Descents, Peaks, and Shuffles of Permutations and Noncommutative Symmetric Functions

Ira M. Gessel

Department of Mathematics
Brandeis University

Workshop on Quasisymmetric Functions
Bannf International Research Station
November 17, 2010

Part 1

Historical Overview

Quasi-symmetric functions grew out of P-partitions. The basic idea is due to MacMahon.

Quasi-symmetric functions grew out of P-partitions. The basic idea is due to MacMahon.

We want to count plane partitions:

$$\begin{array}{ccc} a & \le & b \\ \wedge\!| & & \wedge\!| \\ c & \le & d \end{array}$$

Quasi-symmetric functions grew out of P-partitions. The basic idea is due to MacMahon.

We want to count plane partitions:

$$\begin{array}{ccc} a & \leq & b \\ \wedge\!\mid & & \wedge\!\mid \\ c & \leq & d \end{array}$$

The set of solutions is the disjoint union of the solutions of

$$a \leq b \leq c \leq d$$

and

$$a \leq c < b \leq d.$$

Quasi-symmetric functions grew out of P-partitions. The basic idea is due to MacMahon.

We want to count plane partitions:

$$\begin{array}{ccc} a & \le & b \\ \wedge\!| & & \wedge\!| \\ c & \le & d \end{array}$$

The set of solutions is the disjoint union of the solutions of

$$a \le b \le c \le d$$

and

$$a \le c < b \le d.$$

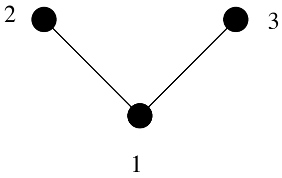
The strict inequalities occur in the descents.

Little was done with this idea until 1970, when Donald Knuth introduced P-partitions for an arbitrary naturally labeled poset, and applied them to counting solid partitions.

Richard Stanley, in his 1971 Ph.D. thesis (published as an AMS Memoir in 1972) studied the general case of P-partitions in great detail.

Some of the basic ideas of P-partitions were independently discovered by Germain Kreweras (1976, 1981).

Stanley considered a refined generating function for P-partitions:



$$f(1) \leq f(2), \quad f(1) \leq f(3)$$

$$\sum_{\substack{f(1)\leq f(2)\\ f(1)\leq f(3)}} x_1^{f(1)} x_2^{f(2)} x_3^{f(3)}$$

$$= \sum_{f(1)\leq f(2)\leq f(3)} x_1^{f(1)} x_2^{f(2)} x_3^{f(3)} + \sum_{f(1)\leq f(3)< f(2)} x_1^{f(1)} x_2^{f(2)} x_3^{f(3)}$$

$$
\begin{aligned}
\sum_{\substack{f(1)\le f(2)\\ f(1)\le f(3)}} & x_1^{f(1)} x_2^{f(2)} x_3^{f(3)} \\
&= \sum_{f(1)\le f(2)\le f(3)} x_1^{f(1)} x_2^{f(2)} x_3^{f(3)} \\
&\quad + \sum_{f(1)\le f(3)< f(2)} x_1^{f(1)} x_2^{f(2)} x_3^{f(3)} \\
&= \frac{1}{(1-x_3)(1-x_2x_3)(1-x_1x_2x_3)} \\
&\quad + \frac{x_2}{(1-x_2)(1-x_2x_3)(1-x_1x_2x_3)}
\end{aligned}
$$

In my 1984 paper I substituted x_j for Stanley's x_i^j. So the quasi-symmetric generating function for the previous example would be

$$\sum_{\substack{f(1)\le f(2)\\ f(1)\le f(3)}} x_{f(1)}x_{f(2)}x_{f(3)} = \sum_{f(1)\le f(2)\le f(3)} x_{f(1)}x_{f(2)}x_{f(3)} + \sum_{f(1)\le f(3)<f(2)} x_{f(1)}x_{f(2)}x_{f(3)}$$

$$= F_{(3)} + F_{(2,1)}.$$

An advantage is that the information contained in this less refined generating function is exactly the multiset of descent sets of the linear extensions of the poset, and if this quasi-symmetric generating function is actually symmetric, we can use the tools of symmetric functions to extract information from it.

An advantage is that the information contained in this less refined generating function is exactly the multiset of descent sets of the linear extensions of the poset, and if this quasi-symmetric generating function is actually symmetric, we can use the tools of symmetric functions to extract information from it.

As Peter McNamara pointed out, Stanley did briefly consider the quasi-symmetric generating function for P-partitions.

Quasi-symmetric functions also appear earlier for special cases of P-partitions in the work of Glânffrwd Thomas, who related them to Baxter algebras (1975, 1977).

Stanley (1984), in studying reduced decompositions of elements of Coxeter groups, defined certain symmetric functions as sums of the fundamental quasi-symmetric functions.

In the mid-1990's, Claudia Malvenuto (1993) and Malvenuto and Christophe Reutenauer (1995), and independently Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thibon (1995) introduced a coproduct on quasi-symmetric functions making QSym into a Hopf algebra, and described the dual Hopf algebra, which Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thibon called the Hopf algebra of noncommutative symmetric functions.

In the mid-1990's, Claudia Malvenuto (1993) and Malvenuto and Christophe Reutenauer (1995), and independently Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thibon (1995) introduced a coproduct on quasi-symmetric functions making QSym into a Hopf algebra, and described the dual Hopf algebra, which Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thibon called the Hopf algebra of noncommutative symmetric functions.

Ehrenborg (1996) introduced the quasi-symmetric generating function for a poset, encoding the flag f-vector.

Part 2

Descents, Peaks, and Shuffles of Permutations and Noncommutative Symmetric Functions

Shuffles

If π and σ are disjoint permutations, let $S(\pi, \sigma)$ be the set of all shuffles of π and σ.

Example:

$$\pi = 1\ 4\ 2 \qquad \sigma = 3\ 7\ 5\ 8$$

Shuffles

If π and σ are disjoint permutations, let $S(\pi, \sigma)$ be the set of all shuffles of π and σ.
Example:

$$\pi = 1\ 4\ 2 \qquad \sigma = 3\ 7\ 5\ 8$$

$$\begin{array}{cccccccc} 1 & 4 & & & & 2 & \\ & & 3 & 7 & 5 & & 8 \end{array}$$

Shuffles

If π and σ are disjoint permutations, let $S(\pi, \sigma)$ be the set of all shuffles of π and σ.
Example:

$$\pi = 1\ 4\ 2 \qquad \sigma = 3\ 7\ 5\ 8$$

$$\begin{array}{ccccccc} 1 & 4 & & & & 2 & \\ & & 3 & 7 & 5 & & 8 \end{array}$$

$$1\ 4\ 3\ 7\ 5\ 2\ 8$$

Shuffles

If π and σ are disjoint permutations, let $S(\pi, \sigma)$ be the set of all shuffles of π and σ.
Example:

$$\pi = 1\ 4\ 2 \qquad \sigma = 3\ 7\ 5\ 8$$

1	4				2	
		3	7	5		8

1 4 3 7 5 2 8

We want to study permutation statistics that are compatible with shuffles.

Permutation statistics

An example

The descent set $D(\pi)$ of $\pi = \pi_1 \cdots \pi_m$ is $\{\, i : \pi_i > \pi_{i+1} \,\}$.

Theorem (Stanley). The number of permutations in $S(\pi, \sigma)$ with descent set A depends only on $D(\pi)$, $D(\sigma)$, and A.

Permutation statistics

An example

The descent set $D(\pi)$ of $\pi = \pi_1 \cdots \pi_m$ is $\{\, i : \pi_i > \pi_{i+1} \,\}$.

Theorem (Stanley). The number of permutations in $S(\pi, \sigma)$ with descent set A depends only on $D(\pi)$, $D(\sigma)$, and A.

Therefore the descent set is an example of a permutation statistic that is shuffle-compatible.

Two permutations are equivalent if they have the same standardization:

$$132 \equiv 253 \equiv 174.$$

A permutation statistic is a function defined on permutations that takes the same value on equivalent permutations.

For example if f is a permutation statistic then

$$f(132) = f(253) = f(174).$$

A permutation statistic stat is shuffle-compatible if it has the property that the the multiset $\{\, \mathrm{stat}(\tau) : \tau \in S(\pi, \sigma) \,\}$ depends only on $\mathrm{stat}(\pi)$ and $\mathrm{stat}(\sigma)$ (and the lengths of π and σ).

A permutation statistic is a descent statistic if it depends only on the descent set. Some important descent statistics:

- the descent set $D(\pi)$
- the descent number $\operatorname{des}(\pi) = \#D(\pi)$
- the major index $\operatorname{maj}(\pi) = \sum_{i \in D(\pi)} i$
- the comajor index $\operatorname{comaj}(\pi) = \sum_{i \in D(\pi)} (n - i)$, where π has length n
- the peak set $P(\pi) = \{\, i : \pi(i-1) < \pi(i) > \pi(i+1) \,\}$
- the peak number $\operatorname{pk}(\pi) = \#P(\pi)$
- the ordered pair $(\operatorname{des}, \operatorname{maj})$

An important permutation statistic that is not a descent statistic is the number of inversions.

All of the above descent statistics are shuffle-compatible. This was proved by Richard Stanley, using P-partitions for des, maj, and (des, maj), and by John Stembridge, using enriched P-partitions, for the peak set and the peak number.

Algebras

Note that for any shuffle-compatible permutation statistic stat we get an algebra $\mathscr{A}_{\text{stat}}$:

First we define an equivalence relation $\equiv_{\text{stat}}$ on permutations by $\pi \equiv_{\text{stat}} \sigma$ if π and σ have the same length and $\text{stat}(\pi) = \text{stat}(\sigma)$. We define $\mathscr{A}_{\text{stat}}$ by taking as a basis all equivalence classes of permutations, with multiplication defined as follows: To multiply two equivalence classes, choose disjoint representatives π and σ of the equivalence classes $[\pi]$ and $[\sigma]$ and define their product to be

$$[\pi][\sigma] = \sum_{\tau \in S(\pi,\sigma)} [\tau].$$

By the definition of a shuffle-compatible permutation statistic, this product is well-defined.

As a simple example, we consider the major index. It is known (from the theory of P-partitions) that if $|\pi| = m$ and $|\sigma| = n$ then

$$\sum_{\tau \in S(\pi,\sigma)} q^{\mathrm{maj}(\tau)} = q^{\mathrm{maj}(\pi)+\mathrm{maj}(\sigma)} \begin{bmatrix} m+n \\ m \end{bmatrix}.$$

As a simple example, we consider the major index. It is known (from the theory of P-partitions) that if $|\pi| = m$ and $|\sigma| = n$ then

$$\sum_{\tau \in S(\pi,\sigma)} q^{\mathrm{maj}(\tau)} = q^{\mathrm{maj}(\pi)+\mathrm{maj}(\sigma)} \begin{bmatrix} m+n \\ m \end{bmatrix}.$$

It follows that the map

$$[\pi] \to q^{\mathrm{maj}(\pi)} \frac{x^m}{m!_q}, \quad \text{where } m = |\pi|,$$

is an isomorphism from the maj algebra $\mathscr{A}_{\mathrm{maj}}$ to an algebra of polynomials (more precisely, polynomials in x whose coefficients are certain rational functions of q).

Big Question: Can we describe all shuffle-compatible permutation statistics and their algebras?

Big Question: Can we describe all shuffle-compatible permutation statistics and their algebras?

Or some of them?

Big Question: Can we describe all shuffle-compatible permutation statistics and their algebras?

Or some of them?

Or at least say something interesting?

Big Question: Can we describe all shuffle-compatible permutation statistics and their algebras?

Or some of them?

Or at least say something interesting?

To make the problem a little easier, we consider only shuffle-compatible descent statistics.

Big Question: Can we describe all shuffle-compatible permutation statistics and their algebras?

Or some of them?

Or at least say something interesting?

To make the problem a little easier, we consider only shuffle-compatible descent statistics.

Any shuffle-compatible descent statistic algebra will be a quotient algebra of the descent set algebra.

Big Question: Can we describe all shuffle-compatible permutation statistics and their algebras?

Or some of them?

Or at least say something interesting?

To make the problem a little easier, we consider only shuffle-compatible descent statistics.

Any shuffle-compatible descent statistic algebra will be a quotient algebra of the descent set algebra.

So let's look at the descent set algebra.

The descent set algebra is isomorphic to the algebra of quasi-symmetric functions. These are formal power series in infinitely many variables that are more general than symmetric functions.Two important bases, indexed by compositions $L = (L_1, L_2, \ldots, L_k)$ where each L_i is a positive integer.

The descent set algebra is isomorphic to the algebra of quasi-symmetric functions. These are formal power series in infinitely many variables that are more general than symmetric functions.Two important bases, indexed by compositions $L = (L_1, L_2, \ldots, L_k)$ where each L_i is a positive integer.

The monomial basis:

$$M_L = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{L_1} x_{i_2}^{L_2} \cdots x_{i_k}^{L_k}.$$

Example:

$$M_{(3,2,3)} = \sum_{i<j<k} x_i^3 x_j^2 x_k^3.$$

The descent set algebra is isomorphic to the algebra of quasi-symmetric functions. These are formal power series in infinitely many variables that are more general than symmetric functions.Two important bases, indexed by compositions $L = (L_1, L_2, \ldots, L_k)$ where each L_i is a positive integer.

The monomial basis:

$$M_L = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{L_1} x_{i_2}^{L_2} \cdots x_{i_k}^{L_k}.$$

Example:

$$M_{(3,2,3)} = \sum_{i<j<k} x_i^3 x_j^2 x_k^3.$$

The fundamental basis: Example:

$$F_{(3,2,3)} = \sum x_{i_1} x_{i_2} \cdots x_{i_8}$$

where

$$\underbrace{i_1 \le i_2 \le i_3}_{3} < \underbrace{i_4 \le i_5}_{2} < \underbrace{i_6 \le i_7 \le i_8}_{3}.$$

There is a bijection between compositions of n and subsets of $[n-1] = \{1, 2, \ldots, n-1\}$:

$$(L_1, \ldots, L_k) \mapsto \{L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{k-1}\}.$$

The inverse map is

$$\{j_1 < j_2 < \cdots < j_{k-1}\} \mapsto (j_1, j_2 - j_1, \ldots, j_{k-1} - j_{k-2}, n - j_{k-1}).$$

If π is a permutation, the descent composition of π is the composition corresponding to the descent set of π; it is the sequence of lengths of the increasing runs of π:

Example:
The descent composition of 1 4 7•2 8•3 6 9 is $(3, 2, 3)$.

Theorem. Let $c^L_{J,K}$ be the number of permutations with descent composition L among the shuffles of a permutation with descent composition J and a permutation with descent composition K. Then

$$F_J F_K = \sum_L c^L_{J,K} F_L.$$

Thus the descent set statistic algebra $\mathscr{A}_{\mathrm{des}}$ is isomorphic to the algebra QSym of quasi-symmetric functions.

Theorem. Let $c^L_{J,K}$ be the number of permutations with descent composition L among the shuffles of a permutation with descent composition J and a permutation with descent composition K. Then

$$F_J F_K = \sum_L c^L_{J,K} F_L.$$

Thus the descent set statistic algebra $\mathscr{A}_{\mathrm{des}}$ is isomorphic to the algebra QSym of quasi-symmetric functions.

So all descent statistic algebras are quotient algebras of QSym (but not conversely).

Now let's look at the descent number algebra. The equivalence classes here correspond to ordered pairs (n, i) where $1 \leq i \leq n$. Let $d_{(l,i),(m,j)}^{(n,k)}$, where $l + m = n$, be the number of permutations with i descents obtained by shuffling a permutation of length l with $i - 1$ descents and a permutation of length m with $j - 1$ descents.

Now let's look at the descent number algebra. The equivalence classes here correspond to ordered pairs (n, i) where $1 \le i \le n$. Let $d^{(n,k)}_{(l,i),(m,j)}$, where $l + m = n$, be the number of permutations with i descents obtained by shuffling a permutation of length l with $i - 1$ descents and a permutation of length m with $j - 1$ descents. Then

$$\binom{\lambda + l - i}{l}\binom{\lambda + m - j}{m} = \sum_k d^{(n,k)}_{(l,i),(m,j)}\binom{\lambda + n - k}{n}.$$

Now let's look at the descent number algebra. The equivalence classes here correspond to ordered pairs (n, i) where $1 \leq i \leq n$. Let $d^{(n,k)}_{(l,i),(m,j)}$, where $l + m = n$, be the number of permutations with i descents obtained by shuffling a permutation of length l with $i - 1$ descents and a permutation of length m with $j - 1$ descents. Then

$$\binom{\lambda + l - i}{l}\binom{\lambda + m - j}{m} = \sum_k d^{(n,k)}_{(l,i),(m,j)} \binom{\lambda + n - k}{n}.$$

Note that both sides are polynomials in λ and that

$$\left\{\binom{\lambda + n - k}{n}\right\}_{k=1,\ldots,n}$$

is a basis for the polynomials in λ of degree at most k that vanish at 0. So the structure constants $d^{(n,k)}_{(l,i),(m,j)}$ describe the expansion of a product of two of these basis elements in these basis elements.

We can look at this algebra in another way. The generating function in λ for the basis polynomial $\binom{\lambda+n-k}{n}$ is

$$\sum_{\lambda=0}^{\infty} \binom{\lambda+n-k}{n} t^{\lambda} = \frac{t^k}{(1-t)^{n+1}}.$$

So

$$\sum_{\lambda=0}^{\infty} \binom{\lambda+l-i}{l} \binom{\lambda+m-j}{m} t^{\lambda} = \frac{\sum_k d_{(l,i),(m,j)}^{(n,k)} t^k}{(1-t)^{n+1}},$$

where $n = l + m$.

We can look at this algebra in another way. The generating function in λ for the basis polynomial $\binom{\lambda+n-k}{n}$ is

$$\sum_{\lambda=0}^{\infty} \binom{\lambda+n-k}{n} t^{\lambda} = \frac{t^k}{(1-t)^{n+1}}.$$

So

$$\sum_{\lambda=0}^{\infty} \binom{\lambda+l-i}{l} \binom{\lambda+m-j}{m} t^{\lambda} = \frac{\sum_k d_{(l,i),(m,j)}^{(n,k)} t^k}{(1-t)^{n+1}},$$

where $n = l + m$. Let us define the Hadamard product $f * g$ of two power series f and g in t by

$$\left(\sum_i a_i t^i\right) * \left(\sum_j b_j t^j\right) = \sum_i a_i b_i t^i.$$

Then we may rewrite our formula as

$$\frac{t^i}{(1-t)^{l+1}} * \frac{t^j}{(1-t)^{m+1}} = \sum_k d_{(l,i),(m,j)}^{(n,k)} \frac{t^k}{(1-t)^{n+1}}$$

The peak number algebra

We can describe the peak number algebra in a similar, but somewhat more complicated way. Instead of the rational functions $t^k/(1-t)^{n+1}$, we have the rational functions

$$P_{(n,j)}(t) = 2^{2j-1}\frac{t^j(1+t)^{n-2j+1}}{(1-t)^{n+1}} = \frac{1}{2}\frac{(1+t)^{n+1}}{(1-t)^{n+1}}\left(\frac{4t}{(1+t)^2}\right)^j,$$

for $1 \le j \le \lfloor (n+1)/2 \rfloor$, corresponding to permutation of length n with $j-1$ peaks. More precisely, the coefficients in the expansion of $P_{l,i}(t) * P_{m,j}(t)$ as a linear combination of $P_{l+m,k}$ are the structure constants for the peak number algebra (Stembridge).

Peaks and descents

Our main result is a common generalization of the descent number algebra and the peak number algebra:

Peaks and descents

Our main result is a common generalization of the descent number algebra and the peak number algebra:

The ordered pair $(\mathrm{des}, \mathrm{pk})$ is shuffle-compatible.

Peaks and descents

Our main result is a common generalization of the descent number algebra and the peak number algebra:

The ordered pair $(\mathrm{des}, \mathrm{pk})$ is shuffle-compatible.

There exists a permutation of length $n > 0$ with $j - 1$ peaks and $k - 1$ descents if and only if $1 \leq j \leq (n+1)/2$ and $j \leq k \leq n + 1 - j$. For such n, j, k let

$$\mathrm{PD}_{n,j,k}(t, y) = \frac{t^j (y + t)^{k-j} (1 + yt)^{n-j-k+1} (1 + y)^{2j-1}}{(1 - t)^{n+1}}$$

Then the structure constants for the $(\mathrm{des}, \mathrm{pk})$ algebra are the same as the structure constants for the rational functions $\mathrm{PD}_{n,j,k}$ under the operation of Hadamard product in t.

Let's see how this result specializes to the known results for descents and peaks separately.

$$\mathrm{PD}_{n,j,k}(t,y) = \frac{t^j(y+t)^{k-j}(1+yt)^{n-j-k+1}(1+y)^{2j-1}}{(1-t)^{n+1}}$$

If we set $y=0$, we get

$$\mathrm{PD}_{n,j,k}(t,0) = \frac{t^k}{(1-t)^{n+1}}$$

and if we set $y=1$, we get

$$\mathrm{PD}_{n,j,k}(t,1) = \frac{t^j(1+t)^{n-2j+1}2^{2j-1}}{(1-t)^{n+1}}$$

How might we prove such a formula?

How might we prove such a formula? Possible approaches are:

- construct the appropriate homomorphism from quasi-symmetric functions by making a substitution for the variables (using Stembridge's enriched P-partitions)
- use noncommutative symmetric functions

I will use noncommutative symmetric functions, which were introduced by Malvenuto (1993), Malvenuto and Reutenauer (1995), and Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thibon (1995).

Let $X_1, X_2, \ldots$ be noncommuting indeterminates, and define

$$\mathbf{h}_n = \sum_{i_1 \le i_2 \le \cdots \le i_n} X_{i_1} X_{i_2} \cdots X_{i_n},$$

with $\mathbf{h}_0 = 1$. The algebra NSym of noncommutative symmetric functions is the algebra generated by the $\mathbf{h}_n$. Alternatively, we could define the $\mathbf{h}_n$ to be noncommuting indeterminates.

Let $X_1, X_2, \ldots$ be noncommuting indeterminates, and define

$$\mathbf{h}_n = \sum_{i_1 \le i_2 \le \cdots \le i_n} X_{i_1} X_{i_2} \cdots X_{i_n},$$

with $\mathbf{h}_0 = 1$. The algebra NSym of noncommutative symmetric functions is the algebra generated by the $\mathbf{h}_n$. Alternatively, we could define the $\mathbf{h}_n$ to be noncommuting indeterminates.

Later on, we'll want to work with noncommutative symmetric functions with coefficients in some algebra over the rationals.

Let $X_1, X_2, \ldots$ be noncommuting indeterminates, and define

$$\mathbf{h}_n = \sum_{i_1 \le i_2 \le \cdots \le i_n} X_{i_1} X_{i_2} \cdots X_{i_n},$$

with $\mathbf{h}_0 = 1$. The algebra NSym of noncommutative symmetric functions is the algebra generated by the $\mathbf{h}_n$. Alternatively, we could define the $\mathbf{h}_n$ to be noncommuting indeterminates.

Later on, we'll want to work with noncommutative symmetric functions with coefficients in some algebra over the rationals.

For any composition $L = (L_1, \ldots, L_k)$, let $\mathbf{h}_L$ be $\mathbf{h}_{L_1} \cdots \mathbf{h}_{L_k}$. Then the $\mathbf{h}_L$ for L a composition of n form a basis for the noncommutative symmetric functions homogeneous of degree n.

We can define a coproduct Δ on NSym (a map NSym $\to$ NSym $\otimes$ NSym satisfying certain properties) by

1. $\Delta \mathbf{h}_n = \sum_{i=0}^{n} \mathbf{h}_i \otimes \mathbf{h}_{n-i}$
2. $\Delta fg = (\Delta f)(\Delta g)$

For example,

$$\begin{aligned}\Delta \mathbf{h}_1 \mathbf{h}_1 &= (\Delta \mathbf{h}_1)(\Delta \mathbf{h}_1) = (1 \otimes \mathbf{h}_1 + \mathbf{h}_1 \otimes 1)(1 \otimes \mathbf{h}_1 + \mathbf{h}_1 \otimes 1) \\ &= 1 \otimes \mathbf{h}_1^2 + 2\,\mathbf{h}_1 \otimes \mathbf{h}_1 + \mathbf{h}_1^2 \otimes 1.\end{aligned}$$

Property 2 means that NSym with the product and coproduct is a bialgebra (in fact it is a Hopf algebra). The coproduct is useful combinatorially, but the product is useful in computing the coproduct.

The dual of any algebra, as a vector space, is a coalgebra in a natural way.

Lemma. (Malvenuto-Reutenauer, Gelfand-Krob-Lascoux-Leclerc-Retakh-Thibon) The dual of the algebra QSym of quasi-symmetric functions is the coalgebra NSym of noncommutative symmetric functions, and the basis M_L of QSym is dual to the basis $\mathbf{h}_L$ of noncommutative symmetric functions.

The dual of any algebra, as a vector space, is a coalgebra in a natural way.

Lemma. (Malvenuto-Reutenauer, Gelfand-Krob-Lascoux-Leclerc-Retakh-Thibon) The dual of the algebra QSym of quasi-symmetric functions is the coalgebra NSym of noncommutative symmetric functions, and the basis M_L of QSym is dual to the basis $\mathbf{h}_L$ of noncommutative symmetric functions.

What does this mean? It means that the structure coefficients are the same: The numbers $c^L_{J,K}$ defined by

$$M_J M_K = \sum_L c^L_{J,K} M_L$$

also satisfy

$$\Delta \mathbf{h}_L = \sum_{J,K} c^L_{J,K} \mathbf{h}_J \otimes \mathbf{h}_K.$$

This is useful because quotient algebras correspond to subcoalgebras of their duals. So a quotient algebra of QSym corresponds to a subalgebra of NSym.

This is useful because quotient algebras correspond to subcoalgebras of their duals. So a quotient algebra of QSym corresponds to a subalgebra of NSym.

In order to use this, we need to find the basis for NSym dual to the fundamental basis for QSym. This is the basis of ribbons defined (by example) by

$$\mathbf{r}_{(3,2,3)} = \sum X_{i_1} X_{i_2} \cdots X_{i_8}$$

where

$$\underbrace{i_1 \le i_2 \le i_3}_{3} > \underbrace{i_4 \le i_5}_{2} > \underbrace{i_6 \le i_7 \le i_8}_{3} .$$

Recall, for comparison, that $F_{(3,2,3)} = \sum x_{i_1} x_{i_2} \cdots x_{i_8}$ where $\underbrace{i_1 \le i_2 \le i_3}_{3} < \underbrace{i_4 \le i_5}_{2} < \underbrace{i_6 \le i_7 \le i_8}_{3}$.

Now let stat be a descent statistic. Then stat gives an equivalence relation on compositions. To show that stat is descent-compatible, we show that the sums $\sum_L \mathbf{r}_L$ over equivalence classes of compositions span a subcoalgebra of NSym:

Now let stat be a descent statistic. Then stat gives an equivalence relation on compositions. To show that stat is descent-compatible, we show that the sums $\sum_L \mathbf{r}_L$ over equivalence classes of compositions span a subcoalgebra of NSym:

Lemma. Let stat be a descent statistic. For each equivalence class α of stat let

$$\mathbf{r}^\alpha = \sum_{L\in\alpha} \mathbf{r}_L.$$

Suppose that for every equivalence class α,

$$\Delta \mathbf{r}^\alpha = \sum_{\beta,\gamma} C^\alpha_{\beta,\gamma}\, \mathbf{r}^\beta \otimes \mathbf{r}^\gamma$$

for some constants $C^\alpha_{\beta,\gamma}$. Then stat is shuffle compatible and the $C^\alpha_{\beta,\gamma}$ are the structure constants for $\mathscr{A}_{\text{stat}}$.

A nonzero element g of a coalgebra is called grouplike if $\Delta g = g \otimes g$. If the coalgebra is a bialgebra, then products of grouplike elements are grouplike, as are inverses of grouplike elements.

A nonzero element g of a coalgebra is called grouplike if $\Delta g = g \otimes g$. If the coalgebra is a bialgebra, then products of grouplike elements are grouplike, as are inverses of grouplike elements.

We can use grouplike elements of NSym to study shuffle-compatible permutation statistics.

For the next lemma we work with noncommutative symmetric functions with coefficient in some $\mathbb{Q}$-algebra R; i.e., we work in $R \otimes_{\mathbb{Q}} \mathrm{NSym}$.

For the next lemma we work with noncommutative symmetric functions with coefficient in some $\mathbb{Q}$-algebra R; i.e., we work in $R \otimes_{\mathbb{Q}} \mathrm{NSym}$.

Lemma. Let stat be a descent statistic. As before, for each equivalence class α, let $\mathbf{r}^\alpha = \sum_{L\in\alpha} \mathbf{r}_L$. Suppose that there exist linearly independent (over $\mathbb{Q}$) $u_\alpha \in R$ such that $\sum_\alpha u_\alpha \mathbf{r}^\alpha$ is grouplike and that there exist constants $C^\alpha_{\beta,\gamma}$ such that

$$u_\beta u_\gamma = \sum_\alpha C^\alpha_{\beta,\gamma} u_\alpha$$

Then stat is shuffle compatible and the $C^\alpha_{\beta,\gamma}$ are the structure constants for the algebra $\mathscr{A}_{\mathrm{stat}}$; so $\mathscr{A}_{\mathrm{stat}}$ is isomorphic to the subalgebra of R spanned by the u_α.

Let $\mathbf{h}(x) = \sum_{n=0}^{\infty} \mathbf{h}_n x^n$. Then $\mathbf{h}(x)$ is grouplike since

$$\begin{aligned}
\Delta \mathbf{h}(x) &= \sum_{n=0}^{\infty} \Delta \mathbf{h}_n(x) \\
&= \sum_{n=0}^{\infty} \sum_{i+j=n} \mathbf{h}_i \otimes \mathbf{h}_j \, x^n \\
&= \sum_{i,j=0}^{\infty} h_i x^i \otimes \mathbf{h}_j x^j \\
&= \mathbf{h}(x) \otimes \mathbf{h}(x)
\end{aligned}$$

Let $\mathbf{h}(x) = \sum_{n=0}^{\infty} \mathbf{h}_n x^n$. Then $\mathbf{h}(x)$ is grouplike since

$$\begin{aligned}
\Delta \mathbf{h}(x) &= \sum_{n=0}^{\infty} \Delta \mathbf{h}_n(x) \\
&= \sum_{n=0}^{\infty} \sum_{i+j=n} \mathbf{h}_i \otimes \mathbf{h}_j \, x^n \\
&= \sum_{i,j=0}^{\infty} h_i x^i \otimes \mathbf{h}_j x^j \\
&= \mathbf{h}(x) \otimes \mathbf{h}(x)
\end{aligned}$$

How do we use this?

As a simple example, we'll see how to show that comaj is shuffle compatible.

As a simple example, we'll see how to show that comaj is shuffle compatible.

We start with the formula

$$K := \prod_{i=0}^{\infty} \mathbf{h}(q^i x) = \sum_{n=0}^{\infty} \sum_{|L|=n} q^{\mathrm{comaj}(L)} \mathbf{r}_L \frac{x^n}{(q)_n},$$

where $(q)_n = (1-q)\cdots(1-q^n)$. which can be proved, for example, using P-partitions. Here each equivalence class α may be represented by a pair (n, i), where α consists of compositions $(L_1, \ldots, L_k)$ of n with comajor index i.Then in the notation of the lemma $u_\alpha = q^i x^n/(q)_n$ so the structure constants for $\mathscr{A}_{\mathrm{comaj}}$ are the coefficients in the expansion of

$$q^i \frac{x^l}{(q)_l} q^j \frac{x^m}{(q)_m}$$

as a linear combination of $q^k x^{l+m}/(q)_{l+m}$.

Since

$$\frac{q^i}{(q)_l}\frac{q^j}{(q)_m} = q^{i+j}\begin{bmatrix} l+m \\ l \end{bmatrix}\frac{1}{(q)_{l+m}},$$

the structure constants are coefficients of q-binomial coefficients.

Where do the Hadamard products come from?

To count permutations by descent number we use the formula

$$K_{\mathrm{des}} := (1 - t\,\mathbf{h}(x))^{-1} = \frac{1}{1-t} + \sum_{n=1}^{\infty} x^n \sum_{|L|=n} \mathbf{r}_L \frac{t^{\mathrm{des}(L)+1}}{(1-t)^{n+1}},$$

To count permutations by descent number we use the formula

$$K_{\text{des}} := (1 - t\,\mathbf{h}(x))^{-1} = \frac{1}{1-t} + \sum_{n=1}^{\infty} x^n \sum_{|L|=n} \mathbf{r}_L \frac{t^{\text{des}(L)+1}}{(1-t)^{n+1}},$$

Unfortunately, K_{des} is not grouplike. But if we expand it in powers of t,

$$K_{\text{des}} = \sum_{k=0}^{\infty} t^k \mathbf{h}(x)^k,$$

the coefficient of t^k is grouplike, so

$$\Delta K_{\text{des}} = \sum_{k=0}^{\infty} t^k\, \mathbf{h}(x)^k \otimes \mathbf{h}(x)^k.$$

If we work in the algebra of quasi-symmetric functions with coefficients in the algebra of power series in t with multiplication by the Hadamard product, then we have

$$\begin{aligned} K_{\mathrm{des}} \otimes K_{\mathrm{des}} &= \left(\sum_{j=0}^{\infty} t^j \mathbf{h}(x)^j\right) \otimes \left(\sum_{k=0}^{\infty} t^k \mathbf{h}(x)^k\right) \\ &= \sum_{j,k} t^j \mathbf{h}(x)^j \otimes t^k \mathbf{h}(x)^k \\ &= \sum_{k} t^k \, \mathbf{h}(x)^k \otimes \mathbf{h}(x)^k = \Delta K_{\mathrm{des}} \end{aligned}$$

Thus in this algebra, K_{des} is grouplike so the descent number algebra is isomorphic to the algebra spanned by

$$\frac{t^j}{(1-t)^{n+1}} x^n, \quad 1 \leq j \leq n$$

where multiplication is the Hadamard product in t.

For the algebra $\mathscr{A}_{(\mathrm{des},\mathrm{pk})}$ we have a completely analogous formula. The key is

$$\left(1 - t\,\mathbf{h}(-yx)^{-1}\mathbf{h}(x)\right)^{-1} = \frac{1}{1-t} + \sum_{n=1}^{\infty} x^n$$

$$\times \sum_{\substack{1\le j\le (n+1)/2 \\ j\le k\le n+1-k}} W_{n,j,k} \frac{t^j (y+t)^{k-j}(1+yt)^{n-j-k+1}(1+y)^{2j-1}}{(1-t)^{n+1}},$$

where $W_{n,j,k}$ is the sum of $\mathbf{r}_L$ over all compositions L of n with $\mathrm{pk}(L) = j-1$ and $\mathrm{des}(L) = k-1$. Just as with $\mathscr{A}_{\mathrm{des}}$, this tells us that $\mathscr{A}_{(\mathrm{pk},\mathrm{des})}$ is the span of the rational functions

$$\frac{t^j (y+t)^{k-j}(1+yt)^{n-j-k+1}(1+y)^{2j-1}}{(1-t)^{n+1}} x^n,$$

where multiplication is the Hadamard product in t.