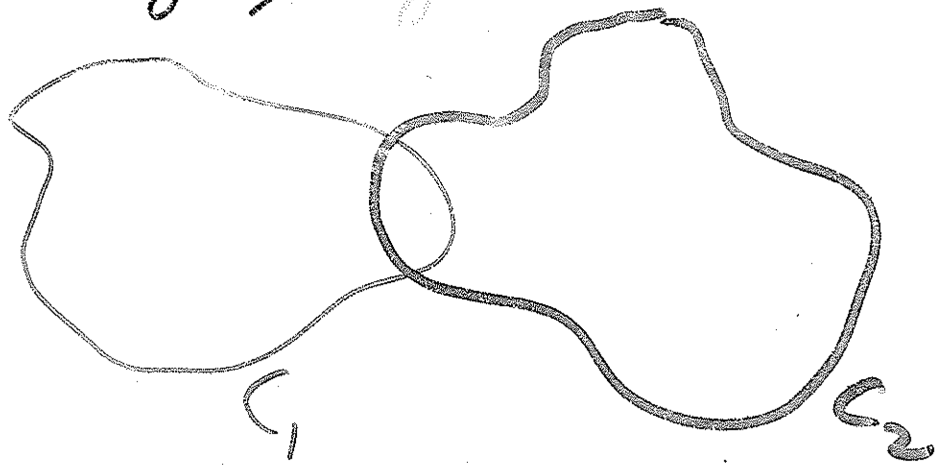
Two-dimensional Shapes and Lemniscates

joint with P. Ebenfelt & H.S. Shapiro

I. Introduction

A "shape" = a simple, smooth, closed curve

(No distinction between shapes obtained from one another by translations and scalings)



How to study the structure of the "space" of shapes?

Hausdorff distance: $h(C_1, C_2) = d_{C_1}(C_2) + d_{C_2}(C_1)$

e.g. $\mathrm{dist}_{C_1}(C_2) = \sup_{z \in C_2} \mathrm{dist}(z, C_1)$.

A.A. Kirillov, D. Mumford - E. Sharon, ...
'87 - '98 2004

based on work of

L. Ahlfors - L. Bers, D. Hamilton, Ch. Bishop,
'60 - Beltrami eq. '90s
"conformal welding"

"shape" $\rightsquigarrow$ "fingerprint" = an orientation preserving diffeo of $\mathbb{T}$ = unit circle.

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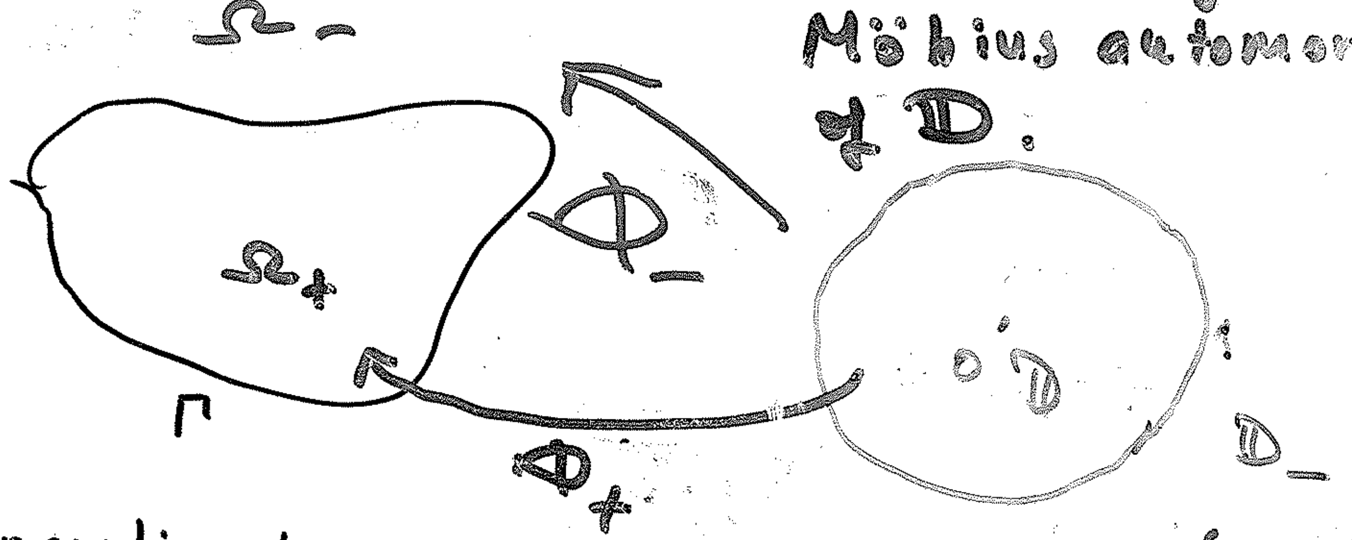
equivalence class of smooth, closed, Jordan curves modulo scalings and translations $\rightsquigarrow$ equivalence class of $\mathrm{Diff}^+(\mathbb{T})$ modulo "multiplication" from the right by a Möbius automorphism of $\mathbb{D}$.



Normalization: $\Phi_-(\infty)=\infty$; $\Phi_-'(\infty)>0$.

Fingerprint of Γ: $k: \mathbb{T} \to \mathbb{T}$, $k = \Phi_-^{-1} \circ \Phi_+$

OR, $k = e^{i\psi}$, $\psi(\theta+2\pi) = \psi(\theta)+2\pi$, $\psi' > 0$.

We obtain:

$$\mathcal{F} : \mathcal{S} \rightsquigarrow \mathrm{Diff}^+(\mathbb{T}) \,/\, \text{Möb}(\mathbb{D})$$

smooth curves / translations, scalings
$\parallel$
Shapes

"fingerprints"

<u>Kirillov's Thm ('87)</u>. $\mathcal{F}$ is a bijection.

<u>NOTE</u>: False (neither 1-1, nor onto) if one replaces Diff^+ by $\mathrm{Homeo}^+(\mathbb{T})$.

Mumford - Sharon: "Constructive" approximation to $\mathcal{F}$, $\mathcal{F}^{-1}$

(i) For $\mathcal{F}$ ... $\Phi_{+,-}$ approximated by Schwarz - Christoffel formula

(ii) For $\mathcal{F}^{-1}$ ... $\Phi_{+,-}$ are found via a series of renormalizations and by solving a Riemann - Hilbert type problem.

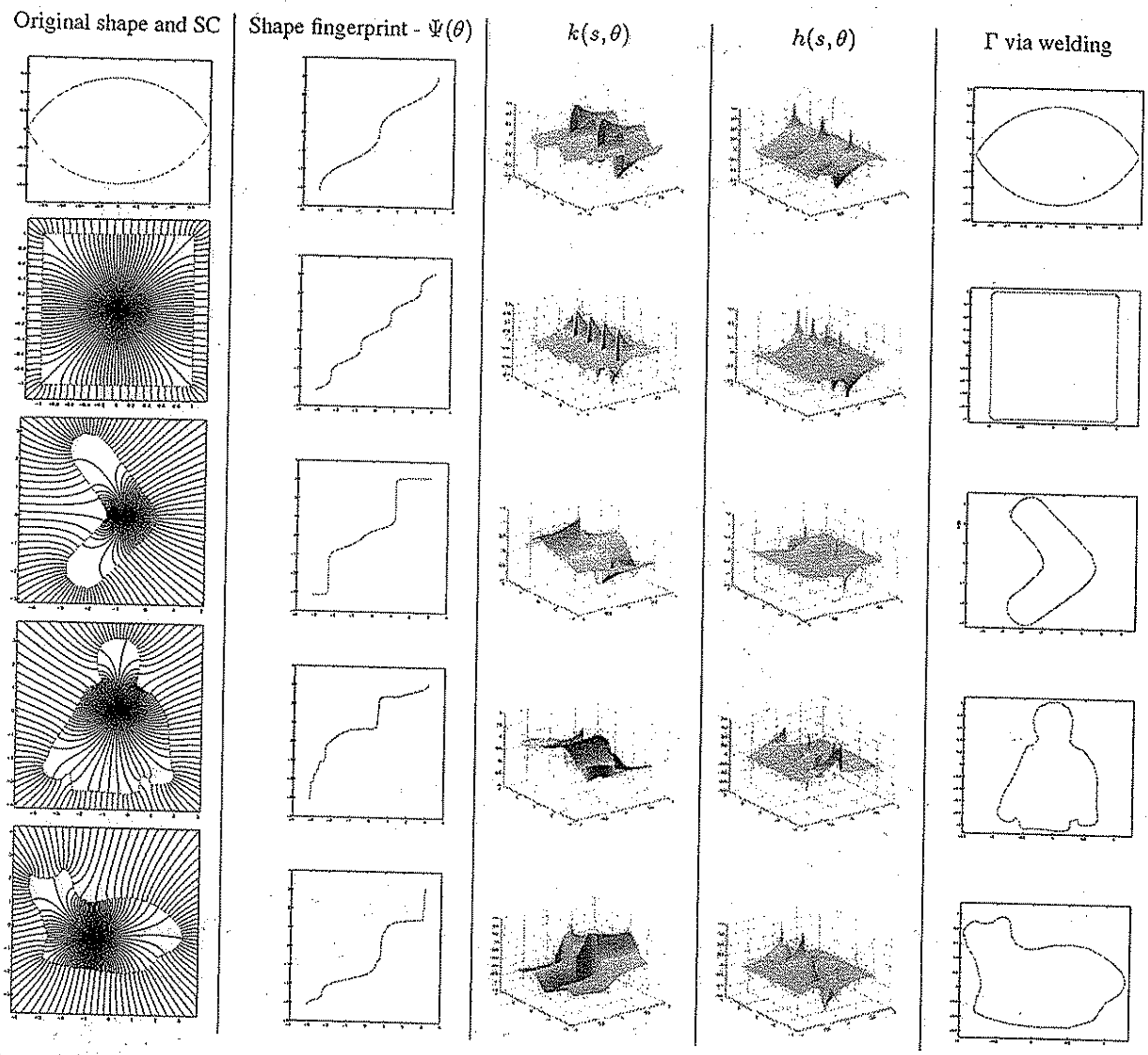


Figure 4: Experimental results. For each shape we present a row of five columns. (From left) 1st column: The conformal mappings Φ_- and Φ_+ carrying the two copies of the unit disc, Δ_- and Δ_+ onto the interior and the exterior of the Γ shape, as explained in Sec. 2.1. The figure illustrates how a homogenous radial grid on Δ_- and Δ_+, made of concentric lines through the origin, is mapped differently into the interior and exterior of the Γ shape. Note the differences in the densities of the radial grid lines, along Γ, between the interior and the exterior maps. This difference in densities along Γ is exactly what is encoded by the diffeomorphisms $\Psi \in G/H$ that match Γ, and is the fingerprint of the shape. 2nd column: Shape fingerprint $\Psi(\theta) : [0, 2\pi] \to [0, 2\pi]$, computed analytically for the first row, and numerically via Schwarz-Christoffel for the last four rows. 3rd column: $k(s,\theta)$, computed over the grid (s_i, θ_j). 4th column: $h(s,\theta)$, conjugated from $k(s,\theta)$ over the grid (s_i, θ_j). 5th column: the shape Γ, as it results from Ψ via welding, up to scale and translation, obtained by drawing $f(s = 0)$ in the complex plane.

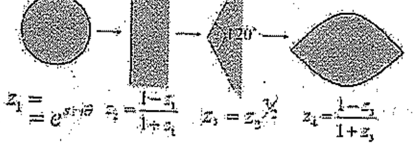
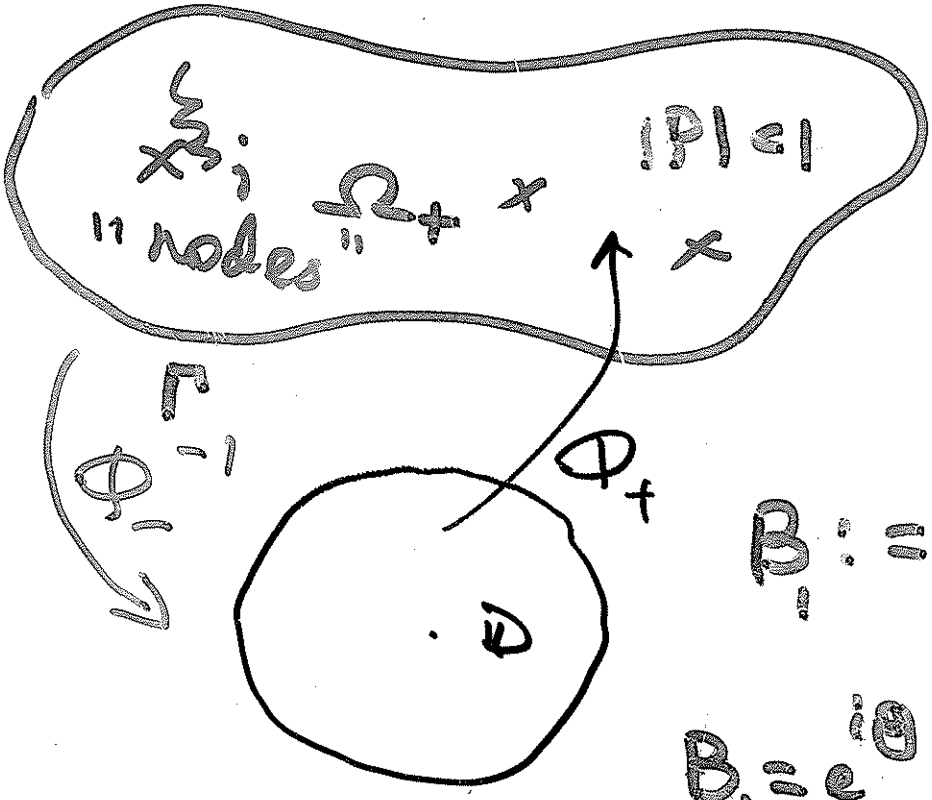


Figure 5: Example: The construction of Φ_- - the conformal mapping of the interior of the unit disc onto the interior of the "eye" shape, presented in steps.

III. Lemniscates

$\Omega_+ := \{ |P| < 1,\ P = \text{polynomial of degree } n \}$



All zeros ξ_j, $j=1..n$ inside Ω_+.

Ω_+ is connected

$$B_1 := P \circ \Phi_+ : \mathbb{D} \to \mathbb{D} \quad (n \text{ to } 1)$$

$$B_1 = e^{i\theta} \prod_1^n \frac{z - a_j}{1 - \bar{a}_j z},$$

$$a_j = \Phi_+^{-1}(\xi_j)$$

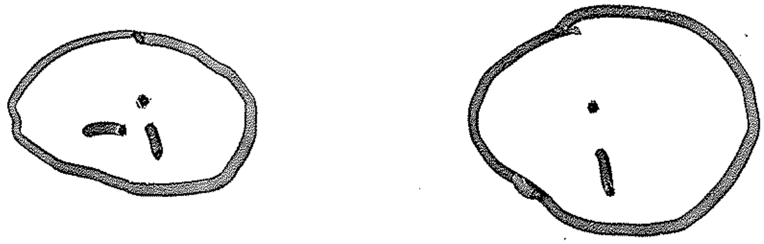
$$\Phi_-^{-1}(w) = \sqrt[n]{P(w)}$$

$$P \circ \Phi_- = c z^n, \quad |c| = 1.$$

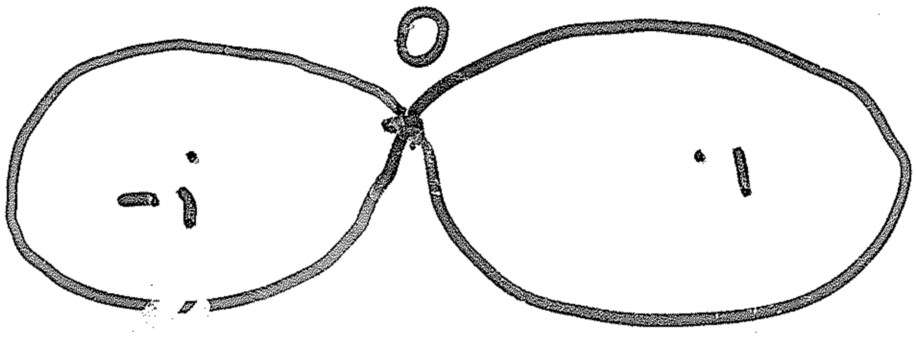
<u>Thm</u> The fingerprint of the lemniscate Γ has the form $k = c\sqrt[n]{B_1(z)}$.

Bernoulli's Lemniscate

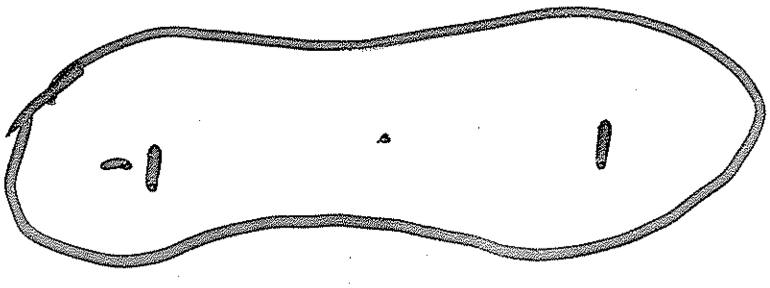
$$|z^2-1| = r^2, \quad r > 0$$



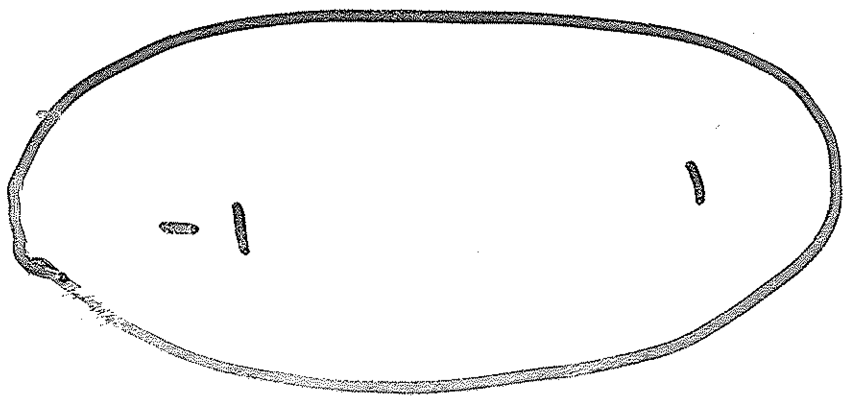
$r < 1$



$r = 1$



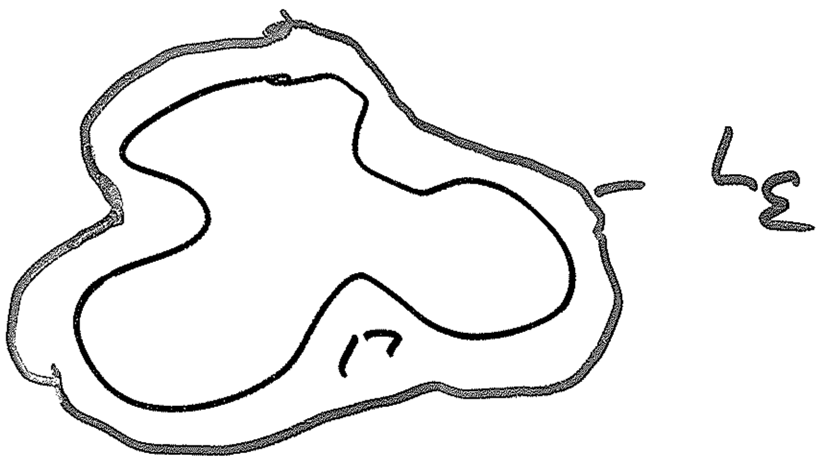
$r < \sqrt{2}$



$r \geq \sqrt{2}$

Hilbert's Theorem (1897)

For any Jordan curve Γ (closed) and any $\varepsilon > 0$, there exists a lemniscate L_ε s.t. L_ε contains Γ in its interior and $h(\Gamma, L_\varepsilon) < \varepsilon$.



Recall: Fingerprint of n-lemniscate

$$k : \mathbb{T} \xrightarrow[\text{Diff}^+]{} \mathbb{T}, \quad k = \sqrt[n]{B(z)},$$

$B = n-$ Blaschke product.

Questions: (i) Are such k dense in $\text{Diff}^+(\mathbb{T})$?

(ii) Does each such k "fingerprint" a lemniscate

IV. Results :

Thm 1 : Algebraic diffeo

$$k = \sqrt[n]{B(z)}\,, \quad B(z) = e^{i\theta} \prod_{1}^{n} \frac{z - a_j}{1 - \bar{a}_j z}$$

$|a_j| < 1$, are dense in $\mathcal{D}iff^{+}(\mathbb{T})$ (in $C^1(\mathbb{T})$-norm).

Thm 2 Every such diffeo k represents the conformal welding associated with a lemniscate $\Gamma := \{ |P| = 1, \deg P = n \}$, where P is a polynomial.

V. "Proofs".

Thm. 1 $\Psi : \mathbb{T} \longrightarrow \mathbb{T}$, $\Psi = e^{i\psi}$,

$\psi(\theta + 2\pi) = \psi(\theta) + 2\pi$, $\psi' > 0$.

To approximate ψ' by

$$\frac{1}{n}\frac{d}{d\theta}\arg B(e^{i\theta}), \quad B = n\text{-B. product}$$

Key: $(*)$ $\displaystyle \frac{d}{d\theta}\left(\frac{1}{n}\arg B(e^{i\theta})\right) = \frac{1}{n}\sum_{j=1}^{n} P(e^{i\theta}, a_j)$

P = Poisson kernel

Approximate ψ' by a positive harmonic polynomial $\rightarrow$ "balayage inward" $\rightarrow$ Poisson formula for exterior $\rightarrow$ $(*)$.

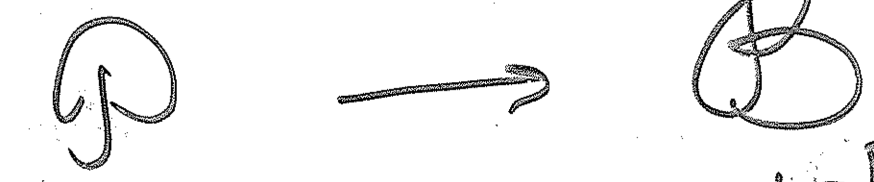
<u>Thm 2</u>

Brouwer's Theorem & Köebe Continuity Method

<u>Brouwer's Thm</u> If $f : \mathbb{R}^N \to \mathbb{R}^N$ is 1-1 and continuous, then f is open.

Applied to

Ω :



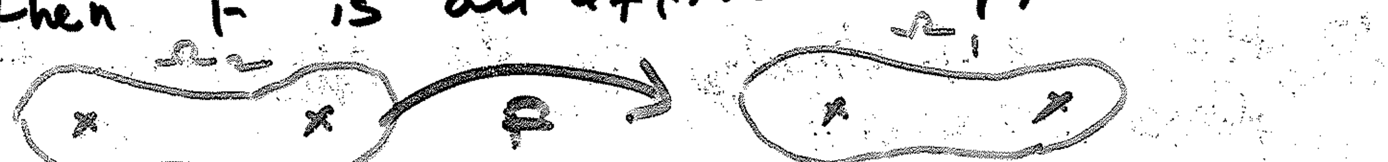
pol's of deg n / affine change of variables

n-B. products / Möb($\mathbb{D}$)

<u>Key</u>: Injectivity of $\mathcal{F}$

("Rigidity Thm")

<u>Thm 3</u> Ω^1, Ω^2 are connected n-lemniscates $\{|P|<1\}$, $\{|Q|<1\}$. If $F : \Omega_2 \to \Omega_1$ is a conformal map that maps nodes into nodes, then F is an affine map, $F = Aw + B$.



Remarks:

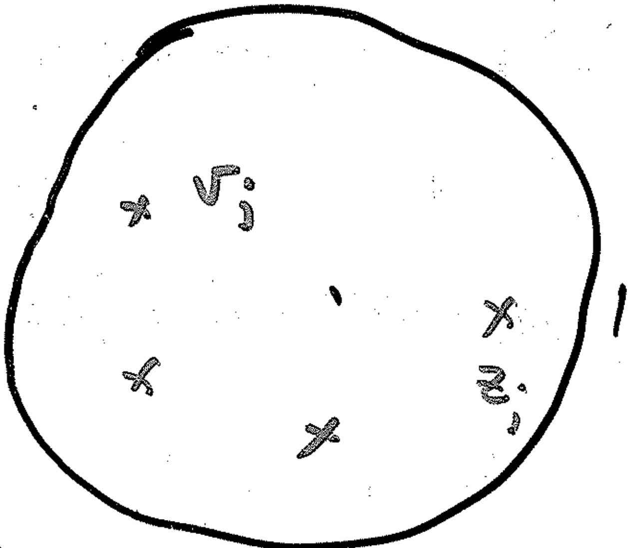
(1) "High ground" to Thm 2.

$B =$ n-B. product, z_j – B's critical points $j = 1, \dots, n-1$

$B'(z_j) = 0$, $j = 1, \dots, n-1$

$v_j = B(z_j)$ – critical values.

$V = \{v_1, \dots, v_n\}$



R. Thom ('65), V. Arnold ('96), B. Shapiro et al ('97 ff.) (goes back to A. Hurwitz 1902):

There exist n^{n-3} equivalence classes (modulo affine change of variables) of pol's of deg n with same set V of crit. values call it $\mathrm{Iso}[V]$

We know: $\mathrm{Iso}[V] \xrightarrow{F} \mathrm{Iso}[B]$ is 1-1.

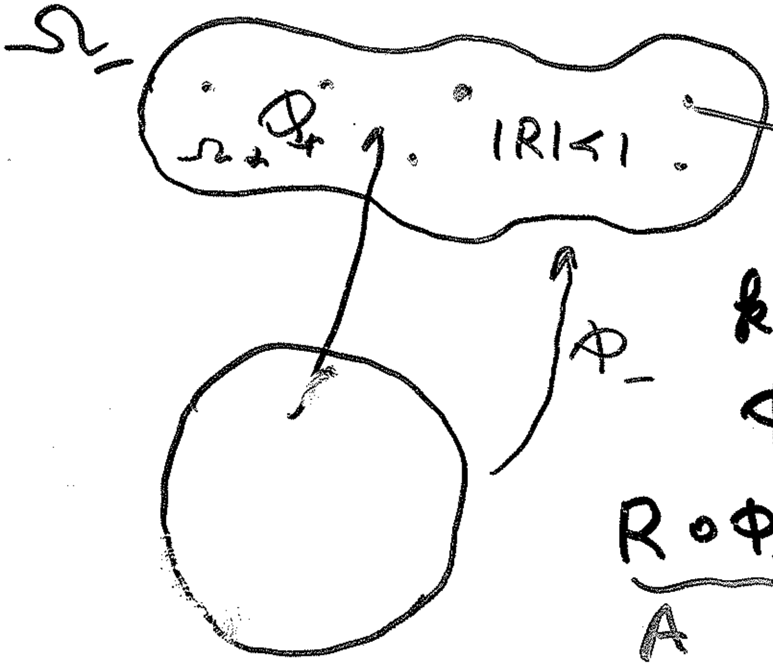
If we knew $\#\mathrm{Iso}[B] = \#\mathrm{Iso}[V]$, F is onto

(2) How effectively to do numerics following this scheme?

Further questions

Rational Lemniscates

R - rational function of degree n.



zeros of R

k = fingerprint

$\Phi_- \circ k = \Phi_+$

$$\underbrace{R\circ\Phi_-}_{A}\circ k = \underbrace{R\circ\Phi_+}_{B}$$

A, B are Blaschke products, yet $B \neq z^n$ as for polynomial lemniscates

Thus, $k = A^{-1}\circ B$

Q. If $k : \mathbb{T} \xrightarrow[\text{diff}]{} \mathbb{T}$, $k = A^{-1}\circ B$, A, B are B. products of degree n, is k a fingerprint of a rational lemniscates?

Obstacle How to characterize analytically $\{|R|<1\} := \Omega_+$ being simply connected & connected? For polynomials: all critical values are in the unit disk.