$\mathcal{H}_\infty$ Control for Descriptor Systems
A Structured Matrix Pencils Approach

Philip Losse
Joint work with Peter Benner, Volker Mehrmann, Lisa Poppe and Timo Reis

AG ModNumDif
Institut für Mathematik
Technische Universität Berlin

28.10.2010
Outline

1. Introduction
2. Modified Optimal $\mathcal{H}_\infty$ Control
2. Suboptimal $\mathcal{H}_\infty$ Control
Introduction

We consider the system

\[
\begin{align*}
E \dot{x} &= Ax + B_1 w + B_2 u, \quad x(t_0) = x^0, \\
z &= C_1 x + D_{11} w + D_{12} u, \\
y &= C_2 x + D_{21} w + D_{22} u,
\end{align*}
\]

\( E, A \in \mathbb{R}^{n,n}, B_i \in \mathbb{R}^{n,m_i}, C_i \in \mathbb{R}^{p_i,n}, \) and \( D_{ij} \in \mathbb{R}^{p_i,m_j}, i, j = 1, 2. \)

- \( E \) may be singular, \( \text{rank}(E) = r \)
- \( \lambda E - A \) regular, i.e. \( \det(\lambda E - A) \) does not vanish identically
- \( x \) descriptor variable, \( w \) disturbance, \( u \) input, \( z \) controlled output, \( y \) measured output
The optimal $\mathcal{H}_\infty$ control problem

Determine a dynamic controller

\[
\begin{align*}
\hat{E}\dot{x} &= \hat{A}x + \hat{B}y, \\
u &= \hat{C}\dot{x} + \hat{D}y,
\end{align*}
\]

with $\hat{E}, \hat{A} \in \mathbb{R}^{N,N}$, $\hat{B} \in \mathbb{R}^{N,p_2}$, $\hat{C} \in \mathbb{R}^{m_2,N}$, $\hat{D} \in \mathbb{R}^{m_2,p_2}$ such that the closed-loop system, formed by the given system combined with the controller, is internally stable and the closed-loop transfer function $T_{zw}(s)$ from $w$ to $z$ is minimized in the $\mathcal{H}_\infty$ norm.
The $\mathcal{H}_\infty$ control problem for descriptor systems has been studied using

- linear matrix inequalities [Rehm/Allgöwer]
- generalized Riccati equations [Takaba/Morihira/Katayama]

Since

- LMIs are non practical for large scale systems
- GREs are facing severe numerical difficulties

we are proposing a matrix pencil approach which relies on the structure preserving computation of deflating subspaces of even matrix pencils, generalizing the results from [Benner/Byers/Mehrmann/Xu '04].

Additionally we would like to use only original system data as long as possible to prevent numerical errors.
Previous Work

The $\mathcal{H}_\infty$ control problem for descriptor systems has been studied using
- linear matrix inequalities [Rehm/Allgöwer]
- generalized Riccati equations [Takaba/Morihira/Katayama]

Since
- LMIs are non practical for large scale systems
- GREs are facing severe numerical difficulties

we are proposing a matrix pencil approach which relies on the structure preserving computation of deflating subspaces of even matrix pencils, generalizing the results from [Benner/Byers/Mehrmann/Xu ’04].

Additionally we would like to use only original system data as long as possible to prevent numerical errors.
The $\mathcal{H}_\infty$ control problem for descriptor systems has been studied using
- linear matrix inequalities [Rehm/Allgöwer]
- generalized Riccati equations [Takaba/Morihira/Katayama]

Since
- LMIs are non practical for large scale systems
- GREs are facing severe numerical difficulties

we are proposing a matrix pencil approach which relies on the structure preserving computation of deflating subspaces of even matrix pencils, generalizing the results from [Benner/Byers/Mehrmann/Xu '04].

Additionally we would like to use only original system data as long as possible to prevent numerical errors.
Two Subproblems

The modified optimal $\mathcal{H}_\infty$ control problem

For the descriptor system let $\Gamma$ be the set of positive real numbers $\gamma$ for which there exists an internally stabilizing dynamic controller such that the transfer function $T_{zw}(s)$ of the closed loop system satisfies

$$\|T_{zw}\|_\infty < \gamma.$$  

In the modified optimal $\mathcal{H}_\infty$ control problem we want to determine $\gamma_{mo} = \inf \Gamma$.

The suboptimal $\mathcal{H}_\infty$ control problem

For a descriptor system and $\gamma \in \Gamma$ with $\gamma > \gamma_{mo}$ determine an internally stabilizing dynamic controller such that the closed loop transfer function satisfies $\|T_{zw}\|_\infty < \gamma$. 
The modified optimal $\mathcal{H}_\infty$ control problem

For the descriptor system let $\Gamma$ be the set of positive real numbers $\gamma$ for which there exists an internally stabilizing dynamic controller such that the transfer function $T_{zw}(s)$ of the closed loop system satisfies

$$\|T_{zw}\|_\infty < \gamma.$$

In the modified optimal $\mathcal{H}_\infty$ control problem we want to determine $\gamma_{mo} = \inf \Gamma$.

The suboptimal $\mathcal{H}_\infty$ control problem

For a descriptor system and $\gamma \in \Gamma$ with $\gamma > \gamma_{mo}$ determine an internally stabilizing dynamic controller such that the closed loop transfer function satisfies $\|T_{zw}\|_\infty < \gamma$. 

6/29 losse@math.tu-berlin.de
Preliminary Assumptions

**A1.** The triple \((E, A, B_2)\) is strongly stabilizable and the triple \((E, A, C_2)\) is strongly detectable.

\((E, A, B_2)\) is called strongly stabilizable, if it is both *finite dynamics stabilizable* i.e. \(\text{rank}[\lambda E - A, B_2] = n\) and *impulse controllable* i.e. \(\text{rank}[E, AS_\infty, B_2] = n\).

\((E, A, C_2)\) is called strongly detectable, if it is both *finite dynamics detectable* i.e. \(\text{rank}[\lambda E^T - A^T, C_2^T] = n\) and *impulse observable* i.e. \(\text{rank}[E^T, A^T T_\infty, C_2^T] = n\).
Preliminary Assumptions

**A1.** The triple \((E, A, B_2)\) is strongly stabilizable and the triple \((E, A, C_2)\) is strongly detectable.

**A2.** \[\text{rank} \begin{bmatrix} A - i\omega E & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2 \text{ for all } \omega \in \mathbb{R}.\]

**A3.** \[\text{rank} \begin{bmatrix} A - i\omega E & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2 \text{ for all } \omega \in \mathbb{R}.\]

**A4.** For matrices \(T_\infty, S_\infty\) with \(\text{Im } S_\infty = \ker E\) and \(\text{Im } T_\infty = \ker E^T\) the rank conditions

\[\text{rank} \begin{bmatrix} T_\infty^T & AS_\infty & T_\infty^T B_2 \\ C_1 S_\infty & D_{12} \end{bmatrix} = n + m_2 - \text{rank} E,\]

\[\text{rank} \begin{bmatrix} T_\infty^T & AS_\infty & T_\infty^T B_1 \\ C_2 S_\infty & D_{21} \end{bmatrix} = n + p_1 - \text{rank} E\]

holds.
Matrix Pencils

Matrix pencils we will use:

\[
\lambda N_H + M_H(\gamma) = \\
\lambda \begin{bmatrix}
0 & -E^T & 0 & 0 & 0 \\
E & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} + \\
\begin{bmatrix}
0 & -A^T & 0 & 0 & -C_1^T \\
-A & 0 & -B_1 & -B_2 & 0 \\
0 & -B_1^T & -\gamma^2 I & 0 & -D_{11}^T \\
0 & -B_2^T & 0 & 0 & -D_{12}^T \\
-C_1 & 0 & -D_{11} & -D_{12} & -I
\end{bmatrix}
\]

and

\[
\lambda N_J + M_J(\gamma) = \\
\lambda \begin{bmatrix}
0 & -E & 0 & 0 & 0 \\
E^T & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} + \\
\begin{bmatrix}
0 & -A & 0 & 0 & -B_1 \\
-A^T & 0 & -C_1 & -C_2 & 0 \\
0 & -C_1 & -\gamma^2 I & 0 & -D_{11} \\
0 & -C_2 & 0 & 0 & -D_{21} \\
-B_1^T & 0 & -D_{11}^T & -D_{21}^T & -I
\end{bmatrix}.
\]

only contain data from the original system.
Even Pencils: \(P(-\lambda)^T = P(\lambda)\).
Deflating Subspaces

Let

\[ X_H(\gamma) = m_1 \begin{bmatrix} X_{H,1}(\gamma) \\ X_{H,2}(\gamma) \\ X_{H,3}(\gamma) \\ X_{H,4}(\gamma) \\ X_{H,5}(\gamma) \end{bmatrix}, \quad X_J(\gamma) = p_1 \begin{bmatrix} X_{J,1}(\gamma) \\ X_{J,2}(\gamma) \\ X_{J,3}(\gamma) \\ X_{J,4}(\gamma) \\ X_{J,5}(\gamma) \end{bmatrix} \]

Deflating Subspaces

Let \( X \in \mathbb{R}^{n,k} \) with full column rank, then \( \text{Im} \ X \) is called **deflating subspace** for the pencil \( \lambda E - A \) if there exists matrices \( Y \in \mathbb{R}^{n,k}, R, U \in \mathbb{R}^{k,k} \) such that

\[(\lambda E - A)X = Y(\lambda R - U).\]

A deflating subspace is called **stable** (**semi-stable**) if all finite eigenvalues of \( \lambda R - U \) are in the open (closed) left half plane.
Deflating Subspaces

Lagrangian Subspaces

Let $\mathcal{J} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$.

- A subspace $\mathcal{L}$ is called isotropic if $x^T \mathcal{J} y = 0$ for all $x, y \in \mathcal{L}$.
- An isotropic subspace with $\dim \mathcal{L} = n$ is called Lagrangian.
Main Result

Theorem

Consider a regular descriptor system of arbitrary index and the even pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$. Suppose that assumptions A1–A4 hold.

Then there exists an internally stabilizing controller such that the transfer function from $w$ to $z$ satisfies $\| T_{zw} \|_\infty < \gamma$ if and only if $\gamma$ is such that the conditions C1–C4 hold.
### Conditions for the General Case

**C1.** The index of both pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$ is at most one.

**C2.** There exists a matrix $X_H(\gamma)$ such that

- **C2.a)** $\text{im } X_H(\gamma)$ is a semi-stable deflating subspace of $\lambda N_H + M_H$;
- **C2.b)** $\text{im } \begin{bmatrix} EX_{H,1}(\gamma) \\ X_{H,2}(\gamma) \end{bmatrix}$ is a $r$-dimensional isotropic subspace of $\mathbb{R}^{2n}$;
- **C2.c)** $\text{rank}(EX_{H,1}(\gamma)) = r$.

**C3.** There exists a matrix $X_J(\gamma)$ such that

- **C3.a)** $\text{im } X_J(\gamma)$ is a semi-stable deflating subspace of $\lambda N_J + M_J$;
- **C3.b)** $\text{im } \begin{bmatrix} E^TX_{J,1}(\gamma) \\ X_{J,2}(\gamma) \end{bmatrix}$ is a $r$-dimensional isotropic subspace of $\mathbb{R}^{2n}$;
- **C3.c)** $\text{rank}(E^TX_{J,1}(\gamma)) = r$.

**C4.** The matrix

$$\mathcal{Y}(\gamma) = \begin{bmatrix} \gamma X_{H,2}(\gamma) EX_{H,1}(\gamma) & X_{H,2}(\gamma) EX_{J,2}(\gamma) \\ X_{J,2}(\gamma) E^TX_{H,2}(\gamma) & \gamma X_{J,2}(\gamma) E^TX_{J,1}(\gamma) \end{bmatrix}.$$  

is positive semidefinite and satisfies $\text{rank}\mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J$.  

---

12/29 losse@math.tu-berlin.de

$\mathcal{H}_\infty$ Control for Descriptor Systems
Conditions for the General Case

C1. The index of both pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$ is at most one.

C2. There exists a matrix $X_H(\gamma)$ such that
   
   C2.a) $\text{im} \ X_H(\gamma)$ is a semi-stable deflating subspace of $\lambda N_H + M_H$;
   
   C2.b) $\text{im} \begin{bmatrix} EX_{H,1}(\gamma) \\ X_{H,2}(\gamma) \end{bmatrix}$ is a $r$-dimensional isotropic subspace of $\mathbb{R}^{2n}$;
   
   C2.c) $\text{rank}(EX_{H,1}(\gamma)) = r$.

C3. There exists a matrix $X_J(\gamma)$ such that
   
   C3.a) $\text{im} \ X_J(\gamma)$ is a semi-stable deflating subspace of $\lambda N_J + M_J$;
   
   C3.b) $\text{im} \begin{bmatrix} E^T X_{J,1}(\gamma) \\ X_{J,2}(\gamma) \end{bmatrix}$ is a $r$-dimensional isotropic subspace of $\mathbb{R}^{2n}$;
   
   C3.c) $\text{rank}(E^T X_{J,1}(\gamma)) = r$.

C4. The matrix

$$\mathcal{Y}(\gamma) = \begin{bmatrix} \gamma X_{H,2}(\gamma)EX_{H,1}(\gamma) & X_{H,2}(\gamma)EX_{J,2}(\gamma) \\ X_{J,2}(\gamma)E^TX_{H,2}(\gamma) & \gamma X_{J,2}(\gamma)E^TX_{J,1}(\gamma) \end{bmatrix}.$$  

is positive semidefinite and satisfies $\text{rank} \mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J$.  

Conditions for the General Case

C1. The index of both pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$ is at most one.

C2. There exists a matrix $X_H(\gamma)$ such that
   
   C2.a) $\text{im } X_H(\gamma)$ is a semi-stable deflating subspace of $\lambda N_H + M_H$;
   
   C2.b) $\text{im } \begin{bmatrix} EX_H,1(\gamma) \\ X_H,2(\gamma) \end{bmatrix}$ is a $r$-dimensional isotropic subspace of $\mathbb{R}^{2n}$;
   
   C2.c) $\text{rank}(EX_H,1(\gamma)) = r$.

C3. There exists a matrix $X_J(\gamma)$ such that
   
   C3.a) $\text{im } X_J(\gamma)$ is a semi-stable deflating subspace of $\lambda N_J + M_J$;
   
   C3.b) $\text{im } \begin{bmatrix} E^T X_J,1(\gamma) \\ X_J,2(\gamma) \end{bmatrix}$ is a $r$-dimensional isotropic subspace of $\mathbb{R}^{2n}$;
   
   C3.c) $\text{rank}(E^T X_J,1(\gamma)) = r$.

C4. The matrix

$$\mathcal{Y}(\gamma) = \begin{bmatrix} \gamma X_{H,2}(\gamma)EX_H,1(\gamma) & X_{H,2}(\gamma)EX_J,2(\gamma) \\ X_{J,2}(\gamma)E^T X_{H,2}(\gamma) & \gamma X_{J,2}(\gamma)E^T X_J,1(\gamma) \end{bmatrix}.$$  

is positive semidefinite and satisfies $\text{rank}\mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J$.  

Losse@math.tu-berlin.de
Conditions for the General Case

**C1.** The index of both pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$ is at most one.

**C2.** There exists a matrix $X_H(\gamma)$ such that

- **C2.a)** $\text{im } X_H(\gamma)$ is a semi-stable deflating subspace of $\lambda N_H + M_H$;
- **C2.b)** $\text{im } \begin{bmatrix} EX_{H,1}(\gamma) \\ X_{H,2}(\gamma) \end{bmatrix}$ is a $r$-dimensional isotropic subspace of $\mathbb{R}^{2n}$;
- **C2.c)** $\text{rank}(EX_{H,1}(\gamma)) = r$.

**C3.** There exists a matrix $X_J(\gamma)$ such that

- **C3.a)** $\text{im } X_J(\gamma)$ is a semi-stable deflating subspace of $\lambda N_J + M_J$;
- **C3.b)** $\text{im } \begin{bmatrix} E^T X_{J,1}(\gamma) \\ X_{J,2}(\gamma) \end{bmatrix}$ is a $r$-dimensional isotropic subspace of $\mathbb{R}^{2n}$;
- **C3.c)** $\text{rank}(E^T X_{J,1}(\gamma)) = r$.

**C4.** The matrix

$$\mathcal{Y}(\gamma) = \begin{bmatrix} \gamma X_{H,2}(\gamma) & X_{H,2}(\gamma) \\ \gamma X_{J,2}(\gamma) & \gamma X_{J,2}(\gamma) \end{bmatrix}$$

is positive semidefinite and satisfies $\text{rank } \mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J$. 

12/29 losse@math.tu-berlin.de
Conditions for the General Case

C1. The index of both pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$ is at most one.

C2. There exists a matrix $X_H(\gamma)$ such that
   - C2.a) $\text{im } X_H(\gamma)$ is a semi-stable deflating subspace of $\lambda N_H + M_H$;
   - C2.b) $\text{im } \begin{bmatrix} E X_{H,1}(\gamma) \\ X_{H,2}(\gamma) \end{bmatrix}$ is a $r$-dimensional isotropic subspace of $\mathbb{R}^{2n}$;
   - C2.c) $\text{rank}(EX_{H,1}(\gamma)) = r$.

C3. There exists a matrix $X_J(\gamma)$ such that
   - C3.a) $\text{im } X_J(\gamma)$ is a semi-stable deflating subspace of $\lambda N_J + M_J$;
   - C3.b) $\text{im } \begin{bmatrix} E^T X_{J,1}(\gamma) \\ X_{J,2}(\gamma) \end{bmatrix}$ is a $r$-dimensional isotropic subspace of $\mathbb{R}^{2n}$;
   - C3.c) $\text{rank}(E^T X_{J,1}(\gamma)) = r$.

C4. The matrix

$\mathcal{Y}(\gamma) = \begin{bmatrix} \gamma X_{H,2}(\gamma) E X_{H,1}(\gamma) & X_{H,2}(\gamma) E X_{J,2}(\gamma) \\ X_{J,2}(\gamma) E^T X_{H,2}(\gamma) & \gamma X_{J,2}(\gamma) E^T X_{J,1}(\gamma) \end{bmatrix}$.

is positive semidefinite and satisfies $\text{rank } \mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J$. 

12/29 losse@math.tu-berlin.de

H∞ Control for Descriptor Systems
Sketch of proof

The proof is mainly based on

- Existence of a preliminary index reducing feedback
  [Bunse-Gerstner/Byers/Mehrmann/Nichols ’99]
- Weierstraß canonical form [Gantmacher ’59]
- Pencil based approach for standard systems
  [Benner/Byers/Mehrmann/Xu ’04]

Neither the computation of the index reducing feedback nor of the Weierstraß canonical form is necessary.
Procedure 1: (Classification of $\gamma$)

**Input:** Data of system, value $\gamma \geq 0$.

**Output:** Decision whether $\gamma < \gamma_{mo}$ or $\gamma \geq \gamma_{mo}$.

1. Form the pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$.
2. Compute the deflating subspace matrices $X_H$ and $X_J$ associated with the eigenvalues in the closed left half plane.
3. IF the dimension of one/both of these subspaces is less than $r$, then $\gamma < \gamma_{mo}$,
   ELSE
   IF the rank of $EX_{H,1}$ and/or $E^T X_{J,1}$ is less than $r$, then $\gamma < \gamma_{mo}$,
   ELSE
   Form the matrix $\hat{Y}$.
   IF $\hat{Y}$ is not symmetric positive semi-definite and/or $\text{rank} \, \hat{Y} < \hat{k}_H + \hat{k}_J$, then $\gamma < \gamma_{mo}$,
   ELSE $\gamma \geq \gamma_{mo}$. 

Computation

- The main part of the algorithm is the computation of the deflating subspaces.
- These subspaces could be computed with the QZ-Algorithm, that however does not take advantage of the special structure of the matrix pencils or its eigensymmetry.
- Therefore we recommend a structure preserving algorithm to compute the eigenvalues and deflating subspaces of the even matrix pencils as has been introduced by [Benner/Byers/Mehrmann/Xu ‘99]...
Spectral Properties

Hamiltonian eigensymmetry

Even pencils exhibit the Hamiltonian eigensymmetry:
if $\lambda$ is a finite eigenvalue of $\mathcal{H} - \lambda \mathcal{S}$, then $\bar{\lambda}, -\lambda, -\bar{\lambda}$ are eigenvalues of $\mathcal{H} - \lambda \mathcal{S}$, too.

Typical Hamiltonian spectrum:
Computation

The main part of the algorithm is the computation of the deflating subspaces.

These subspaces could be computed with the QZ-Algorithm, that however does not take advantage of the special structure of the matrix pencils or its eigensymmetry.

Therefore we recommend a structure preserving algorithm to compute the eigenvalues and deflating subspaces of the even matrix pencils as has been introduced by [Benner/Byers/Mehrmann/Xu ‘99].
Structured real skew-Hamiltonian/Hamiltonian Schur Form \[ \text{[Mehl '99]} \]

Let \( \mathcal{H} - \lambda S \) be a regular real skew-Hamiltonian/Hamiltonian pencil. Under certain conditions on the purely imaginary and infinite eigenvalues there exists an (orthogonal) \( J \)-congruence

\[
J Y^T J^T (\mathcal{H} - \lambda S) Y = \begin{bmatrix} H_{11} & H_{12} \\ 0 & -H_{11}^T \end{bmatrix} - \lambda \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^T \end{bmatrix},
\]

where \( H_{11} \) is quasi-upper triangular, \( S_{11} \) is upper triangular, \( H_{12} \) is symmetric, and \( S_{12} \) is skew-symmetric.

- Not every skew-Hamiltonian/Hamiltonian pencil has such a structured Schur form.
- Embedding in an extended pencil of double size resolves existence problem. \[ \text{[Benner/Byers/Mehrmann/Xu '99]} \]
Structured real skew-Hamiltonian/Hamiltonian Schur Form

Let $\mathcal{H} - \lambda S$ be a regular real skew-Hamiltonian/Hamiltonian pencil. Under certain conditions on the purely imaginary and infinite eigenvalues there exists an (orthogonal) $\mathcal{J}$-congruence

$$\mathcal{J} Y^T \mathcal{J}^T (\mathcal{H} - \lambda S) Y = \begin{bmatrix} H_{11} & H_{12} \\ 0 & -H_{11}^T \end{bmatrix} - \lambda \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^T \end{bmatrix},$$

where $H_{11}$ is quasi-upper triangular, $S_{11}$ is upper triangular, $H_{12}$ is symmetric, and $S_{12}$ is skew-symmetric.

- Not every skew-Hamiltonian/Hamiltonian pencil has such a structured Schur form.
- Embedding in an extended pencil of double size resolves existence problem.  

[Benner/Byers/Mehrmann/Xu '99]
Theorem

Let $\mathcal{H} - \lambda S$ be a real regular skew-Hamiltonian/Hamiltonian pencil, then there exist orthogonal matrices $Q_1, Q_2$ such that

$$Q_1^T \mathcal{H} Q_2 = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix},$$

$$Q_1^T S J Q_1 J^T = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^T \end{bmatrix} \in \mathcal{S}\mathcal{H}_{2n},$$

$$J Q_2^T J^T S Q_2 = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{11}^T \end{bmatrix} \in \mathcal{S}\mathcal{H}_{2n},$$

where $H_{11}, S_{11}, T_{11}$ are upper triangular and $H_{22}^T$ is quasi-upper triangular. The eigenvalues of $\mathcal{H} - \lambda S$ are given by $\pm \Lambda(S_{11}^{-1} H_{11} T_{11}^{-1} H_{22}^T)^{\frac{1}{2}}$. 
Embedding in Extended sH/H-Pencil (I)

Consider a skew-Hamiltonian/Hamiltonian pencil of the form

\[
H - \lambda S = \begin{bmatrix} F & G \\ H & -F^T \end{bmatrix} - \lambda \begin{bmatrix} A & B \\ C & A^T \end{bmatrix}
\]

where \(B\) and \(C\) are skew-symmetric and \(G\) and \(H\) are symmetric.

Now let

\[
B_H = \begin{bmatrix} H & 0 \\ 0 & -H \end{bmatrix}, \quad B_S = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix},
\]

and

\[
\varphi_r = \frac{\sqrt{2}}{2} \begin{bmatrix} l_{2n} & l_{2n} \\ -l_{2n} & l_{2n} \end{bmatrix}, \quad P = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}.
\]

Then

\[
\varphi_r^T B_H \varphi_r = \begin{bmatrix} 0 & H \\ H & 0 \end{bmatrix}, \quad \varphi_r^T B_S \varphi_r = B_S.
\]
Embedding in Extended sH/H-Pencil (I)

Consider a skew-Hamiltonian/Hamiltonian pencil of the form

\[ H - \lambda S = \begin{bmatrix} F & G \\ H & -F^T \end{bmatrix} - \lambda \begin{bmatrix} A & B \\ C & A^T \end{bmatrix} \]

where \( B \) and \( C \) are skew-symmetric and \( G \) and \( H \) are symmetric.

Now let

\[ B_H = \begin{bmatrix} H \\ 0 \\ -H \end{bmatrix}, \quad B_S = \begin{bmatrix} S \\ 0 \\ S \end{bmatrix}, \quad (1) \]

and

\[ \mathcal{Y}_r = \frac{\sqrt{2}}{2} \begin{bmatrix} l_{2n} & l_{2n} \\ -l_{2n} & l_{2n} \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}. \]

Then

\[ \mathcal{Y}_r^T B_H \mathcal{Y}_r = \begin{bmatrix} 0 & H \\ H & 0 \end{bmatrix}, \quad \mathcal{Y}_r^T B_S \mathcal{Y}_r = B_S. \]
Embedding in Extended sH/H-Pencil (II)

Set

\[ B_H^r - \lambda B_S^r := P^T Y_r^T (B_H - \lambda B_S) Y_r P \]

\[ = \begin{bmatrix}
0 & F & 0 & G \\
F & 0 & G & 0 \\
0 & H & 0 & -F^T \\
H & 0 & -F^T & 0
\end{bmatrix} - \lambda \begin{bmatrix}
A & 0 & B & 0 \\
0 & A & 0 & B \\
C & 0 & A^T & 0 \\
0 & C & 0 & A^T
\end{bmatrix} \]
Computation of the Structured Schur Form

With \( \tilde{Q} = \mathcal{P}^T \text{diag}(\mathcal{J} Q_1 \mathcal{J}^T, Q_2) \mathcal{P} \), where \( Q_1, Q_2 \) are as in generalized SURV, we obtain

\[
\mathcal{J} \tilde{Q}^T \mathcal{J}^T B_r \tilde{Q} = \begin{bmatrix}
0 & H_{11} & 0 & H_{12} \\
-H_{22}^T & 0 & H_{12}^T & 0 \\
0 & 0 & 0 & H_{22} \\
0 & 0 & -H_{11}^T & 0
\end{bmatrix} =: \begin{bmatrix}
\tilde{H}_{11} & \tilde{H}_{12} \\
0 & -\tilde{H}_{11}^T
\end{bmatrix},
\]

\[
\mathcal{J} \tilde{Q}^T \mathcal{J}^T B_s \tilde{Q} = \begin{bmatrix}
S_{11} & 0 & S_{12} & 0 \\
0 & T_{11} & 0 & T_{12} \\
0 & 0 & \tilde{S}_{11}^T & 0 \\
0 & 0 & 0 & T_{11}^T
\end{bmatrix} =: \begin{bmatrix}
\tilde{S}_{11} & \tilde{S}_{12} \\
0 & \tilde{S}_{11}^T
\end{bmatrix}.
\]

Re-ordering the structured Schur decomposition \( \implies \)

\[
\begin{bmatrix}
H_{11} & H_{12} \\
0 & -H_{11}^T
\end{bmatrix} - \lambda \begin{bmatrix}
S_{11} & S_{12} \\
0 & S_{11}^T
\end{bmatrix},
\]

where \( \Lambda(H, S) \cap \mathbb{C}^- \subset \Lambda(H_{11}, S_{11}) \).
Structured Schur Form of Embedded sH/H-pencil

Theorem

Let $\mathcal{H} - \lambda S$ be a skew-Hamiltonian/Hamiltonian pencil and consider the extended matrices $\mathcal{B}_\mathcal{H} = \text{diag}(\mathcal{H}, -\mathcal{H})$, $\mathcal{B}_S = \text{diag}(S, S)$.

a) There exist unitary $\mathcal{W}, \mathcal{V}$ such that

$$
\mathcal{W}^T \mathcal{B}_\mathcal{H} \mathcal{V} = \begin{bmatrix}
\mathcal{H}_{11} & \mathcal{H}_{12} \\
0 & \mathcal{H}_{22}
\end{bmatrix},
$$

$$
\mathcal{W}^T \mathcal{B}_S \mathcal{V} = \begin{bmatrix}
S_{11} & S_{12} \\
0 & S_{22}
\end{bmatrix},
$$

where $\mathcal{H}_{11}, S_{11} \in \mathbb{R}^{2n,2n}$ and

$$
\Lambda(\mathcal{B}_S, \mathcal{B}_\mathcal{H}) \cap \mathbb{C}^- \subset \Lambda(S_{11}, \mathcal{H}_{11}),
$$

$$
\Lambda(S_{11}, \mathcal{H}_{11}) \cap \Lambda(\mathcal{B}_S, \mathcal{B}_\mathcal{H}) \cap \mathbb{C}^+ = \emptyset.
$$
Structured Schur Form of Embedded $sH/H$-pencil

**Theorem**

Let $\mathcal{H} - \lambda S$ be a skew-Hamiltonian/Hamiltonian pencil and consider the extended matrices $\mathcal{B}_H = \text{diag}(\mathcal{H}, -\mathcal{H})$, $\mathcal{B}_S = \text{diag}(S, S)$.

a) There exist unitary $W, V$ such that

$$W^T \mathcal{B}_H V = \begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ 0 & \mathcal{H}_{22} \end{bmatrix},$$

$$W^T \mathcal{B}_S V = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}.$$

b) Let $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \in \mathbb{R}^{4n, 2n} = V(:, 1:2n)$, then

$$\text{Def}_{-}(\mathcal{H}, S) \subset \text{range } V_1, \quad \text{Def}_{+}(\mathcal{H}, S) \subset \text{range } V_2.$$

Equality holds if $\not\exists$ eigenvalues $0, \infty$. 

losse@math.tu-berlin.de

$\mathcal{H}_\infty$ Control for Descriptor Systems
Computation of deflating subspaces

- Compute generalized symplectic URV of original pencils
- Embed pencils
- Compute structured Schur forms
- Reorder the eigenvalues
- Extract deflating subspaces from transformation matrices

Our experimental code for a $\gamma$-Iteration relying on this algorithm shows promising results.
We consider the following example [Takaba/Morihira/Katayama, 94], [Rehm/Allgöwer, 98].

\[ E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \]

\[ C_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_{21} = 1 \]

- \((E, A)\) is of index 2.
- goal: find the minimum value \(\gamma\) that satisfies the conditions \(C_1 - C_4\).
- \(\gamma_{opt}\) is calculated as \(\gamma^p = 0.7678\) which is smaller than the calculated values using the LMI approach or the Riccati approach.
Introduction

Modified Optimal $\mathcal{H}_\infty$ Control

Suboptimal $\mathcal{H}_\infty$ Control

The modified optimal $\mathcal{H}_\infty$ control problem

For the descriptor system let $\Gamma$ be the set of positive real numbers $\gamma$ for which there exists an internally stabilizing dynamic controller such that the transfer function $T_{zw}(s)$ of the closed loop system satisfies

$$\|T_{zw}\|_\infty < \gamma.$$ 

In the modified optimal $\mathcal{H}_\infty$ control problem we want to determine $\gamma_{mo} = \inf \Gamma$.

The suboptimal $\mathcal{H}_\infty$ control problem

For a descriptor system and $\gamma \in \Gamma$ with $\gamma > \gamma_{mo}$ determine an internally stabilizing dynamic controller such that the closed loop transfer function satisfies $\|T_{zw}\|_\infty < \gamma$. 

26/29 losse@math.tu-berlin.de

$\mathcal{H}_\infty$ Control for Descriptor Systems
Consider a regular descriptor system of arbitrary index. Suppose that assumptions \( A1-A4 \) hold, \( \gamma > \gamma_{mo} \) and \( \bar{\sigma}(D_{11}) < \gamma \). Then the sub-optimal \( \mathcal{H}_\infty \) control problem has an internally stabilizing controller such that the \( \mathcal{H}_\infty \) norm of the closed loop is less than \( \gamma \) given by:

\[
(-\lambda \hat{E} + \hat{A}) = X_J^T \bar{\Pi}(\lambda) X_H \\
\hat{B} = X_J^T \bar{B}_\Pi \\
\hat{C} = \bar{C}_\Pi X_H \\
\hat{D} = \bar{D}_\Pi
\]
\( \bar{\Pi}(\lambda), \bar{\Pi}_B, \bar{\Pi}_C, \bar{\Pi}_D \) are matrices containing original system data and a \( m_2 \times p_2 \) feedback matrix \( F \) such that \( (E, A + B_2 FC_2) \) is of index one.

- Computation of index reducing feedback necessary
- We also have formulas for the parametrized controller
- Then computation of kernel and cokernel of \( E \) is also necessary
Conclusions

- Existence conditions for $\mathcal{H}_\infty$ controllers in terms of the original system data
- Structure preserving Algorithm for the computation of the deflating subspaces
- Controller formulas in terms of the original system (plus Index reducing Feedback)