Pattern Avoidance and Affine Permutations

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Definition of Affine Permutations

Definition

The group of affine permutations, $\widetilde{S}_n$ is the group of all bijections $\sigma : \mathbb{Z} \to \mathbb{Z}$ such that the following properties hold:

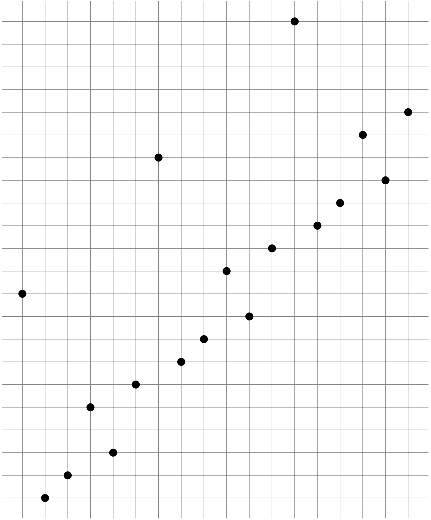
1. $\sigma(i+n) = \sigma(i) + n$ for all $i \in \mathbb{Z}$,
2. $\sum_{i=1}^{n} \sigma(i) = \binom{n+1}{2}$.

We will represent an affine permutation in its one-line notation as the infinite string

$$\cdots \sigma(-1), \sigma(0) \, [\sigma(1), \ldots, \sigma(n)] \, \sigma(n+1), \sigma(n+2) \cdots .$$

Example of an Affine Permutation

$$\sigma = [8, -1, 0, 3, 1, 4] \in \widetilde{S}_6.$$



Coxeter Groups

As a Coxeter group, $\widetilde{S}_n$ is generated by the simple reflections $S = \{s_0, s_1, \ldots, s_{n-1}\}$, where s_i interchanges $i + kn$ and $i + 1 + kn$ for all $k \in \mathbb{Z}$ and leaves all other integers fixed. The relations amongst these generators is summarized in the Coxeter graph.

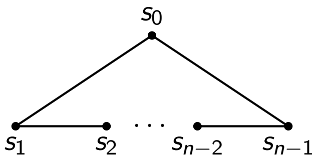


Figure: Coxeter graph for $\widetilde{S}_n$.

Affine Permutations and Symmetric Functions

Affine permutations show up when studying

1. affine Stanley symmetric functions,
2. k-Schur functions and dual k-Schur functions,
3. Macdonald polynomials,
4. Schubert bases for (co)homology of the affine Grassmannian.

(See, e.g., work of Lam, Lapointe, Morse, Shimozono.)

Definition of Pattern Avoidance

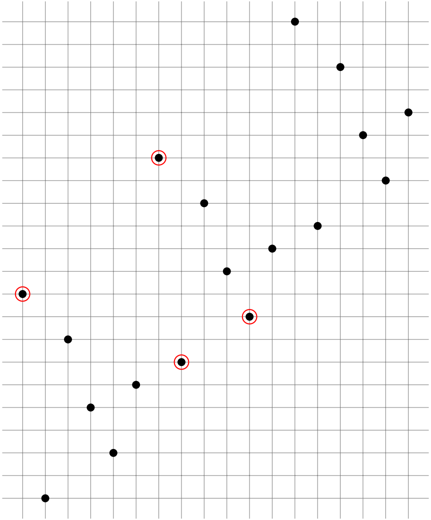
Mimicking the notion of pattern avoidance for non-affine permutations, we make the following definition.

Definition

Let $\sigma \in \widetilde{S}_n$ and $p \in S_m$. We say σ *contains* the pattern p if there exist indices $i_1 < \cdots < i_m$, such that the values $\sigma(i_1), \ldots, \sigma(i_m)$ are in the same relative order as the values $p(1), \ldots, p(m)$. Otherwise, we say σ *avoids* the pattern p.

Example of Pattern Avoidance

$\sigma = [8, -1, 6, 3, 1, 4]$ contains $p = 3412$.



Enumerating Pattern Avoidance

Definition

For a given pattern $p \in S_m$, let $\widetilde{S}_n(p) = \#\left\{\sigma \in \widetilde{S}_n : \sigma \text{ avoids } p\right\}$.

Since $\widetilde{S}_n$ is an infinite group, $\widetilde{S}_n(p)$ may be infinite.

Theorem (Crites 2010)

$\widetilde{S}_n(p)$ is finite if and only if p avoids 321.

Enumeration Results

Theorem (Crites 2010)

1. *If* $\ell(p) = 0$, *then* $\widetilde{S}_n(p) = 0$.
2. *If* $\ell(p) = 1$, *then* $\widetilde{S}_n(p) = 1$.
3. *For* $p \in \{231, 312, 1342, 1423, 2314, 3124\}$, $\widetilde{S}_n(p) = \binom{2n-1}{n}$.

Conjecture

$$\widetilde{S}_n(3142) = \widetilde{S}_n(2413) = \sum_{k=0}^{n-1} \frac{(n-k)}{n} \binom{n-1+k}{k} 2^k$$

$$\widetilde{S}_n(3412) = \widetilde{S}_n(4123) = \widetilde{S}_n(2341) = \frac{1}{3} \sum_{p+q+r=n} \binom{n}{p, q, r}^2$$

Affine Permutation Matrices

Write $\sigma(i) = a_i + b_i n$, where $1 \leq a_i \leq n$. Since σ is a bijection, Property 1 guarantees that $\{a_1, \ldots, a_n\} = \{1, \ldots, n\}$. Property 2 shows that $b_1 + \cdots + b_n = 0$.

Definition

Let $e_\sigma = (m_{ij})_{i,j=1}^n$ be the matrix with $m_{i,a_i} = t^{b_i}$ for $1 \leq i \leq n$, and all other entries 0. Such a matrix is called an *affine permutation matrix*.

Example of an Affine Permutation Matrix

$$\begin{array}{rcccccc} \sigma = [& 8, & -1, & 6, & 3, & 1, & 4 \ ] \\ a_i : & 2 & 5 & 6 & 3 & 1 & 4 \\ b_i : & 1 & -1 & 0 & 0 & 0 & 0 \end{array}$$

$$\updownarrow$$

$$\begin{bmatrix} 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Affine Schubert Varieties

Let

$$\widetilde{G} = \mathrm{GL}_n\left(\mathbb{C}[[t]][t^{-1}]\right),$$

and let

$$\widetilde{B} = \{b \in \mathrm{GL}_n\left(\mathbb{C}[[t]]\right) : \left. b\right|_{t=0} \text{ is upper triangular}\}.$$

Definition

$\widetilde{X} := \widetilde{G}/\widetilde{B}$ is an ind-variety called the *complete affine flag variety*. The corresponding affine Weyl group is $\widetilde{S}_n$.

Affine Schubert Varieties (cont.)

By putting elements of $\widetilde{G}$ in column echelon form, we have the *Bruhat decomposition*

$$\widetilde{X} = \bigsqcup_{\sigma \in \widetilde{S}_n} \widetilde{B}\sigma\widetilde{B}/\widetilde{B}.$$

Definition

1. The *Schubert cell* corresponding to $\sigma \in \widetilde{S}_n$ is $C_\sigma = \widetilde{B}\sigma\widetilde{B}/\widetilde{B}$.
2. The *Schubert variety* corresponding to σ is $\widetilde{X}_\sigma = \overline{C}_\sigma$.

Example of a Schubert Cell

The Schubert cell C_σ, for $\sigma = [8, -1, 6, 3, 1, 4]$:

$$\begin{bmatrix} a + bt & t & c & d & et^{-1} + f & g \\ 0 & 0 & 0 & 0 & t^{-1} & 0 \\ h & 0 & i & j & k & 1 \\ \ell & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The number of free variables in row i is

$$\#\{j : i < j \text{ and } \sigma(i) > \sigma(j)\}.$$

Affine Schubert Varieties (cont.)

As in the classical case, we have the following properties:

$$\widetilde{X}_\tau = \bigcup_{\sigma \leq \tau} \widetilde{X}_\sigma,$$

$$\dim \widetilde{X}_\sigma = \ell(\sigma) = \#\left\{(i,j) : 1 \leq i \leq n, i < j, \sigma(i) > \sigma(j)\right\}.$$

Main Result

Theorem (Billey-Crites 2010)

The affine Schubert variety $\widetilde{X}_\sigma$ is rationally smooth if and only if either σ avoids 3412 and 4231, or σ is a twisted spiral variety.

Since smoothness implies rational smoothness, we also get the following result.

Corollary

If σ contains 3412 or 4231, then $\widetilde{X}_\sigma$ is not smooth.

Rational Smoothness

Definition

The *Poincaré polynomial* for $\tau \in \widetilde{S}_n$ is given by

$$P_\tau(q) = \sum_{\sigma \leq \tau} q^{\ell(\sigma)}.$$

Definition

A variety X is *rationally smooth* if, for each $x \in X$, the singular cohomology $H^i(X, X \backslash \{x\}, \mathbb{Q}) = 0$ for $i \neq 2 \dim X$, and is one-dimensional when $i = 2 \dim X$.

Theorem (Carrell-Peterson)

$\widetilde{X}_\sigma$ is rationally smooth if and only if $P_\sigma(q)$ is palindromic.

Spiral Permutations

Definition

Pick any $1 \leq i \leq n$. Starting at s_i, proceed clockwise or counterclockwise $k(n-1)$ steps, building a word from right to left at each step. The resulting affine permutation is called a *spiral permutation*.

Example

$$\sigma = s_2 s_1 s_0 s_3 s_2 s_1 s_0 s_3 s_2 = [-2, 11, 0, 1] \in \widetilde{S}_4.$$

Twisted Spiral Permutations

Definition

A *twisted spiral permutation* is obtained by taking a spiral permutation starting at s_i, and multiplying on the right by the long element of the maximal parabolic subgroup generated by $S \backslash \{s_i\}$.

Example

$$\sigma \cdot s_3 s_0 s_3 s_1 s_0 s_3 = [-3, -4, 15, 2].$$

Comparison with Classical Case

For non-affine permutations, smoothness and rational smoothness are equivalent.

Theorem (Lakshmibai-Sandhya)

Let $\sigma \in S_n$. Then X_σ is (rationally) smooth if and only if σ avoids 3412 and 4231.

Conjecture

Let $\sigma \in \widetilde{S}_n$. Then $\widetilde{X}_\sigma$ is smooth if and only if σ avoids 3412 and 4231.