0-Hecke algebra actions on coinvariants and flags

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  - another interpretation of the (q,t)-ribbon number,
  - a folumar involving maj, inv and inverse descent.

New result I

$$\mathcal{F}(H_n(0)) \overset{\text{Krob-Thibon}}{=\!=} \sum_{w \in \mathfrak{S}_n} F_{D(w^{-1})}$$

$q \rightarrow 1$ $\qquad$ $t \rightarrow 1$

$$\mathcal{F}_q(H_n(0)) \overset{\text{Krob-Thibon}}{=\!=} \sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} F_{D(w^{-1})}$$

$q \leftrightarrow t$

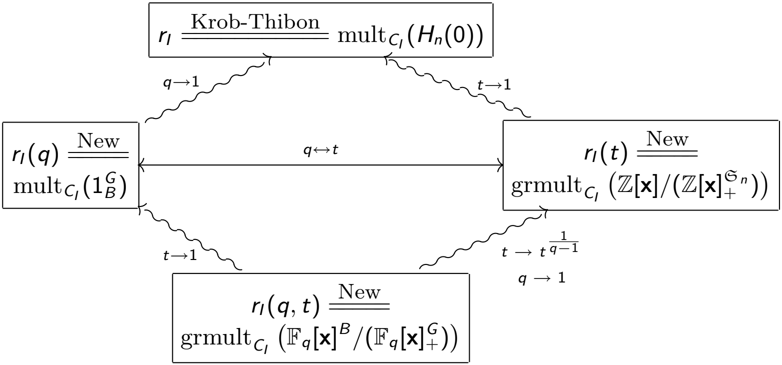
(Foata-Schützenberger)

$$\mathcal{F}_t(\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})) \overset{\text{New}}{=\!=} \sum_{w \in \mathfrak{S}_n} t^{\mathrm{maj}(w)} F_{D(w^{-1})}$$

$t \rightarrow 1$ $\qquad$ $q \rightarrow 1$

$$\mathcal{F}_{q,t}\left(\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})\right) \overset{\text{New}}{=\!=} \sum_{w \in \mathfrak{S}_n} t^{\mathrm{maj}(w)} q^{\mathrm{inv}(w)} F_{D(w^{-1})}$$

New result II



Compositions

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- The *descent set* of I is $D(I) = \{i_1, i_1 + i_2, \ldots, i_1 + \cdots + i_{k-1}\}$.
- e.g.

$$\begin{aligned} & I = (3, 1, 2, 2) \text{ is a composition of } 8 \\ \leftrightarrow \quad & D(I) = \{3, 3+1, 3+1+2\} = \{3, 4, 6\} \subseteq [7]. \end{aligned}$$

Ribbon number

- $$r_I = \#\{w \in \mathfrak{S}_n : \mathrm{Des}(w) = D(I)\}.$$

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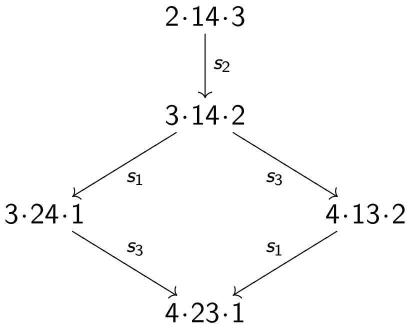
- $$r_{(1,2,1)} = 4! \begin{vmatrix} \frac{1}{1!} & \frac{1}{(1+2)!} & \frac{1}{(1+2+1)!} \\ 1 & \frac{1}{2!} & \frac{1}{(2+1)!} \\ 0 & 1 & \frac{1}{1!} \end{vmatrix} = 5.$$

Descent class of $I = (1, 2, 1)$



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- $$\begin{aligned} r_{(1,2,1)}(q) &= [4]_q! \begin{vmatrix} \frac{1}{[1]!_q} & \frac{1}{[1+2]!_q} & \frac{1}{[1+2+1]!_q} \\ 1 & \frac{1}{[2]!_q} & \frac{1}{[2+1]!_q} \\ 0 & 1 & \frac{1}{[1]!_q} \end{vmatrix} \\ &= q^2 + q^3 + 2q^4 + q^5 \end{aligned}$$

  where
  $$[n]!_q = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)(1-q)\cdots(1-q)}.$$

t-ribbon number

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t-ribbon number

- $$r_I(t) = \sum_{w:\mathrm{Des}(w^{-1})=D(I)} t^{\mathrm{maj}(w)}.$$
- Equidistribution of maj and inv on inverse descent classes (Foata-Schützenberger 1978) implies
$$r_I(t) \xrightarrow{t\to q} r_I(q).$$

(q, t)-ribbon number (Reiner-Stanton 2010)

- $$r_I(q, t) = \sum_{w : \mathrm{Des}(w) = D(I)} \mathrm{wt}(w; q, t).$$

(q,t)-ribbon number (Reiner-Stanton 2010)

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- If $n!_{q,t} = (1-t^{q^n-1})(1-t^{q^n-q})\cdots(1-t^{q^n-q^{n-1}})$ then

$$\begin{aligned}
r_{(1,2,1)}(q,t) =& 4!_{q,t}\begin{vmatrix} \frac{1}{1!_{q,t}} & \frac{1}{(1+2)!_{q,t}} & \frac{1}{(1+2+1)!_{q,t}} \\ 1 & \frac{1}{2!_{q,t^q}} & \frac{1}{(2+1)!_{q,t^q}} \\ 0 & 1 & \frac{1}{1!_{q,t^{q^3}}} \end{vmatrix} \\
=& \frac{(1-t^{q^4-1})(1-t^{q^4-q})(1-t^{q^4-q^2})}{(1-t^{q-1})(1-t^{q^3-q})(1-t^{q^3-q^2})} - \frac{1-t^{q^4-1}}{1-t^{q-1}} \\
&- \frac{(1-t^{q^4-1})(1-t^{q^4-q})(1-t^{q^4-q^2})}{(1-t^{q^3-1})(1-t^{q^3-q})(1-t^{q^3-q^2})} + 1.
\end{aligned}$$

More examples

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- $$r_{(1,2)}(q) = q + q^2$$
- $$r_{(1,2)}(t) = t + t^2$$
- $$r_{(1,2)}(q,t) = t^{q-1} + t^{2(q-1)} + \cdots + t^{(q+q^2)(q-1)}.$$

Coxeter complex $\Delta(W, S)$ of $W = \mathfrak{S}_n$

- *Partial flags of subsets of* $[n]$ ordered by containment.

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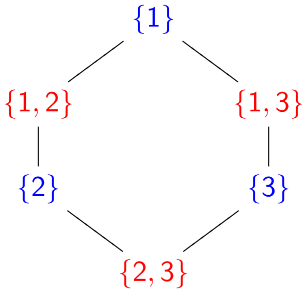
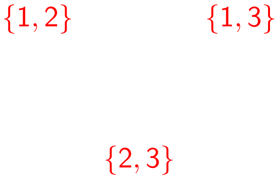


$I = \{1, 2\}$

Rank selection:



$I = \{2\}$

Tits building $\Delta(G, B)$ of $G = GL(n, \mathbb{F}_q)$

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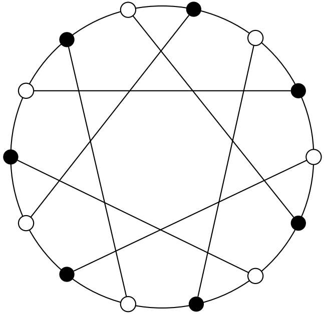

Homology representations

- $$\chi^I = \textit{top homology of } \Delta(W,S)_I$$
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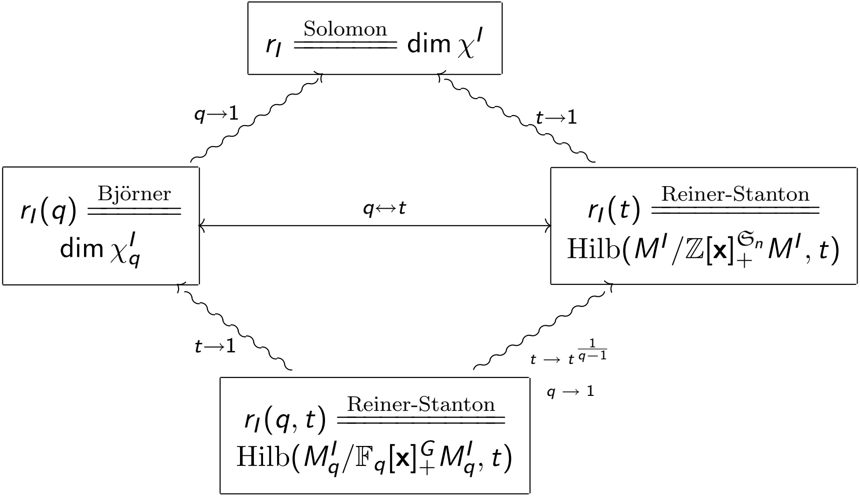
- $$M^I = \mathrm{Hom}_{\mathbb{Z}W}\left(\chi^I, \mathbb{Z}[\mathbf{x}]\right), \text{ a } \mathbb{Z}[\mathbf{x}]^W\text{-module}$$
  $$M^I_q = \mathrm{Hom}_{\mathbb{F}_q G}\left(\chi^I_q, \mathbb{F}_q[\mathbf{x}]\right), \text{ an } \mathbb{F}_q[\mathbf{x}]^G\text{-module}$$

Interpreting ribbon numbers



0-Hecke algebra

- Our goal is to show two *similar pictures* obtained from representations of 0-Hecke algebra $H_n(0)$ of type A (0-*deformation* for $\mathbb{Z}[\mathfrak{S}_n]$).

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- $H_n(0)$ is generated (over $\mathbb{Z}$) by $T_1, \ldots, T_{n-1}$ with relations

$$\begin{cases} T_i^2 = -T_i, & 1 \le i \le n-1, \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, & 1 \le i \le n-2, \\ T_i T_j = T_j T_i, & |i-j| > 1. \end{cases}$$

Representation theory of $H_n(0)$ (Norton 1979)

- *Simple modules* $C_I \leftrightarrow$ compositions I of n:

$$T_i = \begin{cases} -1, & \text{if } i \in D(I), \\ 0, & \text{if } i \notin D(I). \end{cases}$$

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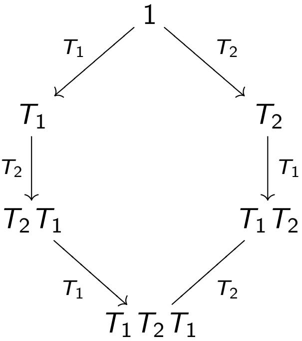
- $$H_n(0) = \bigoplus_I P_I.$$

$H_3(0)$



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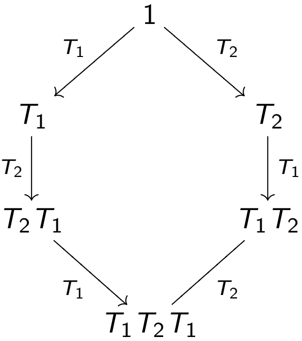
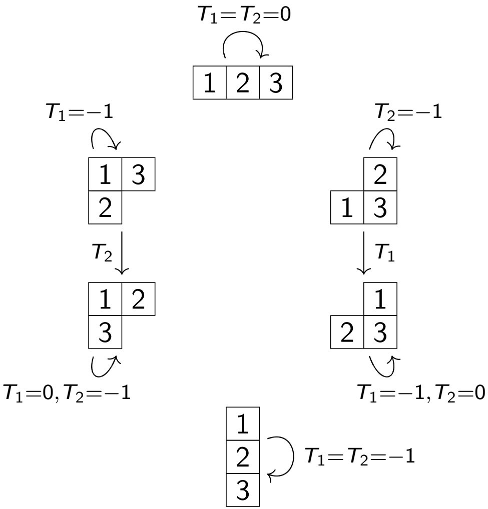
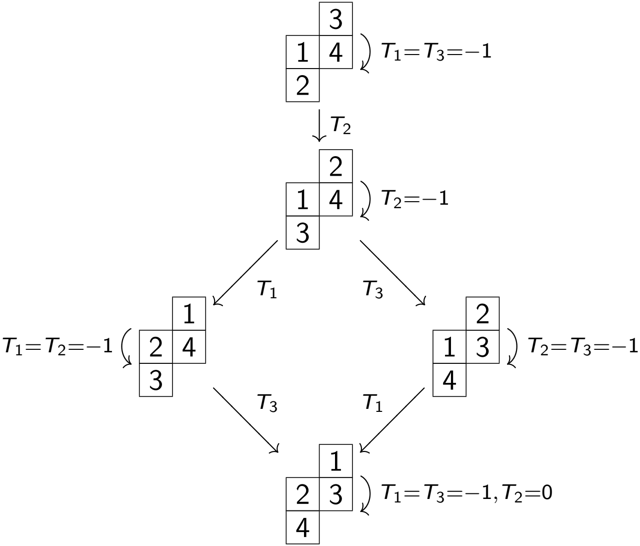


1
T1
T2
T1
T2
T2
T1
T2 T1
T1 T2
T1
T2
T1 T2 T1



T1=T2=0
1 2 3
T1=−1
1 3
2
T2
1 2
3
T1=0, T2=−1
T2=−1
2
1 3
T1
1
2 3
T1=−1, T2=0
1
2
3
T1=T2=−1

$P_{(1,2,1)}$



3
1 4
2
T1=T3=−1
T2
2
1 4
3
T2=−1
T1
T3
T1=T2=−1
1
2 4
3
2
1 3
4
T2=T3=−1
T3
T1
1
2 3
4
T1=T3=−1, T2=0

Frobenius characteristic (Krob-Thibon 1997)

- *QSym* has a basis consisting of *Quasi-ribbon functions*

$$F_I = \sum_{J \text{ refining } I} M_J.$$

e.g. $F_{212} = M_{212} + M_{1112} + M_{2111} + M_{11111}$.

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$$\mathcal{F} : \bigoplus_{n \geq 0} G_0(H_n(0)) \quad \xrightarrow{\sim} \quad QSym,$$

$$M = \bigoplus_{i=1}^{k} [C_{I_i}] \quad \mapsto \quad F_{I_1} + \cdots + F_{I_k}.$$

$n = 3$

$$\begin{aligned} \mathcal{F}_q(H_3(0)) &= F_{(3)} + (q+q^2)F_{(1,2)} + (q+q^2)F_{(2,1)} + q^3 F_{(1,1,1)} \\ \xrightarrow{q\to 1} \mathcal{F}(H_3(0)) &= F_{(3)} + 2F_{(1,2)} + 2F_{(2,1)} + F_{(1,1,1)}. \end{aligned}$$

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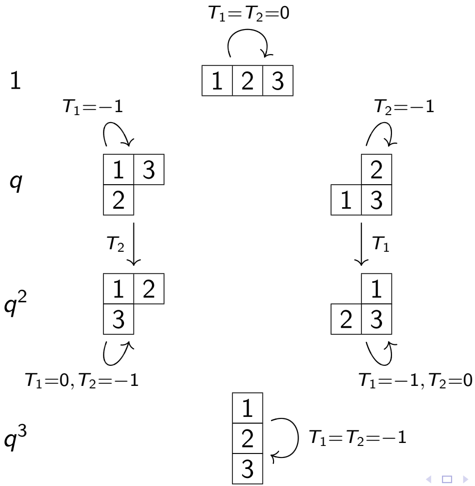

$\mathcal{F}_q(H_n(0))$

$$\begin{aligned}
\mathcal{F}_q(H_n(0)) &= \sum_{w\in\mathfrak{S}_n} q^{\mathrm{inv}(w)} F_{\mathrm{Des}(w^{-1})} \\
&= \sum_I r_I(q) F_I \\
\xrightarrow{q\to 1} \mathcal{F}(H_n(0)) &= \sum_I r_I F_I .
\end{aligned}$$

Demazure Operators

- $H_n(0)$ acts on $\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})$ by $T_i = \overline{\pi}_i = \partial_i x_i - 1$, where

$$\partial_i = \frac{1 - s_i}{x_i - x_{i+1}}, \ s_i = (i, i+1).$$

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  - $\overline{\pi}_2 f = x_1^3 \cdot 0 \cdot x_4^4$,
  - $\overline{\pi}_3 f = x_1^3 x_2 (-x_3 x_4^4 - x_3^2 x_4^3 - x_3^3 x_4^2)$.

Coinvariant algebra $\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})$

Theorem (H.)

(i) $\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})$ *is a graded version of* $H_n(0)$*:*

$$\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n}) = \bigoplus_I N_I \cong H_n(0)$$

where $N_I \cong P_I \subseteq H_n(0)$.

(ii) It has a basis of certain Demazure atoms whose leading terms give the well-known descent basis (Garsia 1980).

Descent basis (Garsia 1980)

- *Descent monomial* for a permutation w is

$$x_w = \prod_{i \in \mathrm{Des}(w)} (x_{w(1)} \cdots x_{w(i)}).$$

e.g. $35 \cdot 2 \cdot 14: \; x_3 x_5 \cdot x_3 x_5 x_2 = x_3^2 x_5^2 x_2.$

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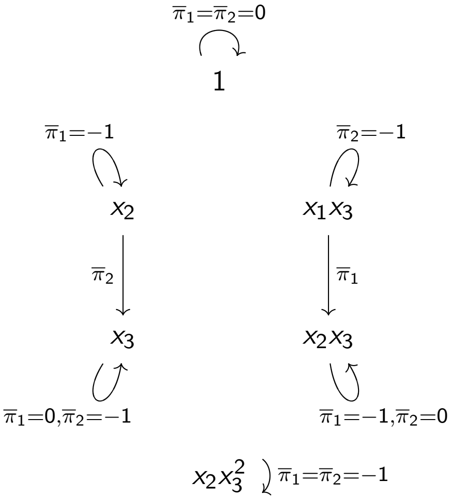
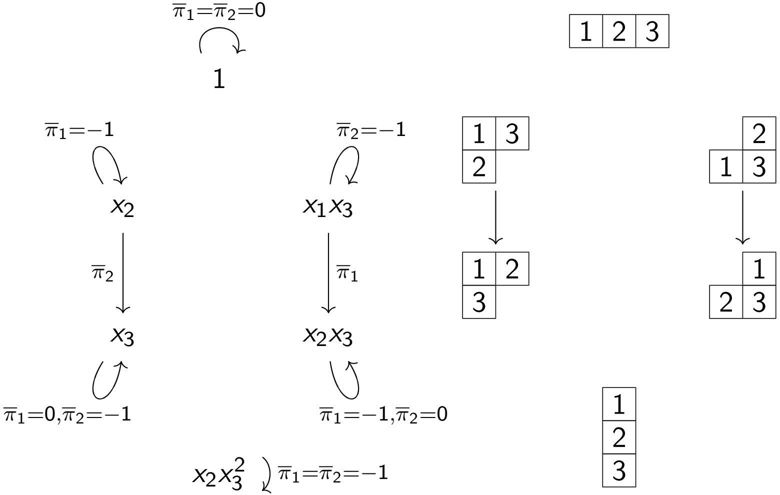
- *Descent basis* for $\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})$:

$$\{\overline{x}_w : w \in \mathfrak{S}_n\}.$$

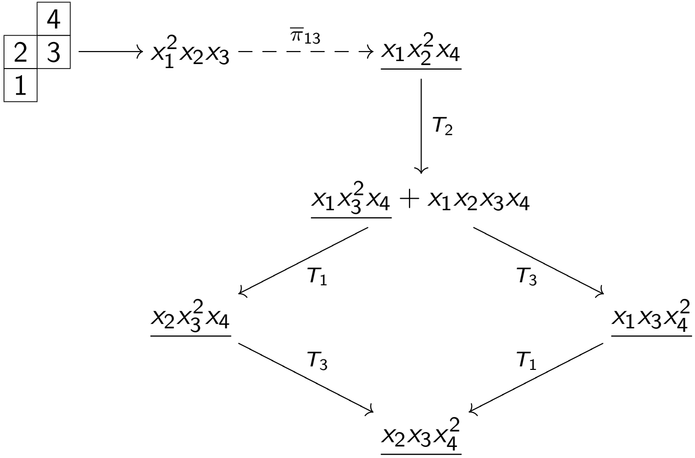
$\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_3})$



$\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_3})$

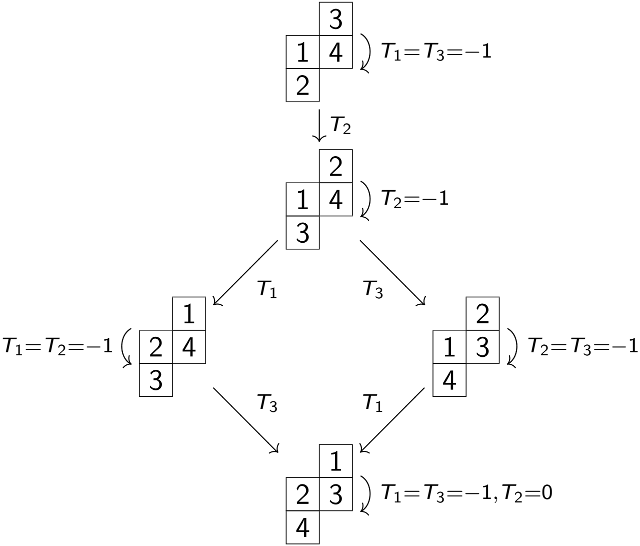


$N_{(1,2,1)}$



4
2 3
1
$x_1^2 x_2 x_3$
$\overline{\pi}_{13}$
$x_1 x_2^2 x_4$
T_2
$x_1 x_3^2 x_4 + x_1 x_2 x_3 x_4$
T_1
T_3
$x_2 x_3^2 x_4$
$x_1 x_3 x_4^2$
T_3
T_1
$x_2 x_3 x_4^2$

$P_{(1,2,1)}$



3
1 4
2
T1=T3=−1
T2
2
1 4
3
T2=−1
T1
T3
1
2 4
3
T1=T2=−1
2
1 3
4
T2=T3=−1
T3
T1
1
2 3
4
T1=T3=−1, T2=0

New result I

$$\mathcal{F}(H_n(0)) \stackrel{\text{Krob-Thibon}}{=\!=\!=} \sum_{w\in\mathfrak{S}_n} F_{D(w^{-1})}$$

$q\to 1$ $t \to 1$

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$q \leftrightarrow t$

(Foata-Schützenberger)

$$\mathcal{F}_t(\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})) \stackrel{\text{New}}{=\!=\!=} \sum_{w\in\mathfrak{S}_n} t^{\mathrm{maj}(w)} F_{D(w^{-1})}$$

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- $H_n(0)$ hence acts on $1_B^G \otimes \mathbb{F}_q$ and the coinvariant algebra

$$S^B/(S_+^G) \cong \mathrm{Hom}_{\mathbb{F}_q[G]}(1_B^G \otimes \mathbb{F}_q, S)$$

where $S = \mathbb{F}_q[x_1, \ldots, x_n]$.

"Inclusion-Exclusion" for simple factors

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- Lemma:

$$\mathcal{F}_t(Q) = \sum_I c_I(Q) F_I,$$

where

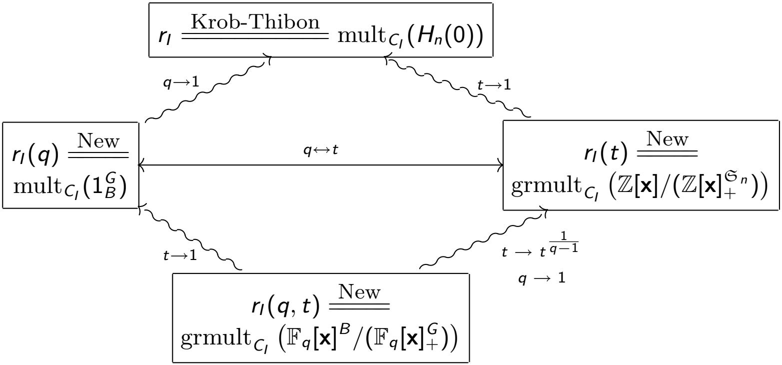
$$c_I(Q) = \sum_{J : D(J) \subseteq D(I)} (-1)^{\ell(I,J)} \mathrm{Hilb}(Q_J, t).$$

Simple factors of 1_B^G and $S^B/(S_+^G)$

Theorem (H.)

$$\begin{aligned} \mathcal{F}\left(1_B^G \otimes \mathbb{F}_q\right) &= \sum_I r_I(q) F_I, \\ \mathcal{F}_t\left(S^B/(S_+^G)\right) &= \sum_I r_I(q,t) F_I. \end{aligned}$$

New result II



When is $\mathcal{F}(M)$ symmetric?

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- By the RSK correspondence one has

$$\mathcal{F}(1_B^G) = \sum_{\lambda \vdash n} q^{b(\lambda)} \frac{[n]!_q}{\prod_{u \in \lambda} [h_u]_q} s_\lambda.$$

where h_u is the hook length of u in λ and $b(\lambda) = \sum_i (i-1)\lambda_i$.

Decomposing 1_B^G and $S^B/(S_+^G)$ as $H_n(0)$-modules?

- For $n = 3$, $q = 2, 3, 5, 7$, computations show that

$$1_B^G \cong P_3 \oplus (P_{21} \oplus P_{12})^{\oplus \binom{q+1}{2}} \oplus (P_{111})^{\oplus q^3} .$$

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Other questions

- *Algebraic* proof for the equidistributivity of inv and maj on inverse descent classes?

New result I

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$$\mathcal{F}_q(H_n(0)) \overset{\text{Krob-Thibon}}{=\!=} \sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} F_{D(w^{-1})}$$

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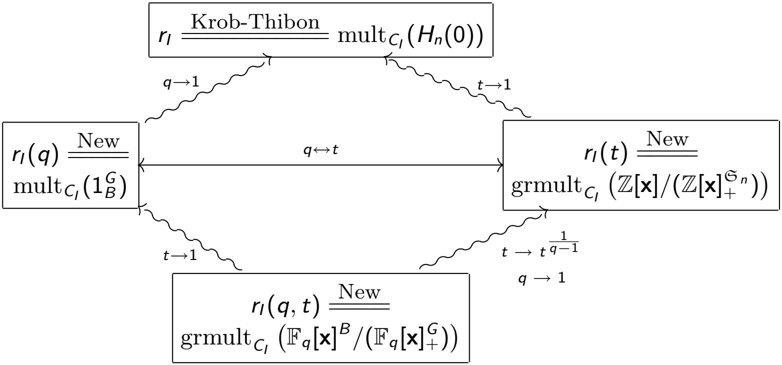

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- *Algebraic* proof for the equidistributivity of inv and maj on inverse descent classes?
- Any q-analogue for $\overline{\pi}_i$?
- Generalizing to finite *Coxeter groups* and *groups with BN-pairs*: what's the action on the coinvariant algebra of W?

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  - a basis of certain Demazure atoms, with leading terms being descent monomials.

Thank you!