Theory and Applications of Matrices described by Patterns (10w5024)

Richard Brualdi (University of Wisconsin-Madison),
Shaun Fallat (University of Regina),
Leslie Hogben (Iowa State University),
Bryan Shader (University of Wyoming),
Pauline van den Driessche (University of Victoria)

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1 Participants

Brualdi, Richard	University of Wisconsin-Madison
Cavers, Michael	University of Regina
Cioabă, Sebastian	University of Delaware
Deaett, Louis	University of Victoria
Fallat, Shaun	University of Regina
Forster, Jürgen	ITS Informationstechnik Service GmbH, Germany
Gregory, David	Queens University
Grout, Jason	Drake University
Haemers, Willem H.	Tilburg University
Hogben, Leslie	Iowa State University
Kim, In-Jae	Minnesota State University
Kirkland, Steve	Hamilton Institute, Ireland
Martin, Ryan	Iowa State University
McDonald, Judi	Washington State University
Nikiforov, Vladimir	University of Memphis
Olesky, Dale	University of Victoria
Shader, Bryan	University of Wyoming
Sherstov, Alexander	Microsoft Research (New England)
Srinivasan, Venkatesh	University of Victoria
Tsatsomeros, Michael	Washington State University
van den Driessche, Pauline	University of Victoria
Vander Meulen, Kevin	Redeemer University College

2 Overview of the Field

Over the past several decades, linear algebra, combinatorics, and graph theory have grown into substantial and central disciplines in mathematics. These areas not only blend a range of mathematical tools, but also enrich the vitality of mathematics by connecting many branches of mathematics to various significant applications. Furthermore, the interaction between linear algebra, graph theory, and combinatorics has developed into a true discipline "combinatorial matrix theory" that continues to attract bright young people. Pattern classes of matrices represent an important focused subset of this discipline. Examples of applications that fall within this focus are models of problems in economics, mathematical biology (especially ecological food webs), statistical mechanics, and communication complexity, where the signs rather than the magnitudes of interactions are known.

The problems discussed at the workshop are related to determining, for three types of patterns, what spectra are allowed by a given pattern. Using $*$ to denote a nonzero real number, the three types of patterns are:

- symmetric $(0, *)$ patterns (corresponding to graphs) describing symmetric real matrices;
- general $(0, *)$ patterns (corresponding to digraphs or to bipartite graphs) describing real matrices; and
- $(+, -, 0)$ patterns (corresponding to signed digraphs or to signed bipartite graphs) describing real matrices.

Specific problems that are relevant to all three types of patterns include minimum rank problems, identification of spectrally and inertially arbitrary patterns, pattern-nonsingular matrices, and concepts and questions related to specific matrices associated with the pattern.

Problems related to a general $(0, *)$ pattern include the nonnegative inverse eigenvalue problem for a given pattern; minimum rank of a (rational) nonnegative matrix for a given pattern; and real (or rational) nonnegative factorizations based on the $(0, *)$ patterns of the factors. An open problem related to $(+, -, 0)$ patterns is potential stability, which has important applications in dynamical systems.

Two main objectives of the workshop were:

1. To bring together researchers with differing perspectives, including those in other areas of the mathematical and computational sciences that use the concepts of pattern matrices, but often with differing terminology.
2. To simultaneously examine several specific problems from the perspective of the three pattern types and to cross-apply techniques from one type of pattern to others.

These objectives were accomplished as follows. The workshop provided a single forum in which recent important results and applications were disseminated, and from which new collaborations have emerged. In addition, the workshop enabled researchers in combinatorial matrix theory to keep abreast of developments central to their own interests and exposed them to the array of recent activity and new applications taking place in this important and emerging area. It also enabled researchers in applied areas (e.g., communication complexity) to become familiar with theoretical results that have application.

Recently connections have been identified between minimum rank results and communication complexity, where minimum rank is called sign-rank or dimension complexity. While researchers in this field have recognized the importance of linear algebra, many are unaware of recent results on the minimum rank problem; analogously, researchers working on minimum rank are often unaware of recent developments involving sign-rank in communication complexity. The workshop devoted substantial time and effort to the goal of bridging this gap, and there is no doubt that new collaborations and lines of communication were created. It is of course too early to tell what will come from these collaborations.

Although many researchers on minimum rank problems are aware of the strategies used to attack the problem for each type of pattern, group discussion of the similarities and differences in the approaches to the minimum rank problem across the three types of patterns was initiated leading to a more unified approach.

Participants at this workshop also studied the eigenvalues of certain matrices associated with a graph, including the adjacency matrix and the Laplacian matrix (and a newly introduced skew-adjacency matrix of a graph), with an emphasis on obtaining information to determine (or assess the likelihood) that two graphs are non-isomorphic.

This BIRS workshop on matrices described by patterns was the first workshop devoted to this subject since the very successful “Spectra of families of matrices described by graphs, digraphs, and sign patterns” workshop held at the American Institute of Mathematics (AIM) Oct. 23-27, 2006, which led to ten publications. The focus of both this BIRS workshop and the AIM workshop were inspired by the emergence of pattern matrices as a dominant theme in the successful 2-day workshop “Directions in Combinatorial Matrix Theory” held at BIRS May 6-8, 2004. This is an area of substantial and growing interest. This BIRS workshop provided an opportunity to build on these activities and brought together a more scientifically diverse group. It is expected to be a significant catalyst for continued progress and development of the area. In addition, it may provide an impetus for the organization of future special sessions dedicated to this subject.

As evidenced by the detailed program of the workshop, it welcomed participation from junior researchers, and facilitated the building of collaborations between junior and senior researchers by actively involving them in collaborative research through the use of focused research groups. The organizers promoted an informal atmosphere to the proceedings, with time for casual discussions and research collaborations. The day plan was designed to promote the sharing of information, identification of open problems, and active research on a small number of such problems as selected by the group.

3 Structure of the workshop

This workshop used a focused collaborative research group structure that is designed to build new mathematical collaboration around specific mathematical goals. While a typical workshop may offer some free time for existing collaborators to work, the use of research groups organized at the workshop fosters new collaborations and full inclusion of junior/less well-known participants. The design used also allows more time for research and has fewer talks than a typical workshop, and is scheduled dynamically, that is, in response to developments during the workshop. The schedule as it actually occurred can be found at `http://www.public.iastate.edu/~lhogben/BIRSschedule.html#sched`. Here we give an overview and explain the purpose of the workshop structure and schedule.

The goal for much of the workshop (Monday, Tuesday, Thursday, part of Friday) was to identify specific problems related to the two themes of minimum rank/communication complexity and spectral graph theory, form small research groups to attack specific problems, and actually begin work on the problems chosen. Monday was devoted to minimum rank of matrices described by a pattern and connections to communication complexity; overview talks were given on these topics in the morning and early afternoon. Although the original intent had been to have all the talks in the morning and select problems in the afternoon, the morning was rather full, and we realized that splitting into small groups so early might split researchers by background, contrary to the intent to bring different perspectives together. Furthermore, forming minimum rank/communication complexity research groups on Monday would complicate forming research groups for problems in spectral graph theory on Tuesday. Thus (scheduling dynamically), the lunch break was taken before the last talk on communication complexity, and the remainder of the afternoon was spent in discussion trying to bridge minimum rank and communication complexity.

On Tuesday morning there were overview talks on two areas of spectral graph theory. With all this background in place, Tuesday afternoon began with the creation of a list of open problems on these two themes; that list appears in Section 5. Having participants create such a list is integral in forming research groups. After generating the list, the participants selected problems and began work in three groups, each containing a mix of junior and senior researchers and participants with varying mathematical perspectives. The three groups continued their work on Thursday, and presented reports on progress on Friday morning. These reports appear in Section 6. Note that each group in fact explored more than one of the questions as new ideas emerged. It is expected the groups will continue their work via e-mail, visits, and at meetings that several members attend.

Wednesday morning was devoted to a talk by Dale Olesky surveying the current state of work on potential stability. The talk was followed by a discussion of promising approaches. Wednesday afternoon was left free, and Wednesday evening was used to showcase junior researchers in the field. The speakers, titles, and abstracts are listed in Section 4.2.

In preparation for the workshop, a bibliography focussing on the specific topics of our workshop was compiled and circulated to participants. The bibliography grew throughout the workshop, and the final bibliography is now available to all the participants.

4 Presentation Highlights

In keeping with our twin goals (1) to engage both established and junior researchers, and (2) to encourage discussion, interaction, and collaboration between researchers with diverse scientific perspective, there were two types of talks:

1. Talks giving an overview of current developments in the field by established researchers.

2. Research reports by junior researchers.

4.1 Overview talks

The intent of the overview talks was to bring people with different perspectives on the problem together and to facilitate collaboration. These talks were held primarily on the mornings of the first three days, and fell into three groups.

On Monday, L. Hogben and B. Shader spoke on the minimum rank matrices described by a graph, digraph, or sign pattern, and V. Srinivasan and J. Forster on communication complexity and the role of matrix theory and sign patterns in its development.

On Tuesday, problems in spectral graph theory were introduced by W. Haemers, speaking on graphs determined by their spectra, and V. Nikiforov speaking on spectral properties of Hermitian matrices with applications to matrix patterns.

On Wednesday, D. D. Olesky surveyed potential stability of sign patterns. Slides of Olesky's talk are available at `http://www.public.iastate.edu/~lhogben/PotentialStability.pdf` and a video is at `http://www.birs.ca/birspages.php?task=eventvideos&event_id=10w5024`.

4.1.1 Minimum rank and communication complexity

A graph, digraph, or signed digraph describes the zero-nonzero or sign pattern of a family of matrices. The matrices may be symmetric, positive semidefinite, or not necessarily symmetric, and the diagonal entries may be free or constrained, depending on the type of (di)graph or pattern. A minimum rank problem is to determine the minimum among the ranks of the matrices in one of these families; the determination of maximum nullity is equivalent. Considerable progress has been made on the minimum rank problem for the family of symmetric matrices described by a simple graph (free diagonal), although the problem is far from solved. The techniques for the symmetric minimum rank problem were surveyed, and extensions to digraphs and sign patterns were discussed. The minimum rank problem has been completely solved for all types of tree patterns.

The sign-rank of a pattern is defined to be the minimum rank of a real matrix with that sign pattern. The sign-rank of a matrix is a measure of the robustness of the rank of a matrix with that sign pattern under sign preserving perturbations. Techniques were presented for constructing $(+,-)$ patterns of low sign-rank, including the use of Vandemonde matrices, and the following related facts:

$$\text{If the } m \times n\ (+,-) \text{ pattern } P \text{ has at most } k \text{ sign changes per row, then sign-rank } P \leq k+1. \tag{1}$$

$$\text{If the } m \times n\ (+,-) \text{ pattern } P \text{ has at least } k \text{ sign changes per row, then sign-rank } P \leq n-k. \tag{2}$$

For a function $f : X \times Y \to \{\pm 1\}$, the communication complexity of f is the minimum number of bits that needs to be exchanged by two parties, Alice and Bob, to compute f when Alice is given an input $x \in X$ and Bob is given an input $y \in Y$ respectively. Communication complexity is a central area of research in theoretical computer science with many interesting applications. Formal definitions of various notions of communication complexity were presented and some techniques used to prove lower bounds results in this area were surveyed. Of particular interest in the context of this workshop is unbounded error randomized communication complexity, as it is is known to be bounded below by $\log_2(\text{sign-rank}(A))$ where $A = [a_{ij}]$ is the matrix whose (x, y) entry is $\text{sgn}(f(x, y))$.

J. Forster's bound for an $m \times n$ $(+, -)$ pattern P is

$$\text{sign-rank}(P) \geq \frac{\sqrt{mn}}{\|M\|}$$

where M is the $(1, -1)$ matrix with sign pattern P, and $\|M\|$ is the spectral norm. Forster's bound gives

$$\text{sign-rank}(\text{sgn}(H)) \geq \frac{\sqrt{n \cdot n}}{\|H\|} = \frac{n}{\sqrt{n}} = \sqrt{n}$$

where H is an $n \times n$ Hadamard matrix and $\text{sgn}(H)$ is its sign pattern. An outline of the proof and extensions of these results were presented.

4.1.2 Spectral graph theory

Willem Haemers' talk dealt with the question: "Which graphs are determined by their spectrum?". For various kind of matrices associated with a graph (e.g., the adjacency matrix and the Laplacian matrix), this is a difficult but intriguing question. It is conjectured that the answer is that almost all graphs are spectrally determined (for the adjacency and the Laplacian matrix), but it has only been proved in a very few cases. On the other hand, there are many graphs that are known to not be determined by their spectrum. The talk surveyed some history, recent developments, and interesting open problems concerning the above question.

Vladimir Nikiforov spoke on the spectra of Hermitian matrix properties, where a *Hermitian matrix property* is a class of Hermitian matrices closed under permutations of the index set. His talk presented two types of Hermitian matrix properties and discussed some general theorems about their spectra.

First, a Hermitian matrix property $\mathcal{P}$ is called *hereditary* if $A \in \mathcal{P}$ implies that every principal submatrix of A is in $\mathcal{P}$. Many natural classes of Hermitian matrices are in fact hereditary, e.g., the positive definite matrices, or all Hermitian matrices with least eigenvalue exceeding -2010. Second, a Hermitian matrix property $\mathcal{P}$ is called *multiplicative* if $A \in \mathcal{P}$ implies that $A \otimes J_p \in \mathcal{P}$, where p can be any positive integer, J_p stands for the all ones $p \times p$ matrix and $\otimes$ denotes the Kronecker product. Note that "positive semidefinite" is a multiplicative property, but "positive definite" is not.

When a hereditary or multiplicative property consists of matrices with bounded entries, some fairly general theorems about their extremal eigenvalues can be proved. The motivation for this approach comes from graph theory and computer science.

4.1.3 Potential stability

A *sign pattern* $\mathcal{S} = [s_{ij}]$ is a matrix with entries in $\{+, -, 0\}$ and its associated *sign pattern class* is

$$Q(\mathcal{S}) \ = \ \{A = [a_{ij}] : a_{ij} \in \mathbb{R} \text{ and } \operatorname{sgn} a_{ij} = s_{ij} \text{ for all } i, j\}.$$

An $n \times n$ sign pattern $\mathcal{S}$ is *potentially stable* if $\mathcal{S}$ allows (negative) stability; i.e., if there exists a real matrix $A \in Q(\mathcal{S})$ for which each eigenvalue of A has negative real part. The problem of specifying necessary and sufficient conditions for potential stability has remained unsolved for over forty years, and this talk summarized progress by many researchers, including recent developments. Conditions were given that are either necessary or sufficient for potential stability for general sign patterns. In addition, necessary and sufficient conditions were given for potential stability for some sign patterns having a directed graph that is a tree, including those for which the directed graph is a star. Complete lists of all potentially stable tree sign patterns are known for $n = 2, 3, 4$. Techniques for constructing potentially stable sign patterns were described, and open problems concerning potential stability were given.

4.2 Research reports

On Wednesday evening a session of short talks was held to showcase young researchers in both linear algebra and communication complexity (followed by a modest reception hosted by the organizers in the Corbett Hall Lounge to encourage and honor these young people). The titles and abstracts of these reports follow.

Louis Deaett, University of Victoria, The rank of a matrix and the girth of its graph

In the case of a positive semidefinite matrix, a result of Moshe Rosenfeld provides a lower bound on the rank when the graph of the matrix is triangle-free. We'll show a new proof of this bound. We also consider the possibility of generalizing the bound under a stronger condition on the girth of the graph.

Jason Grout, Drake University, Computing bounds for minimum rank with Sage

I will explain and give examples of a suite of functions in Sage that use a number of bounds from the literature to compute minimum rank bounds in Sage. The suite also includes a lookup table of minimum ranks for all graphs with fewer than 8 vertices. This program is currently being formatted for inclusion in Sage, and will then be in every copy of Sage, enhancing Sage's comprehensive graph functionality. These functions are a collaborative work between Laura DeLoss, Tracy Hall, Josh Lagrange, Tracy McKay, Jason Smith, Geoff Tims, and myself, and were initially developed in Leslie Hogben's early graduate research class at Iowa State University. For an earlier version of the program, see `http://arxiv.org/abs/0812.1616`.

Alexander Sherstov, Microsoft Research (New England), Sign-Rank and the Polynomial Hierarchy in Communication Complexity

The *sign-rank* of a matrix $A = [A_{ij}]$ with ± 1 entries is the least rank of a real matrix $B = [B_{ij}]$ with $A_{ij}B_{ij} > 0$ for all i, j. We obtain the first exponential lower bound on the sign-rank of a matrix computable by the polynomial hierarchy in communication complexity. Namely, let $f(x, y) = \bigwedge_{i=1}^{m} \bigvee_{j=1}^{m^2} (x_{ij} \wedge y_{ij})$. We show that the matrix $[f(x, y)]_{x,y}$ has sign-rank $\exp(\Omega(m))$. This in particular implies that $\Sigma_2^{cc} \not\subseteq \mathsf{UPP}^{cc}$, which solves an open problem in communication complexity posed by Babai, Frankl, and Simon (1986).

Our result additionally implies a lower bound in learning theory. Specifically, let $\phi_1, \ldots, \phi_r : \{0, 1\}^n \to \mathbb{R}$ be functions such that every DNF formula $f : \{0, 1\}^n \to \{-1, +1\}$ of polynomial size has the representation $f \equiv \operatorname{sgn}(a_1\phi_1 + \cdots + a_r\phi_r)$ for some reals $a_1, \ldots, a_r$. We prove that then $r \geq \exp(\Omega(n^{1/3}))$, which essentially matches an upper bound of $\exp(\tilde{O}(n^{1/3}))$ due to Klivans and Servedio (2001).

Finally, our work yields the first exponential lower bound on the size of *threshold-of-majority* circuits computing a function in AC^0. This generalizes and strengthens the results of Krause and Pudlák (1997). Joint work with Alexander Razborov (University of Chicago).

Sebastian Cioabă, University of Delaware, On decompositions of complete hypergraphs

In this talk, I will study the minimum number of complete r-partite r-uniform hypergraphs needed to partition the edges of the complete r-uniform hypergraph on n vertices. This problem is the hypergraph extension of the classical Graham-Pollak theorem.

Michael Cavers, University of Regina, On the energy of graphs

The concept of the energy of a graph was defined by Ivan Gutman in 1978 and originates from theoretical chemistry. To determine the energy of a graph, we essentially add up the eigenvalues (in absolute value) of the adjacency matrix of a graph. Recently, the Laplacian energy, distance energy, incidence energy, signless Laplacian energy and normalized Laplacian energy has received much interest. We will look at these different types of energies and see how they are affected by the structure of a graph. In the past ten years, there have been more than 150 papers published on graph energy, and it continues to be a highly researched topic by pure mathematicians and theoretical chemists alike.

In-Jae Kim, Minnesota State University, On eventual positivity

An $n \times n$ real matrix A is said to be eventually positive if there exists a positive integer k_0 such that $A^k > 0$ (entrywise positive) for all positive integers $k \geq k_0$. An $n \times n$ sign pattern $\mathcal{A}$ is potentially eventually positive (PEP) if $\mathcal{A}$ has a realization that is eventually positive. In this talk, some necessary or sufficient conditions for a sign pattern to be PEP are given. In addition, it is shown that the minimum number of positive entries in a PEP sign pattern is $n + 1$. Joint work with A. Berman, M. Catral, L. M. DeAlba, A. Elhashash, F. J. Hall, L. Hogben, D. D. Olesky, P. Tarazaga, M. J. Tsatsomeros, P. van den Driessche.

5 Open problems

On Tuesday afternoon the whole group discussed open questions in minimum rank and communication complexity (related to connections between the fields) and spectral graph theory (related to the co-spectral graphs and classes of Hermitian matrices). A subset of problems was selected, groups were formed, and work began that afternoon. On Wednesday, following Olesky's talk on potential stability, the whole group brainstormed open questions in potential stability related to the recent developments that had been surveyed. Lists of questions generated in the problem session and questions from the overview talks on minimum rank and communication complexity, and on spectral graph theory are given below.

5.1 Open questions in minimum rank and communication complexity

1. Is there a Forster-type bound for $(0,+)$ patterns? For information on Forster's bound, see Section 4.1.1. More generally, can one use sign-rank techniques for $(+,-)$ patterns on $(0,+)$ patterns? Note that it is easy to transform a $(0,1)$ matrix M to a $(1,-1)$ matrix M' via a rank one perturbation, but this does not immediately give information of the relationship between the sign-rank/minimum rank of sgn(M) and sign-rank/minimum rank of sgn(M').

2. Is there a Forster-type bound for $(0,+,-)$ patterns? More generally, can one use sign-rank techniques for $(+,-)$ patterns on $(0,+,-)$ patterns?

3. For a symmetric $(+,-)$ pattern, if we consider only symmetric matrices having the given pattern, is there a (higher) Forster-type bound?

4. Can we find a family with significantly higher symmetric minimum rank/sign-rank than the Hadamard patterns?

5. For a symmetric $(+,-)$ pattern, if we consider only symmetric matrices having the given pattern, is there an analog of (1) (see Section 4.1.1)?

6. What can be said about minimum rank and sign-rank of $(+,-,?)$ patterns where ? denotes $+$, $-$, or 0?

7. It is known that the minimum rank of a simple graph, mr(G) satisfies

$$\mathrm{mr}(G) \leq |G| - \kappa(G)$$

   where $\kappa(G)$ is the vertex connectivity of G, and in fact a positive semidefinite matrix can be found to realize the upper bound. Is there an analog of this result for $(+,-)$ patterns?

8. Find $(+,-)$ patterns of large sign rank.

9. The rigidity function $R_A(r)$ of a real $n \times n$ matrix A is the minimum number of entries needed to be changed in order to bring the rank down to r (i.e. the Hamming distance to a rank r matrix). Hadamard matrices seem to have large rigidity. Can one constuct an explicit family of matrices A such that $R_A(r) \geq (n-r)^2$?

10. What can be said about sign-rank of $(0,+,-)$ patterns with other properties such as allowing orthogonality, allowing eventual positivity, allowing eventual nonnegativity, allowing nilpotence, spectrally arbitrary patterns?

11. (The δ-conjecture) Let $\delta(G)$ and $\kappa(G)$ denote the minimum degree and vertex connectivity of a graph G with n vertices. It is known that the minimum rank of G is at most $2(n-\delta(G))$ and at most $n-\kappa(G)$. It is conjectured that the minimum rank of G is at most $n-\delta(G)$.

12. Construct a "large" family of "dense" graphs G having minimum rank greater than $\frac{1}{7}|G|$ (for $|G|$ large, almost all graphs have minimum rank greater than $\frac{1}{7}|G|$).

5.2 Open questions on spectral graph theory

1. Given a graph G with adjacency matrix A, can non-isomorphic graphs be distinguished by examining the spectra of the family of skew adjacency matrices obtained by signing the nonzero entries of A so as to produce skew symmetric matrices, in all possible ways? This was answered negatively by Group 3.
2. Find additional *interesting* families of connected graphs that are determined by their spectra.
3. (Inverse Eigenvalue Problem for Adjacency Matrix) What are the possible eigenvalues of the adjacency matrix of a graph? What if extra zeros can be added to the spectrum? Consider a small number of nonzero eigenvalues as a special case.
4. (Inverse Eigenvalue Problem for Laplacian) What are the possible eigenvalues of the Laplacian matrix of a graph? What if extra zeros can be added to the spectrum? Consider a small number of nonzero eigenvalues as a special case.
5. What are the possible inertias of the adjacency matrix of a graph?
6. What are the possible inertias of the Laplacian matrix of a graph.
7. Investigate the maximum absolute value of an eigenvalue $\lambda \neq \pm d$ of a d-regular graph.

6 Group reports

6.1 Group 1

Group members: Richard Brualdi, Jürgen Forster, Jason Grout, Leslie Hogben, Ryan Martin, Bryan Shader, Sasha Sherstov, Venkatesh Srinivasan and Pauline van den Driessche.

This group's work was motivated by the following problem:

> Given an $m \times n$ $(+,-,?)$-pattern $P = [p_{ij}]$, determine the smallest k such that there exist vectors $u_1, u_2, \ldots, u_k \in \mathbb{R}^m$ and $v_1, \ldots, v_k \in \mathbb{R}^n$ with $\text{sgn}(u_i^T v_j) = p_{ij}$ whenever $p_{ij} \neq ?$.

In communication terms, one can view P as describing a "partial problem", $f : S \rightarrow \{+.-\}$, where $S = \{(i,j) : p_{ij} \neq ?\}$. Thus, Alice and Bob know a priori that they only need to be able to determine $f((i,j))$ for $(i,j) \in S$. In matrix completion terms, we are asking for the smallest sign-rank of a $(0,+,-)$ pattern obtained from P by allowing the ?s to be $+$ or $-$, or even $+$, $-$, or 0.

This led the group to the problem of characterizing $(+.-)$ sign-patterns that have small or large minimum rank, where some observations and initial results were obtained. It is a standard result from qualitative matrix theory that the $m \times n$ $(+,-)$ sign-pattern P has minimum rank m if and only if it contains a matrix that is sign-equivalent to Λ_m, where Λ_m is the $m \times 2^{m-1}$ matrix whose columns are all the $m \times 1$ vectors with entries $+$ or $-$ and whose first coordinate is $+$. It is easy to characterize $(+,-)$ sign-patterns with minimum rank 1, both in terms of the sign changes in a column (P has minimum rank 1 if and only if it is sign-scalable to the matrix of all +s) and in terms of a forbidden configuration (P has minimum rank 1 if and only if A does not contain a matrix that is sign-equivalent to Λ_2). A more difficult task is to characterize $(+,-)$ sign-patterns of minimum rank at most 2. The group was able to establish both a characterization in terms of sign-changes, and a forbidden configuration characterization involving Λ_3. Characterizing $(+,-)$ sign-patterns with minimum rank at most 3 was recognized as a daunting task as a result of Peter Shor implies that this problem is NP-complete. However, the group began the process of relating $(+,-)$ sign-patterns of minimum rank at most 3 to realizable rank 3 oriented matroids. The Pappus configuration was used to construct a sign-pattern of minimum rank 4 and not containing any matrix sign-equivalent to Λ_4. Further research along these lines is on-going with the goal of classifying the $(+,-)$ sign-patterns of minimum rank 3 and "small" order, and will involve the study of pseudo-line configurations from the theory of oriented matroids.

The discussions led to the development of several algorithms for studying the minimum rank of $(+,-)$ patterns. These algorithms have been implemented in Sage and will be made available to the mathematical community.

In addition, the group worked, and continues to work, on using probabilistic methods (e.g., the Regularity Lemma) to aid in the study of $(+,-)$ sign-patterns with certain forbidden configurations (e.g., Λ_k).

The group also plans to work on the following questions:

- Is there a combinatorial interpretation of Forster's theorem?
  Or is there a weaker result than Forster's theorem that gives a combinatorial reason for a $(+,-)$ sign-pattern to have large minimum rank?
- Can one characterize the $m \times n$ $(+,-)$ sign-patterns of minimum rank $m-1$, or $m-2$?

6.2 Group 2

Group members: Louis Deaett, In-Jae Kim, Vladimir Nikiforov, Dale Olesky, Kevin Vander Meulen,

The meeting introduced us to a result of Jürgen Forster that provides a lower bound of $\sqrt{n}$ on the sign-rank of an $n \times n$ Hadamard matrix. This result led to the first linear lower bound on the unbounded-error probabilistic communication complexity of a natural family of Boolean functions. We have begun investigating new avenues for establishing upper and lower bounds on the sign-rank of Hadamard matrices and of other $\pm$ patterns in general. We have already managed to improve a known upper bound on the sign-rank of large Hadamard matrices. We expect that by refining our construction we will be able to improve this upper bound further. Other upper bound techniques involve analysis of the extreme numbers of "sign changes" occurring within rows of the matrix. In the context of Hadamard matrices we have discovered provable limits on the power of this method. On the lower bound side, beyond the analytic methods of Forster few other techniques are known, and these seem limited to combinatorial arguments based on the presence of large L-matrices. A simple counting argument shows that even for a Hadamard matrix of order as small as 64, the lower bound provided by these L-matrices cannot match the lower bound of Forster. Hence, no combinatorial technique of reasonable power seems to be known, and the combinatorial behavior of sign-rank in general seems wide open for investigation. Such a combinatorial understanding could lead to new techniques for proving upper and lower bounds. Our group includes researchers at different levels of experience who have not previously collaborated. We have begun to explore how new combinatorial tools could lead to new techniques for improving upper and lower bounds related to sign-rank and communication complexity. We intend to continue this fruitful collaboration.

6.3 Group 3

Group members: Michael Cavers, Sebastian Cioabă, Shaun Fallat, David Gregory, Willem Haemers, Steve Kirkland, Judi McDonald, Michael Tsatsomeros

The group's work centered on eigenvalues for graphs. We began with an investigation of the following inverse eigenvalue question for graphs: given a collection of real numbers, how can it be determined whether or not it is the non-zero part of the adjacency spectrum of some graph. It was noted that without the caveat of looking for the non-zero part of the spectrum, that question is quite difficult, since for example, a resolution of that problem would settle the existence question for certain projective planes. However, being given the freedom to add zeros to a candidate spectrum provides considerable leeway. After some consideration of the solution of the corresponding inverse eigenvalue problem for nonnegative integer matrices, the group decided to move in another direction.

Motivated by an interest in trying to find ways to distinguish between graphs that have cospectral adjacency matrices, the group considered the following family of skew adjacency matrices: given a graph G with adjacency matrix A, the family of skew adjacency matrices for G is formed by signing the nonzero entries of A so as to produce skew symmetric matrices, in all possible ways. It was thought possible that perhaps a pair of adjacency cospectral graphs might be distinguished by looking at the spectra of the corresponding skew adjacency matrices. This possibility was quickly refuted, when the group observed that any pair of adjacency cospectral trees must also share the same skew spectrum.

This last observation led the group to address the problem of characterizing the graphs for which all members of the family of skew adjacency matrices have a common spectrum. A conjecture was formulated on that problem, and work on the resolution of that conjecture is ongoing.

7 Outcomes of the Meeting

We thank BIRS for supporting this workshop, which all participants found valuable and stimulating.

It is expected that the three working groups or subsets thereof will continue their collaborations on the problems identified, which will likely lead to publications.

Such papers may be submitted to the *Linear Algebra and its Applications* Special Issue on the occasion of the Workshop at the Banff International Research Station titled: "Theory and Applications of Matrices described by Patterns." (January 31 - February 5, 2010). Papers within the scope of the Workshop are solicited from all interested whether or not a participant in the Workshop. The deadline for submission of papers is October 1, 2010. Papers for submission should be sent to one of the four special editors, Shaun Fallat, Leslie Hogben, Bryan Shader, or Pauline van den Driessche. They will be subject to normal refereeing procedures according to *LAA* standards. The editor-in-chief responsible for this special issue is Richard A. Brualdi.

Several surveys introducing the themes of the workshop are also planned. Shaun Fallat and Leslie Hogben will update and broaden their 2007 survey on the problem of minimum rank of a graph to digraphs, sign patterns, and positive semidefinite minimum rank. Venkatesh Srinivasan will survey linear algebraic methods and problems in communication complexity with an emphasis on minimum rank/sign-rank problems.

A webpage associated with the workshop, including the schedule, abstracts, and the slides of D. D. Olesky's Potential Stability talk, is available at `http://www.public.iastate.edu/~lhogben/BIRSschedule.html`.