$$q_{n,t} = \frac{\max_{i \leq n} \frac{\max(X_i, W_{n,t})}{\max(Y_i, W_{n,t})} + \max_{i \leq n} \frac{\max(Y_i, W_{n,t})}{\max(X_i, W_{n,t})} - 2}{\max_{i \leq n} \frac{\max(X_i, W_{n,t})}{\max(Y_i, W_{n,t})} \times \max_{i \leq n} \frac{\max(Y_i, W_{n,t})}{\max(X_i, W_{n,t})} - 1}.$$ 

Then $2n\{1 - e^{-1/W_{n,t}}\}q_{n,t} \xrightarrow{\mathcal{L}} \chi^2_4$.

Random thresholds at work

- The limit distribution does not depend on the power transformation index $t$. This is an important property in practice. One does not need to deal with $W_{n,t}$.

- For example, suppose we simulate a value $u_1$ from $W_{n,t_1}$ for a pre-specified $t_1$. Then for any $s_1 > 0$, $u_1^{s_1}$ can be used as a value simulated from $W_{n,s_1*t_1}$, i.e. we can set $W_{n,t}$ as some pre-specified value $u_n$, for example the sample 100th percentiles.
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Testing procedure

- For a given significance level $\alpha$ and an appropriate chosen $u_n$, if
  \[ 2n\{1 - \exp(-1/u_n)\}q_{u_n} > \chi^2_{4;\alpha}, \]
  $H_0$ of (2) is rejected, and we conclude there exists extremal dependence between two random variables of interest. Here $\chi^2_{4;\alpha}$ is the upper $\alpha$ percentile of a $\chi^2$ distributed random variable with 4 degrees of freedom.

- If $H_0$ of (2) is rejected, (6) is an estimate of $\lambda$.

Remarks

- Pearson’s sample correlation coefficient and TQCC are asymptotically independent. Zhang, Qi, and Ma (2010).

- TQCC is $\sqrt{n}$ convergence under the alternative hypothesis of bivariate Gumbel copula. Wang (2010).
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Simulation and performance comparison

**Model specifications**

1. Componentwise maxima of bivariate normal random variables with \( \rho = 0.2, 0.4, 0.6, 0.8 \).
2. Example 2.1 in this talk revisited with \( \rho = 0.2, 0.4, 0.6, 0.8 \).
4. Resnick’s example \((1/U, 1/(1 - U))\).
5. Product of two random variables: \( X_i = E_i * Z_i, \ Y_i = E_i * Z_i' \).
6. Bivariate \( t \) distribution example. d.f. = 4. \( \rho = 0.8 \).

**Test statistics and threshold levels**

- Hüsler and Li (2009) test (HL T), top 25% order statistics
- TQCC, at levels 80%:0.025:97.5%
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### Empirical Type I errors and powers

#### Sample size $n=300$

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#### Sample size $n=500$

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## Empirical Type I errors and powers

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### Empirical Type I errors and powers

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What do we learn from these three tests

- HL test is conservative.
- Madogram test is too aggressive. Type I errors are not controlled within the pre-specified nominal levels.
- TQCC based test seems acceptable.
The precipitation data

- The data are daily precipitation totals covering period 1950-1999 over 5873 stations in the continental USA (excluding Alaska and Hawaii). The data units are tenths of a millimeter.
- The data are the same as used by Smith, Grady, and Hegerl (2007), and by Shamseldin, Smith, Sain, Mearns, and Cooley (2008).
- The data are first fitted to GEVs, and then transformed to unit Fréchet margins.

\[
H(x; \xi, \mu, \psi) = \exp\left[-\left\{1 + \frac{\xi(x - \mu)}{\psi}\right\}_{+}^{-1/\xi}\right],
\]

(12)

to local maxima of observations, where \(\mu\) is a location parameter, \(\psi > 0\) is a scale parameter, and \(\xi\) is a shape parameter.
Figure 7: Fitted tail shape parameter values for all 5873 station time series. The left panel shows the distribution of fitted shape parameter values. The right panel plots fitted shape parameter values to US map using krigging.

- Precipitations appear to be non-stationarity, spatial clusters, and asymmetry over all stations.
- Precipitations over stations near Mexican bay region and stations near Atlantic ocean and along North Carolina coast have heavier tails than precipitations over other stations have.
Overall extremal dependency across all stations

Figure 8: Maximal extremal precipitation dependencies between one station and the rest of stations on the same day.

- Each individual station $s_i$, at least 80% of paired stations $(s_i, s_j)$, $j = 1, \ldots, 5873; j \neq i$ are rejecting the null hypothesis of extremal independence.
- The maximal extremal dependencies decay as time goes by.
Table 1: The 10 largest tail quotient correlation coefficients and their corresponding stations information.

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Figure 9: **Maximal extremal precipitation dependencies between Station Healdsburg and the rest of stations on the same day.**

We can see that each of these plots itself can be viewed as a skewed and long tailed distribution. This phenomenon suggests flooding can be anywhere which shares smaller extremal dependencies with other locations.
The most recent flooding areas

The largest tail quotient correlation coefficients and their corresponding stations information.

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Nonlinear quotient correlation for less extremes

\[ q_n(g) = \frac{g\left(\frac{\max(u_n, X_1)}{\max(u_n, Y_1)}, \ldots, \frac{\max(u_n, X_n)}{\max(u_n, Y_n)}\right) + g\left(\frac{\max(u_n, Y_1)}{\max(u_n, X_1)}, \ldots, \frac{\max(u_n, Y_n)}{\max(u_n, X_n)}\right) - 2}{g\left(\frac{\max(u_n, X_1)}{\max(u_n, Y_1)}, \ldots, \frac{\max(u_n, X_n)}{\max(u_n, Y_n)}\right) \times g\left(\frac{\max(u_n, Y_1)}{\max(u_n, X_1)}, \ldots, \frac{\max(u_n, Y_n)}{\max(u_n, X_n)}\right) - 1}, \]

where \(g(z_1, \ldots, z_n)\) gives the \(k\)th largest value, or the \(p\)th percentile, of \(\{z_1, \ldots, z_n\}\) such that \(g(z_1, \ldots, z_n) \geq 1\).

Note that the shape of nonlinear dependence correlations show a bell shaped curve.
Linear versus nonlinear (extreme)

Asymptotic independence of Pearson’s correlation and the quotient correlation: Zhang, Qi and Ma (2010)
We hope:

- TQCC is a sample based alternative to Pearson’s correlation coefficient.
- TQCC and NQCC can be applied to many applications in which as long as one uses Pearson’s correlation coefficient.
- TQCC may be used to evaluate climate model performance and to guide model building.
- TQCC may result in true sparsity in very large correlation matrices.
Thank You!