Self-adjoint Differential-Algebraic and Difference Operators and their Application

Lena Scholz
(joint work with Volker Mehrmann)

Institut für Mathematik
Technische Universität Berlin

BIRS Workshop on Control and Optimization with Differential-Algebraic Constraints
Banff, 25th-29th October 2010
Motivation

1. Motivation

2. Optimal Control of DAE Systems
   - Continuous-time Linear-Quadratic Optimal Control Problem
   - Discrete-time Linear-Quadratic Optimal Control Problem

3. Linear Self-adjoint Operators
   - Self-adjoint Differential-Algebraic Operators
   - Self-adjoint Difference Operators

4. Structure Preserving Discretization

5. Conclusion
Linear-Quadratic Optimal Control Problem

- Minimizing a quadratic cost functional

\[ J(x, u) = \frac{1}{2} \int_{t_0}^{t_f} (x^T W x + 2x^T S u + u^T R u) \, dt, \]

with \( W = W^T \in \mathbb{R}^{n,n} \), \( S \in \mathbb{R}^{n,m} \) and \( R = R^T \in \mathbb{R}^{m,m} \)

- subject to the system dynamics given by the descriptor system

\[ E \dot{x} + Ax + Bu = 0, \quad x(t_0) = 0, \]

with \( E, A \in \mathbb{R}^{n,n}, B \in \mathbb{R}^{n,m} \),

- \( x(t) \in \mathbb{R}^n \) state vector, \( u(t) \in \mathbb{R}^m \) control input vector.

- Goal: determine optimal controls \( u \in U = C^0(\mathbb{I}, \mathbb{R}^m) \).
**Necessary conditions for optimality**

Let \( u_* \) define the minimal solution and let \( x_* \) be the corresponding trajectory, i.e., the solution of

\[
E \dot{x}(t) + Ax(t) + Bu_*(t) = 0, \quad x(t_0) = 0.
\]

Then there exists a costate function \( \zeta(t) \), such that \((x_*(t), \zeta(t), u_*(t))\) satisfy the Euler-Lagrange boundary value problem:

\[
\begin{bmatrix}
0 & E & 0 \\
-E^T & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\zeta}(t) \\
\dot{x}(t) \\
\dot{u}(t)
\end{bmatrix}
+ \begin{bmatrix}
0 & A & B \\
A^T & W & S \\
B^T & S^T & R
\end{bmatrix}
\begin{bmatrix}
\zeta(t) \\
x(t) \\
u(t)
\end{bmatrix}
= 0,
\]

with boundary conditions \( x(t_0) = 0 \) and \( E^T \zeta(t_f) = 0 \).
Even matrix pencils

The associated matrix pair

\[(\mathcal{N}, \mathcal{M}) = \left( \begin{bmatrix} 0 & E & 0 \\ -E^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A & B \\ A^T & W & S \\ B^T & S^T & R \end{bmatrix} \right) \]

is a so-called even matrix pair, i.e.,

\[\mathcal{N} = -\mathcal{N}^T \text{ and } \mathcal{M} = \mathcal{M}^T,\]

since the associated linear matrix polynomial

\[\mathcal{P}(\lambda) = \lambda \mathcal{N} + \mathcal{M}\]

is an even polynomial

\[\mathcal{P}(\lambda) = \lambda \mathcal{N} + \mathcal{M} = (-\lambda)(-\mathcal{N}^T) - \mathcal{M}^T = \mathcal{P}^T(-\lambda).\]
Reduced Euler Lagrange equations

If $E$ and $R$ are invertible then we obtain the equivalent reduced Euler-Lagrange system

$$
\begin{bmatrix}
\dot{x} \\
\dot{\xi}
\end{bmatrix} + \begin{bmatrix}
F & G \\
H & -F^T
\end{bmatrix} \begin{bmatrix}
x \\
\xi
\end{bmatrix} = 0, \quad x(t_0) = 0, \quad \xi(t_f) = 0,
$$

with $\xi = -E^T\zeta$ and with the Hamiltonian matrix

$$
\begin{bmatrix}
F & G \\
H & -F^T
\end{bmatrix} = \begin{bmatrix}
E^{-1}(A - BR^{-1}S^T) & E^{-1}BR^{-1}B^TE^{-T} \\
W - SR^{-1}S^T & -(E^{-1}(A - BR^{-1}S^T))^T
\end{bmatrix}
$$

In general:

- Even matrix pencils generalize Hamiltonian matrices.
- Even matrix pencils have Hamiltonian spectrum plus possibly some extra infinite eigenvalues or singular parts.
Discretization of Hamiltonian systems

- The discretization of an Hamiltonian system

  \[ \dot{x} = \mathcal{H} x, \quad \text{with} \quad \mathcal{H} J = (\mathcal{H} J)^T, \quad J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \]

  with symplectic integration methods yields a discrete system

  \[ x_{i+1} = S x_i, \quad x_i \approx x(t_i) \quad \text{for some} \quad t_i \in [t_0, t_f] \]

  with symplectic iteration matrix \( S \), i.e., \( S^T J S = J \).

- Using symplectic methods the total energy of the system (i.e., the Hamiltonian function of the dynamical system) and the symplecticity of the flow is preserved.
Palindromic Matrix Polynomials

- A matrix polynomial

\[ P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i \]

of degree \( k \), where \( A_i \in \mathbb{R}^{n,n} \), is said to be **palindromic** if

\[ \lambda^k P^T(1/\lambda) = P(\lambda), \]

i.e., if

\[ A_{k-i}^T = A_i \quad \text{for} \quad i = 0, \ldots, k. \]

- Palindromic matrix polynomials generalize symplectic matrices.
- The spectrum of a palindromic polynomial is symmetric w.r.t. the unit circle and if 0 is an eigenvalue then also \( \infty = \frac{1}{0} \).
Example

- For an Hamiltonian system
  \[ \dot{x} = \mathcal{H}x \]
  a discretization with the **implicit midpoint rule** yields
  \[
  (I_n - \frac{h}{2}\mathcal{H})x_{i+1} = (I_n + \frac{h}{2}\mathcal{H})x_i,
  \]
  \[
  x_{i+1} = (I_n - \frac{h}{2}\mathcal{H})^{-1}(I_n + \frac{h}{2}\mathcal{H})x_i = Sx_i,
  \]
  with **symplectic matrix**
  \[ S = (\sigma I_n - \mathcal{H})^{-1}(\sigma I_n + \mathcal{H}) \text{ for } \sigma = \frac{2}{h}. \]

- Discretization of an even system
  \[ N\dot{x} + Mx = 0, \quad N = -N^T, \quad M = M^T, \]
  with the implicit midpoint rule yields
  \[
  (N + \frac{h}{2}M)x_{i+1} + (-N + \frac{h}{2}M)x_i = 0,
  \]
  i.e., a **palindromic difference equation**.
# Generalization of Hamiltonian/Symplectic Structures

## Continuous Time

<table>
<thead>
<tr>
<th>Hamiltonian system</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{x} = \mathcal{H}x, \quad \mathcal{H}J = (\mathcal{H}J)^T )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Even system</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{N}\dot{x} + \mathcal{M}x = 0, \quad \mathcal{N} = -\mathcal{N}^T, \quad \mathcal{M} = \mathcal{M}^T )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Self–adjoint system</th>
</tr>
</thead>
</table>
| \( \mathcal{N}(t)\dot{x} + \mathcal{M}(t)x = 0, \quad \mathcal{N} = -\mathcal{N}^T \)
| \( \mathcal{M} = (\mathcal{M} - \mathcal{N})^T \) |

## Discrete Time

<table>
<thead>
<tr>
<th>Symplectic system</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_{i+1} = Sx_i, \quad S^TJS = J )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Palindromic system</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P^T x_{i+1} + Px_i = 0 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(Discrete) self–adjoint system</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P^T(i + 1)x_{i+1} + Q(i)x_i + P(i)x_{i-1} = 0, \quad Q(i) = Q^T(i) )</td>
</tr>
</tbody>
</table>
Outline

1 Motivation

2 Optimal Control of DAE Systems
   - Continuous-time Linear-Quadratic Optimal Control Problem
   - Discrete-time Linear-Quadratic Optimal Control Problem

3 Linear Self-adjoint Operators
   - Self-adjoint Differential-Algebraic Operators
   - Self-adjoint Difference Operators

4 Structure Preserving Discretization

5 Conclusion
Outline

1. Motivation

2. Optimal Control of DAE Systems
   - Continuous-time Linear-Quadratic Optimal Control Problem
   - Discrete-time Linear-Quadratic Optimal Control Problem

3. Linear Self-adjoint Operators
   - Self-adjoint Differential-Algebraic Operators
   - Self-adjoint Difference Operators

4. Structure Preserving Discretization

5. Conclusion
The linear-quadratic optimal control problem

- Minimize the quadratic cost functional

\[ J(x, u) = \frac{1}{2} \int_{t_0}^{t_f} (x^T W(t)x + x^T S(t)u + u^T R(t)u) \, dt, \]

where \( W = W^T \in C^0(\mathbb{I}, \mathbb{R}^{n,n}) \), \( S \in C^0(\mathbb{I}, \mathbb{R}^{n,m}) \), \( R = R^T \in C^0(\mathbb{I}, \mathbb{R}^{m,m}) \).

- subject to the constraint

\[ E(t) \dot{x} + A(t)x + B(t)u = f(t), \quad x(t_0) = 0, \]

where \( E \in C^1(\mathbb{I}, \mathbb{R}^{n,n}) \), \( A \in C^0(\mathbb{I}, \mathbb{R}^{n,n}) \), \( B \in C^0(\mathbb{I}, \mathbb{R}^{n,m}) \), \( f \in C^0(\mathbb{I}, \mathbb{R}^{n}) \)
sufficiently smooth.
Reduced problem

- For control problems of the form

\[
E(t)\dot{x} + A(t)x + B(t)u = f(t), \quad x(t_0) = 0,
\]

- a behavior approach by introducing \( z = [x^T, u^T]^T \) leads to

\[
\mathcal{E}(t)\dot{z} + A(t)z = f(t),
\]

with \( \mathcal{E}(t) = \begin{bmatrix} E(t) & 0 \end{bmatrix} \), \( A(t) = \begin{bmatrix} A(t) & B(t) \end{bmatrix} \)

- Using derivative arrays we obtain a reduced system:

\[
\begin{bmatrix}
\hat{E}_1(t) \\
0
\end{bmatrix} \dot{z} + \begin{bmatrix}
\hat{A}_1(t) \\
\hat{A}_2(t)
\end{bmatrix} z = \begin{bmatrix}
\hat{f}_1(t) \\
\hat{f}_2(t) \\
\hat{f}_3(t)
\end{bmatrix},
\]

\( \hat{d} \) differential equations

\( \hat{a} \) algebraic equations

\( \hat{u} \) consistency equations

We assume from now on that the system is regular and given in reduced form.
Theorem (Kunkel & Mehrmann ’08)

Consider the linear quadratic DAE optimal control problem with a consistent initial condition. Suppose that the system is strangeness-free as a behavior system. If \((x, u) \in X \times U\) is a solution to this optimal control problem, then there exists a Lagrange multiplier function \(\zeta \in C_{E+E}(I, \mathbb{R}^n)\) with

\[
C_{E+E}(I, \mathbb{R}^n) = \left\{ x \in C^0(I, \mathbb{R}^n) \mid E^+Ex \in C^1(I, \mathbb{R}^n) \right\}.
\]

such that \((x, \zeta, u)\) satisfy the optimality boundary value problem

\[
E \frac{d}{dt}(E^+Ex) + (A - E \frac{d}{dt}(E^+E))x + Bu = f, \quad (E^+Ex)(t_0) = 0, \\
-E^T \frac{d}{dt}(EE^+\zeta) + Wx + Su + (A - EE^+\dot{E})^T\zeta = 0, \quad (EE^+\zeta)(t_f) = 0, \\
S^T x + Ru + B^T\zeta = 0.
\]
The differential-algebraic operator

If the coefficients are sufficiently smooth then the differential-algebraic operator corresponding to the boundary value problem is given by

\[
\begin{bmatrix}
0 & E(t) & 0 \\
-E^T(t) & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\frac{d}{dt} + \begin{bmatrix}
0 & A(t) & B(t) \\
A^T(t) - \dot{E}^T(t) & W(t) & S(t) \\
B^T(t) & S^T(t) & R(t)
\end{bmatrix}.
\]

The associated DAE operator is formally self-adjoint in $L_2$.

Analogous linear operators are obtained for higher order optimal control problems.
Outline

1 Motivation

2 Optimal Control of DAE Systems
   - Continuous-time Linear-Quadratic Optimal Control Problem
   - Discrete-time Linear-Quadratic Optimal Control Problem

3 Linear Self-adjoint Operators
   - Self-adjoint Differential-Algebraic Operators
   - Self-adjoint Difference Operators

4 Structure Preserving Discretization

5 Conclusion
Discrete-time Linear-Quadratic Optimal Control Problem

Minimize the cost functional

\[ J(x, u) = \frac{1}{2} \sum_{j=0}^{\infty} \left( x_j^T W x_j + 2x_j^T S u_j + u_j^T R u_j \right) \]

subject to

\[ E x_{j+1} + A x_j + B u_j = 0, \quad j = 0, 1, \ldots \]

with given starting value \( x_0 \in \mathbb{R}^n \) and coefficient matrices \( W = W^T \in \mathbb{R}^{n,n}, S \in \mathbb{R}^{n,m}, R = R^T \in \mathbb{R}^{m,m} \) and \( E, A \in \mathbb{R}^{n,n}, B \in \mathbb{R}^{n,m} \).

- Classical case: \( \hat{R} = \begin{bmatrix} W & S \\ S^T & R \end{bmatrix} \) symm.pos.def., \( E \) nonsingular.
- Discrete-time \( H_\infty \) control: \( \hat{R} \) indefinite or singular.
- Descriptor system: \( E \) singular.
Maximum Principle

- Introducing Lagrange multipliers $m_j = [-\nu_j^T - \tilde{\nu}_j^T]^T$ with $\nu_j \in \mathbb{R}^n$ and $\tilde{\nu}_j \in \mathbb{R}^{(k-1)n}$ and applying the Pontryagin maximum principle.

- This leads to the two-point boundary value problem

$$
\begin{bmatrix}
0 & E & 0 \\
A^T & 0 & 0 \\
B^T & 0 & 0
\end{bmatrix}
\begin{bmatrix}
m_{j+1} \\
x_{j+1} \\
u_{j+1}
\end{bmatrix} +
\begin{bmatrix}
0 & A & B \\
E^T & W & S \\
0 & S^T & R
\end{bmatrix}
\begin{bmatrix}
m_j \\
x_j \\
u_j
\end{bmatrix} = 0,
$$

with original initial condition and terminal condition $\lim_{j \to \infty} E^T m_j = 0$. 

Transformation into Palindromic form

- Shift the first block row one step downwards and introduce another boundary value $x_{-1} = 0$ to obtain

$$
\begin{bmatrix}
0 & 0 & 0 \\
A^T & 0 & 0 \\
B^T & 0 & 0
\end{bmatrix}
\begin{bmatrix}
m_{j+1} \\
x_{j+1} \\
u_{j+1}
\end{bmatrix}
+ 
\begin{bmatrix}
0 & E & 0 \\
E^T & W & S \\
0 & S^T & R
\end{bmatrix}
\begin{bmatrix}
m_j \\
x_j \\
u_j
\end{bmatrix}
+ 
\begin{bmatrix}
0 & A & B \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
m_{j-1} \\
x_{j-1} \\
u_{j-1}
\end{bmatrix}
= 0.
$$
Transformation into Palindromic form

This can be extended to variable coefficients

\[
\begin{bmatrix}
0 & 0 & 0 \\
A_j^T & 0 & 0 \\
B_j^T & 0 & 0
\end{bmatrix}
\begin{bmatrix}
m_{j+1} \\
x_{j+1} \\
u_{j+1}
\end{bmatrix}
+ \begin{bmatrix}
0 & E_j & 0 \\
E_j^T & W_j & S_j \\
0 & S_j^T & R_j
\end{bmatrix}
\begin{bmatrix}
m_j \\
x_j \\
u_j
\end{bmatrix}
+ \begin{bmatrix}
0 & A_{j-1} & B_{j-1} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
m_{j-1} \\
x_{j-1} \\
u_{j-1}
\end{bmatrix}
= 0.
\]

This corresponds to a self-adjoint difference operator in \( \ell^2 \).
Outline

1 Motivation

2 Optimal Control of DAE Systems
   - Continuous-time Linear-Quadratic Optimal Control Problem
   - Discrete-time Linear-Quadratic Optimal Control Problem

3 Linear Self-adjoint Operators
   - Self-adjoint Differential-Algebraic Operators
   - Self-adjoint Difference Operators

4 Structure Preserving Discretization

5 Conclusion
Outline

1 Motivation

2 Optimal Control of DAE Systems
   - Continuous-time Linear-Quadratic Optimal Control Problem
   - Discrete-time Linear-Quadratic Optimal Control Problem

3 Linear Self-adjoint Operators
   - Self-adjoint Differential-Algebraic Operators
   - Self-adjoint Difference Operators

4 Structure Preserving Discretization

5 Conclusion
Linear DAE operators

Consider a linear $k$-th order differential-algebraic operator

$$
\mathcal{L} : \mathbb{X} \to \mathbb{Y}, \quad x \mapsto \mathcal{L}x = \sum_{i=0}^{k} A_i(t)x^{(i)},
$$
on $\mathbb{I} = [t_0, t_f]$ with sufficiently smooth matrix-valued functions $A_i \in C(\mathbb{I}, \mathbb{R}^{n,n})$ for $i = 0, \ldots, k$ acting on the Hilbert space

$$
L^2(\mathbb{I}, \mathbb{R}^n) := \left\{ x : \mathbb{I} \to \mathbb{R}^n \bigg| \int_{\mathbb{I}} \| x(t) \|^2 dt \text{ exists and is finite} \right\}
$$
with standard $L^2$-inner product

$$
\langle x, y \rangle = \int_{t_0}^{t_f} x^T(t)y(t)dt \quad \text{for all } x, y \in L_2(\mathbb{I}, \mathbb{R}^n).
$$
and function spaces $\mathbb{X} \subset L^2(\mathbb{I}, \mathbb{R}^n)$ (domain of $\mathcal{L}$), $\mathbb{Y} \subset L^2(\mathbb{I}, \mathbb{R}^n)$. 
Reduced Form

Assume that the matrix pencil

$$(A_k(t), A_{k-1}(t), \ldots, A_0(t))$$

is regular (i.e. $\det(P(\lambda))$ does not vanish identically) and given in reduced form

$$
\begin{pmatrix}
A_{k,1}(t) & 0 & 0 & \cdots & 0 \\
0 & A_{k-1,1}(t) & A_{k-1,2}(t) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & A_0(t) \\
0 & 0 & \cdots & \cdots & A_0(t)_{k+1}(t)
\end{pmatrix}
$$

with pointwise nonsingular matrix

$$
\begin{bmatrix}
A_{k,1}(t) \\
A_{k-1,2}(t) \\
\vdots \\
A_{0,k+1}(t)
\end{bmatrix}
$$
The Adjoint Operator

Definition

For a linear differential operator $\mathcal{L} : X \to Y$ the adjoint operator $\mathcal{L}^* : Y^* \to X^*$ is the operator with domain

$$Y^* = \mathcal{D}(\mathcal{L}^*) = \{ y \in Y | \exists z \in X^* \text{ with } \langle \mathcal{L}x, y \rangle = \langle x, z \rangle \ \forall x \in X \},$$

i.e., for all $y \in Y^*$ we define $\mathcal{L}^*y$ such that

$$\langle \mathcal{L}x, y \rangle = \langle x, \mathcal{L}^*y \rangle \text{ for all } x \in X.$$

An operator $\mathcal{L}$ is said to be self-adjoint if $Y^* = X$ and $\mathcal{L}^* = \mathcal{L}$.

Lemma

- The adjoint operator is unique and $(\mathcal{L}^*)^* = \mathcal{L}$.
- $\mathcal{L}_1, \mathcal{L}_2$ self-adjoint, $\lambda \in \mathbb{R} \implies \mathcal{L}_1 + \mathcal{L}_2$ and $\lambda \mathcal{L}_1$ self-adjoint.
Integration by Parts

For $x \in \mathbb{X}$ and $y \in \mathbb{Y}^*$ we have

$$\langle \mathcal{L}x, y \rangle = \int_{\mathbb{I}} \sum_{i=0}^{k} (x^{(i)})^T A_i^T y \, dt = \sum_{i=0}^{k} \int_{\mathbb{I}} (x^{(i)})^T A_i^T y \, dt.$$

Integration by parts of the terms $(x^{(i)})^T A_i^T y$ yields

$$\int_{\mathbb{I}} (x^{(i)})^T A_i^T y \, dt = b_i(x, y) + (-1)^i \int_{\mathbb{I}} x^T (A_i^T y)^{(i)} \, dt$$

with boundary term

$$b_i(x, y) = \sum_{j=0}^{i-1} (-1)^j (x^{(i-j-1)})^T (A_i^T y)^{(j)} \bigg|_{t_0}^{t_f}.$$

Thus, formally the adjoint operator is given by

$$\mathcal{L}^* y = \sum_{i=0}^{k} (-1)^i \frac{d^i}{dt^i} (A_i^T y).$$
Boundary Conditions

- The domain $\mathcal{X}$ defines boundary conditions for $\mathcal{L}$, while $\mathcal{Y}^*$ defines adjoint boundary conditions for $\mathcal{L}^*$.

Define $\mathcal{X}$, $\mathcal{Y}^*$ such that the boundary terms $b_i(x, y)$ vanish.

- we consider the function spaces

$$\mathcal{X} = \{ x \in C^0(\mathbb{I}, \mathbb{R}^n) \mid A_i^+ A_i x \in C^i(\mathbb{I}, \mathbb{R}^n), \quad B_i(x, t_0) = 0, \quad i = 1, \ldots, k \},$$

$$\mathcal{Y} = C^0(\mathbb{I}, \mathbb{R}^n),$$

with homogeneous boundary conditions

$$B_i(x, t_0) = 0, \quad i = 1, \ldots, k,$$
The Adjoint Operator

**Theorem**

A linear operator \( \mathcal{L} : \mathbb{X} \to \mathbb{Y} \) with regular matrix tuple \((A_k, \ldots, A_0)\) in reduced form and boundary conditions

\[
B_i(x, t_0) = \{(A_i^+ A_i)^{(\ell)} x^{(i-j-1)} \mid t_0 = 0, \text{ for } j = 0, \ldots, i - 1, \ell = 0, \ldots, j\},
\]

has a unique adjoint operator \( \mathcal{L}^* : \mathbb{Y}^* \to \mathbb{X}^* \) with

\[
\mathbb{X}^* = C^0(\mathbb{I}, \mathbb{R}^n), \quad \mathbb{Y}^* = \{y \in C^0(\mathbb{I}, \mathbb{R}^n) \mid A_i A_i^+ y \in C^i(\mathbb{I}, \mathbb{R}^n), B_i^*(y, t_f) = 0, i = 1, \ldots, k\}
\]

and boundary terms

\[
B_i^*(y, t_f) = \{(A_i A_i^+)^{(\ell)} y^{(j-\ell)} \mid t_f = 0, \text{ for } j = 0, \ldots i - 1, \ell = 0 \ldots, j\}
\]

that is given by \( \mathcal{L}^* y = \sum_{i=0}^{k} (-1)^i \frac{d^i}{dt^i}(A_i^T y) \).
Example

- Considering a linear first order differential-algebraic operator

\[
\mathcal{L}x = A_1 \dot{x} + A_0 x,
\]

with sufficiently smooth matrix-valued functions \( A_1, A_0 \in C(\mathbb{I}, \mathbb{R}^{n,n}) \)

- and homogeneous initial condition

\[
(A_1^+ A_1 x)(t_0) = 0.
\]

- Then, the adjoint operator is of the form

\[
\mathcal{L}^* x = -\frac{d}{dt}(A_1^T y) + A_0^T y = -A_1^\dot{y} + (A_0^T - \dot{A}_1^T)y,
\]

- with homogeneous end condition

\[
(A_1 A_1^+ y)(t_f) = 0.
\]
The adjoint operator $L^*$ can be written as

$$L^*y = \sum_{i=0}^{k} (-1)^i \frac{d^i}{dt^i} (A_i^T y) = \sum_{i=0}^{k} (-1)^i \sum_{j=0}^{i} \binom{i}{j} (A_i^T)^{(j)} y^{(i-j)}.$$ 

For self-adjointness we need $L = L^*$ and therefore the formal conditions for self-adjointness are

$$A_\ell = \sum_{i=0}^{k} (-1)^i \binom{i}{i-\ell} (A_i^T)^{(i-\ell)} = \sum_{i=\ell}^{k} (-1)^i \binom{i}{\ell} (A_i^T)^{(i-\ell)}$$

for $\ell = 0, \ldots, k$ using that $\binom{i}{j} = 0$ for $j < 0$. 
Self-adjoint DAE Operators

Theorem

A differential-algebraic operator \( \mathcal{L} \) with regular matrix tuple \((A_k, \ldots, A_0)\) in reduced form, sufficiently smooth \( A_i \in C^i(\mathbb{I}, \mathbb{R}^{n,n}) \) and

\[
\mathbb{X} = \{ x \in C^0(\mathbb{I}, \mathbb{R}^n) | A_i^+ A_i x \in C^i(\mathbb{I}, \mathbb{R}^n), B_i(x, t_0) = B_i^*(x, t_f) = 0 \},
\]

\[
\mathbb{Y} = C^0(\mathbb{I}, \mathbb{R}^n),
\]

is self-adjoint if and only if

\[
A_\ell = \sum_{i=\ell}^{k} (-1)^i \binom{i}{\ell} (A_i^T)^{i-\ell} \quad \text{for } \ell = 0, \ldots, k.
\]
Self-adjoint DAE Operators

**Theorem**

A differential-algebraic operator $\mathcal{L}$ with regular matrix tuple $(A_k, \ldots, A_0)$ in reduced form, sufficiently smooth $A_i \in C^i(\mathbb{I}, \mathbb{R}^{n,n})$ and

$$X = \{ x \in C^0(\mathbb{I}, \mathbb{R}^n) \mid A_i^+ A_i x \in C^i(\mathbb{I}, \mathbb{R}^n), \quad B_i(x, t_0) = B_i^*(x, t_f) = 0 \},$$

$$Y = C^0(\mathbb{I}, \mathbb{R}^n),$$

is self-adjoint if and only if

$$A_\ell = \sum_{i=\ell}^{k} (-1)^i \binom{i}{\ell} (A_i^T)^{(i-\ell)} \quad \text{for } \ell = 0, \ldots, k.$$

- An operator with constant coefficients is formally self-adjoint if

$$A_\ell = (-1)^\ell A_\ell^T \quad \text{for } \ell = 0, \ldots, k.$$
Even/Odd Order Splitting

Theorem

Any formally self-adjoint operator $\mathcal{L}x$ is a sum of operators of the form

$$
\mathcal{L}_{2\nu}x = (P_{2\nu}x^{(\nu)})^{(\nu)},
$$

$$
\mathcal{L}_{2\nu-1}x = \frac{1}{2}[(Q_{2\nu-1}x^{(\nu-1)})^{(\nu)} + (Q_{2\nu-1}x^{(\nu)})^{(\nu-1)}],
$$

with matrix valued functions

- $P_{2\nu} = P_{2\nu}^T \in C^\nu(\mathbb{I}, \mathbb{R}^{n,n})$ and
- $Q_{2\nu-1} = -Q_{2\nu-1}^T \in C^\nu(\mathbb{I}, \mathbb{R}^{n,n})$ for $\nu = 0, \ldots, \mu$,

whereby $\mu = \frac{k}{2}$ if $k$ is even and $\mu = \frac{k+1}{2}$ if $k$ is odd.
Even/Odd Order Splitting

**Theorem**

Any formally self-adjoint operator $\mathcal{L}x$ is a sum of operators of the form

\[
\mathcal{L}_{2\nu}x = (P_{2\nu}x^{(\nu)})^{(\nu)},
\]

\[
\mathcal{L}_{2\nu-1}x = \frac{1}{2}[(Q_{2\nu-1}x^{(\nu-1)})^{(\nu)} + (Q_{2\nu-1}x^{(\nu)})^{(\nu-1)}],
\]

with matrix valued functions

- $P_{2\nu} = P_{2\nu}^T \in C^\nu(\mathbb{I}, \mathbb{R}^{n,n})$ and
- $Q_{2\nu-1} = -Q_{2\nu-1}^T \in C^\nu(\mathbb{I}, \mathbb{R}^{n,n})$ for $\nu = 0, \ldots, \mu$,
- whereby $\mu = \frac{k}{2}$ if $k$ is even and $\mu = \frac{k+1}{2}$ if $k$ is odd.

A self-adjoint operator is in **canonical form** if it is given by

\[
\mathcal{L}x = \begin{cases} 
\sum_{\nu=0}^{r} \mathcal{L}_{2\nu}x + \sum_{\nu=1}^{r} \mathcal{L}_{2\nu-1}x, & \text{if } m \text{ is even, } r = \frac{m}{2}, \\
\sum_{\nu=0}^{r-1} \mathcal{L}_{2\nu}x + \sum_{\nu=1}^{r} \mathcal{L}_{2\nu-1}x, & \text{if } m \text{ is odd, } r = \frac{m+1}{2}.
\end{cases}
\]
Example

Consider a second order differential-algebraic operator

\[ \mathcal{L}_2 x = A_2 \ddot{x} + A_1 \dot{x} + A_0 x, \]

with boundary conditions

\[
B_1(x, t_0) = A_1^+ A_1 x|_{t_0} = 0,
\]

\[
B_2(x, t_0) = \left\{ A_2^+ A_2 \dot{x}|_{t_0} = 0, A_2^+ A_2 x|_{t_0} = 0, (A_2^+ A_2)^{(1)} x|_{t_0} = 0 \right\}.
\]

Then the corresponding adjoint operator given by

\[ \mathcal{L}_2^* y = \frac{d^2}{dt^2}(A_2^T y) - \frac{d}{dt}(A_1^T y) + A_0^T y, \]

with boundary conditions

\[
B_1^*(y, t_f) = A_1 A_1^+ y|_{t_f} = 0,
\]

\[
B_2^*(y, t_f) = \left\{ A_2 A_2^+ y|_{t_f} = 0, A_2 A_2^+ \dot{y}|_{t_f} = 0, (A_2 A_2^+)^{(1)} y|_{t_f} = 0 \right\}.
\]
Example (continued)

- The operator is self-adjoint if and only if
  \[ A_2 = A_2^T, \quad A_1 = (2\dot{A}_2 - A_1)^T, \quad \text{and} \quad A_0 = (\ddot{A}_2 - \dot{A}_1 + A_0)^T, \]
  and all of the above boundary conditions hold.

- A self-adjoint second order operator \( \mathcal{L}_2 \) can be written as
  \[
  \mathcal{L}_2 x = \frac{d}{dt} (P_2 \dot{x}) + P_0 x + \frac{1}{2} \frac{d}{dt} (Q_1 x) + \frac{1}{2} Q_1 \dot{x},
  \]
  with
  - \( P_2 = A_2 = P_2^T \),
  - \( Q_1 = A_1 - \dot{A}_2 = -Q_1^T \), and
  - \( P_0 = A_0 - \frac{1}{2} \dot{A}_1 + \frac{1}{2} \ddot{A}_2 = P_0^T \).
Outline

1 Motivation

2 Optimal Control of DAE Systems
   • Continuous-time Linear-Quadratic Optimal Control Problem
   • Discrete-time Linear-Quadratic Optimal Control Problem

3 Linear Self-adjoint Operators
   • Self-adjoint Differential-Algebraic Operators
   • Self-adjoint Difference Operators

4 Structure Preserving Discretization

5 Conclusion
Difference Operators

- Consider the Hilbert space

\[ \ell^2(\mathbb{Z}) := \left\{ (x_i)_{i \in \mathbb{Z}}, \ x_i \in \mathbb{R}^n \ \middle| \ \sum_{i \in \mathbb{Z}} \| x_i \|^2 < \infty \right\}, \]

with the inner product

\[ \langle x, y \rangle = \sum_{i \in \mathbb{Z}} x_i^T y_i, \quad \text{for } x = (x_i)_{i \in \mathbb{Z}}, \ y = (y_i)_{i \in \mathbb{Z}}. \]

- Linear $k$th-order difference operator $L_d : X_d \rightarrow Y_d$ is given by

\[ L_d x = \sum_{j=0}^{k} A_j(i) x_{i+j} = 0, \quad \text{for all } i \in I \subset \mathbb{Z} \]

with $A_j(i) \in \mathbb{R}^{n,n}$ for all $i \in I_0 = \{0, 1, \ldots, N\} \subset I$ and function spaces $X_d, Y_d \subset \ell^2(\mathbb{Z})$. 
Adjoint Difference Operator

- The adjoint is defined via the relation $\langle \mathcal{L}_d x, y \rangle = \langle x, \mathcal{L}_d^* y \rangle$, and the operator is self-adjoint if $\langle \mathcal{L}_d x, y \rangle = \langle x, \mathcal{L}_d y \rangle$.
- Since, formally, the adjoint of a forward shift operator is always a backward shift no difference operator of order $k \geq 1$ defined in this way can be self-adjoint.
- Alternative: define linear difference operators of even order $k = 2\mu$

\[
\mathcal{L}_d x = \sum_{j=0}^{k} A_j(i) x_{i-\mu+j} = 0, \quad \text{for all } i \in I,
\]

with $A_j(i) \in \mathbb{R}^{n,n}, j = 0, \ldots, k$ defined for all $i \in I_0$.

- e.g. for $k = 2$:

\[
\mathcal{L}_d x = A_2(i) x_{i+1} + A_1(i) x_i + A_0(i) x_{i-1} = 0, \quad \text{for all } i \in I.
\]
Summation by Parts

\[
\langle \mathcal{L}_d \mathbf{x}, \mathbf{y} \rangle = \sum_{i=0}^{N} \sum_{j=0}^{k} x_{i-\mu+j}^T A_j^T(i) y_i
\]

\[
= \sum_{i=0}^{N} x_i^T \sum_{j=0}^{k} A_{k-j}^T(i - \mu + j) y_{i-\mu+j} + B(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathcal{L}_d^* \mathbf{y} \rangle,
\]

With boundary term \( B(\mathbf{x}, \mathbf{y}) \) given by

\[
B(\mathbf{x}, \mathbf{y}) = \sum_{j=0}^{\mu-1} \left[ \sum_{i=0}^{\mu-1-j} x_{i-\mu+j}^T A_j^T(i) y_i - x_i^T A_{k-j}^T(i - \mu + j) y_{i-\mu+j} \right.
\]

\[
+ \sum_{i=N+1}^{N+\mu-j} x_i^T A_{k-j}^T(i - \mu + j) y_{i-\mu+j} - x_{i-\mu+j}^T A_j^T(i) y_i \left. \right].
\]
The Adjoint Difference Operator

**Theorem**

Consider a difference operator $\mathcal{L}_d$ even order $k = 2\mu$ with regular matrix tuple $(A_k, \ldots, A_0)$ in reduced form and function spaces

$$X_d = \{ \mathbf{x} = (x_i)_{i \in \mathcal{I}}, \ x_i \in \mathbb{R}^n \mid B_j(\mathbf{x}) = 0 \text{ for } j = 0, \ldots, \mu - 1 \} \subset \ell^2(\mathbb{Z}),$$

$$Y_d = \{ \mathbf{y} = (y_i)_{i \in \mathcal{I}}, \ y_i \in \mathbb{R}^n \} \subset \ell^2(\mathbb{Z}),$$

where $\mathcal{I} = \{-\mu, \ldots, N + \mu\}$ and

$$B_j(\mathbf{x}) = \{ A^+_{k-j}(i-\mu+j)A_{k-j}(i-\mu+j)x_i = 0, \ i = N + 1, \ldots, N + \mu - j, \ A^+_j(i)A_j(i)x_{i-\mu+j} = 0, \ i = 0, \ldots, \mu - 1 - j \}.$$
Theorem (continued)

Then the adjoint operator $\mathcal{L}_d^*$ with function spaces

\[
\mathcal{X}_d^* = \{ (x_i)_{i \in \mathcal{I}}, x_i \in \mathbb{R}^n \}, \\
\mathcal{Y}_d^* = \left\{ (y_i)_{i \in \mathcal{I}}, y_i \in \mathbb{R}^n \mid B_j^*(y) = 0 \text{ for } j = 0, \ldots, \mu - 1 \right\}
\]

and

\[
B_j^*(y) = \{ A_{k-j}(i - \mu + j)A_{k-j}^+(i - \mu + j)y_{i-\mu+j} = 0, \ i = 0, \ldots, \mu - 1 - j, \\
A_j(i)A_j^+(i)y_i = 0, \ i = N + 1, \ldots, N + \mu - j \}
\]

is given by

\[
\mathcal{L}_d^*y = \sum_{j=0}^{k} A_{k-j}^T(i - \mu + j)y_{i-\mu+j}.
\]
Example

For a second order linear difference operator given by

\[ \mathcal{L}_d x = A_2(i)x_{i+1} + A_1(i)x_i + A_0(i)x_{i-1} \]

with boundary conditions

\[ B_0(x) = \begin{cases} A_2^+(N)A_2(N)x_{N+1} = 0, & A_0^+(0)A_0(0)x_{-1} = 0 \end{cases} \]

the adjoint operator is given by

\[ \mathcal{L}^*_d y = A_0^T(i + 1)y_{i+1} + A_1^T(i)y_i + A_2^T(i - 1)y_{i-1}. \]

with boundary conditions

\[ B_0^*(y) = \begin{cases} A_2(-1)A_2^+(-1)y_{-1} = 0, & A_0(N + 1)A_0^+(N + 1)y_{N+1} = 0 \end{cases} \]
Self-adjoint difference operator

**Theorem**

An even order difference operator $\mathcal{L}_d$ is self-adjoint if and only if

$$X_d = \{ x = (x_i)_{i \in I}, x_i \in \mathbb{R}^n | B_j(x) = B_j^*(x) = 0 \text{ for all } j = 0, \ldots, \mu - 1 \}$$

and

$$A_j(i) = A_{k-j}^T(i + j - \mu) \quad \text{for all } j = 0, \ldots, k, \ i \in \mathcal{I}_0 = \{0, \ldots, N\}.$$
Theorem

An even order difference operator $\mathcal{L}_d$ is self-adjoint if and only if

$$X_d = \{ \mathbf{x} = (x_i)_{i \in \mathcal{I}}, x_i \in \mathbb{R}^n | B_j(\mathbf{x}) = B_j^*(\mathbf{x}) = 0 \text{ for all } j = 0, \ldots, \mu - 1 \}$$

and

$$A_j(i) = A_{k-j}^T(i + j - \mu) \quad \text{for all } j = 0, \ldots, k, \ i \in \mathcal{I}_0 = \{0, \ldots, N\}.$$ 

For constant coefficients the conditions for self-adjointness are

$$A_j = A_{k-j}^T \quad \text{for} \quad j = 0, \ldots, k$$

and a self-adjoint difference operator is given in palindromic form

$$\mathcal{L}_d \mathbf{x} = A_0 x_{i-\mu} + A_1 x_{i-\mu+1} + \cdots + A_\mu x_i + \cdots + A_1^T x_{i+\mu-1} + A_0^T x_{i+\mu}.$$
Example (cont.)

For a second order linear difference operator the conditions for self-adjointness are for \( i = 0, \ldots, N \)

\[
A_0(i) = A_2^T(i - 1),
A_1(i) = A_1^T(i).
\]

Second order self-adjoint difference operator:

\[
\mathcal{L}_d x = A_0^T(i + 1)x_{i+1} + A_1(i)x_i + A_0(i)x_{i-1},
\]

with \( A_1(i) = A_1^T(i) \) for all \( i \in I_0 \) and boundary conditions

\[
B_0(x) = \{ A_0(N + 1)A_0^+(N + 1)x_{N+1} = 0, \ A_0^+(0)A_0(0)x_{-1} = 0 \}\]
Is this the right definition of self-adjointness?

- Our definition corresponds to the case of self-adjoint difference equations of the form

\[ \mathcal{L}_d \mathbf{x} = \Delta [P_i \Delta x_{i-1}] + Q_i x_i = 0, \quad P_i = P_i^T, \quad Q_i = Q_i^T \]

with forward difference operator \( \Delta x_i = x_{i+1} - x_i \).

- In our case we also have that \( \mathcal{L}^{**} = \mathcal{L}_d \).

- Drawback: for odd order difference operators there exists no self-adjoint operator corresponding to the above definition.
Alternative Formulation for Odd Order

Consider the Hilbert space of sequences with index set
\[ \mathcal{B} = \{ \ldots, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots \} \]

\[ \ell^2(\mathcal{B}) = \left\{ (x_b)_{b \in \mathcal{B}}, \ x_b \in \mathbb{R}^n \mid \sum_{b \in \mathcal{B}} \|x_b\|_2^2 < \infty \right\}, \]

with the inner product
\[ \langle x, y \rangle = \sum_{b \in \mathcal{B}} x_b^T y_b \quad \text{for all } x, y \in \ell^2(\mathcal{B}). \]

As before we have
\[ \mathcal{L}_d x = \sum_{j=0}^{k} A_j(i) x_{i - \frac{k}{2} + j}, \quad \mathcal{L}_d^* y = \sum_{j=0}^{k} A^T_{k-j}(i - \frac{k}{2} + j) y_{i - \frac{k}{2} + j}, \]

and a difference operator \( \mathcal{L}_d \) is self-adjoint if and only if
\[ A_j(i) = A^T_{k-j}(i + j - \frac{k}{2}). \]

i.e. for \( k = 1 \) a self-adjoint operator is given by
\[ \mathcal{L}_d x = A^T_0(i + \frac{1}{2}) x_{i + \frac{1}{2}} + A_0(i) x_{i - \frac{1}{2}}. \]
Operators in the Optimal Control Setting

Theorem

If the coefficient matrices are sufficiently smooth then, under the additional condition that

\[(EE^+ \zeta)(t_0) = 0 \text{ and } (EE^+ x)(t_f) = 0,\]

the differential-algebraic operator associated with the necessary optimality system for the linear-quadratic optimal control problem is self-adjoint.
Theorem

Under the condition that

$$x_{-1} = 0 \text{ and } m_{N+1} = 0,$$

the linear difference operator corresponding to the boundary value problem of the optimality system for the discrete-time optimal control problem is formally self-adjoint in $\ell_2$. 
Outline

1 Motivation

2 Optimal Control of DAE Systems
   - Continuous-time Linear-Quadratic Optimal Control Problem
   - Discrete-time Linear-Quadratic Optimal Control Problem

3 Linear Self-adjoint Operators
   - Self-adjoint Differential-Algebraic Operators
   - Self-adjoint Difference Operators

4 Structure Preserving Discretization

5 Conclusion
Central Finite Differences

- The $n$th-order central difference is given by

$$
\delta^n[x](t) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} x(t + \left(\frac{n}{2} - j\right)h)
$$

for some discretization stepsize $h$ such that

$$
\frac{d^n x(t)}{dt^n} = \frac{\delta^n[x](t)}{h^n} + O(h^2).
$$

- For odd $n$ in the central difference $h$ is multiplied by non-integers.

- This problem may be avoided by taking the average of $\delta^n[x](t - \frac{h}{2})$ and $\delta^n[x](t + \frac{h}{2})$. We denote this average by

$$
\bar{\delta}^n[x](t) = \frac{1}{2} \left( \delta^n[x](t - \frac{h}{2}) + \delta^n[x](t + \frac{h}{2}) \right).
$$
Finite Differences Discretization

Theorem
Consider a self-adjoint differential operator in canonical form (i.e. as sum of even/odd order operators). A discretization using \( \bar{\delta}^n[.] (t_i) \) for odd derivatives of order \( n \) and \( \delta^n[.] (t_i) \) for even derivatives of order \( n \) leads to a self-adjoint difference operator of even order.

Proof:
E.g. for a self-adjoint second order operator given in canonical form

\[
\mathcal{L}_2 x = \frac{d}{dt} (P_1 \dot{x}) + \frac{1}{2} \left[ \frac{d}{dt} (Q_1 x) + Q_1 \dot{x} \right] + P_0 x,
\]

with \( P_1 = P_1^T \), \( Q_1 = -Q_1^T \), \( P_0 = P_0^T \) we get the discretized system

\[
\mathcal{L}_2 x(t_i) \approx \bar{\delta} [P_1 \bar{\delta} [x]](t_i) + \frac{1}{2} \left[ \bar{\delta} [Q_1 x](t_i) + Q_1(t_i) \bar{\delta} [x](t_i) \right] + P_0(t_i) x(t_i)
\]

\[
= \frac{1}{4} \left\{ P_{1,i+1} x_{i+2} + [Q_{1,i+1} + Q_{1,i}] x_{i+1} + [4P_{0,i} - P_{1,i+1} - P_{1,i-1}] x_i + [-Q_{1,i-1} - Q_{1,i}] x_{i-1} + P_{1,i-1} x_{i-2} \right\}.
\]
Outline

1 Motivation

2 Optimal Control of DAE Systems
   - Continuous-time Linear-Quadratic Optimal Control Problem
   - Discrete-time Linear-Quadratic Optimal Control Problem

3 Linear Self-adjoint Operators
   - Self-adjoint Differential-Algebraic Operators
   - Self-adjoint Difference Operators

4 Structure Preserving Discretization

5 Conclusion
Conclusions and open problems

- Linear quadratic optimal control problems lead to self-adjoint DAE operators.
- Self-adjointness of a systems is a more appropriate structure that can also be dealt with in the variable coefficient or singular case.
- We have given a proper definition of self-adjointness of differential and difference operators.
- In order to preserve the structure continuous-time systems should be discretized in such a way that self-adjointness is preserved.
- What is the right discretization of continuous time self-adjoint operators that yield discrete time self-adjoint operators?
Thank you very much for your attention.