HERMITIAN COMPACT SCHEMES FOR THE NAVIER-STOKES EQUATIONS

Jean-Pierre Croisille

Univ. Paul Verlaine-Metz

Banff, Nov. 22-26, 2010

Joint work with

M. Ben-Artzi, Hebrew University, Jerusalem
D. Fishelov, Tel-Aviv Academic College of Engineering and School of Mathematical Sciences, Tel Aviv University

Outline

1. The Pure Streamfunction Formulation of the Navier-Stokes equations
2. Compact finite-difference schemes for biharmonic problems
3. Fast resolution procedure
4. Compact finite-difference schemes for the Navier-Stokes equation

Navier-Stokes equations in 2D

Velocity-pressure formulation:

Find $u(x,t) \in \mathbb{R}^2$, $p(x,t) \in \mathbb{R}$ solutions of

$$(NS)\begin{cases} u_t + u.\nabla u - \nu\Delta u + \nabla p = 0, & x \in \Omega \subset \mathbb{R}^2, t > 0 \\ \text{div}\, u = 0, \quad x \in \Omega\ ,\ t > 0 \\ u = 0\ ,\ x \in \partial\Omega\ ,\ t > 0 \\ u(x,0) = u_0(x), \quad x \in \Omega \end{cases}$$

Streamfunction formulation:

- $u = (-\psi_y, \psi_x) = \nabla^{\perp}\psi$, $\nabla \wedge u = \Delta\psi$. The streamfunction ψ evolves according to

$$\partial_t(\Delta\psi) + (\nabla^{\perp}\psi)\cdot\nabla(\Delta\psi) - \nu\Delta^2\psi = 0\ , \quad x \in \Omega\ , \quad t > 0$$

  (Landau-Lifschitz, Fluid Dynamics).

- The boundary conditions are given for all points $(x,y) \in \partial\Omega$,

$$\begin{cases} \psi(x,y,t) = 0 & \text{no-leak condition + gauge condition} \\ \frac{\partial\psi}{\partial n}(x,y,t) = 0 & \text{tangential velocity given} \end{cases} \quad (1)$$

  Initial data: $\psi_0(x,y) = \psi(x,y,t)|_{t=0}$, $\quad (x,y) \in \Omega$.

Navier-Stokes equations in 2D

Velocity-pressure formulation:

Find $u(x,t) \in \mathbb{R}^2$, $p(x,t) \in \mathbb{R}$ solutions of

$$(NS) \begin{cases} u_t + u.\nabla u - \nu\Delta u + \nabla p = 0, & x \in \Omega \subset \mathbb{R}^2, t > 0 \\ \operatorname{div} u = 0, \quad x \in \Omega \;,\; t > 0 \\ u = 0 \;,\; x \in \partial\Omega \;,\; t > 0 \\ u(x,0) = u_0(x), \quad x \in \Omega \end{cases}$$

Streamfunction formulation:

- $u = (-\psi_y, \psi_x) = \nabla^{\perp}\psi$, $\nabla \wedge u = \Delta\psi$. The streamfunction ψ evolves according to

  $$\partial_t(\Delta\psi) + (\nabla^{\perp}\psi)\cdot\nabla(\Delta\psi) - \nu\Delta^2\psi = 0 \;,\quad x \in \Omega \;,\quad t > 0$$

  (Landau-Lifschitz, Fluid Dynamics).

- The boundary conditions are given for all points $(x,y) \in \partial\Omega$,

  $$\begin{cases} \psi(x,y,t) = 0 & \text{no-leak condition + gauge condition} \\ \frac{\partial\psi}{\partial n}(x,y,t) = 0 & \text{tangential velocity given} \end{cases} \quad (1)$$

  Initial data: $\psi_0(x,y) = \psi(x,y,t)|_{t=0}, \quad (x,y) \in \Omega$.

Navier-Stokes equations in 2D

Velocity-pressure formulation:

Find $u(x,t) \in \mathbb{R}^2$, $p(x,t) \in \mathbb{R}$ solutions of

$$(NS) \begin{cases} u_t + u.\nabla u - \nu \Delta u + \nabla p = 0, & x \in \Omega \subset \mathbb{R}^2, t > 0 \\ \text{div}\, u = 0, \quad x \in \Omega \ , \ t > 0 \\ u = 0 \ , \ x \in \partial\Omega \ , \ t > 0 \\ u(x,0) = u_0(x), \quad x \in \Omega \end{cases}$$

Streamfunction formulation:

- $u = (-\psi_y, \psi_x) = \nabla^{\perp}\psi$, $\nabla \wedge u = \Delta\psi$. The streamfunction ψ evolves according to

$$\partial_t(\Delta\psi) + (\nabla^{\perp}\psi) \cdot \nabla(\Delta\psi) - \nu\Delta^2\psi = 0 \ , \quad x \in \Omega \ , \quad t > 0$$

(Landau-Lifschitz, Fluid Dynamics).

- The boundary conditions are given for all points $(x, y) \in \partial\Omega$,

$$\begin{cases} \psi(x,y,t) = 0 \ \text{ no-leak condition + gauge condition} \\ \frac{\partial\psi}{\partial n}(x,y,t) = 0 \ \text{ tangential velocity given} \end{cases} \quad (1)$$

Initial data: $\psi_0(x,y) = \psi(x,y,t)|_{t=0}, \quad (x,y) \in \Omega.$

Hermitian Derivative Operator

Definition

Suppose given $(u_i)_{i\in\mathbb{Z}}$. The hermitian derivative is $(u_{x,i})_{i\in\mathbb{Z}}$ given by

$$\frac{1}{6}u_{x,i-1} + \frac{2}{3}u_{x,i} + \frac{1}{6}u_{x,i+1} = \frac{u_{i+1} - u_{i-1}}{2h}, \quad i \in \mathbb{Z} \tag{2}$$

Finite Difference form

Can be rewritten as

$$\sigma_x u_{x,i} = \delta_x u_i, \quad i \in \mathbb{Z} \tag{3}$$

where σ_x, δ_x are

$$\sigma_x u_i = \frac{1}{6}u_{i-1} + \frac{2}{3}u_i + \frac{1}{6}u_{i+1}, \quad \delta_x u_i = \frac{u_{i+1} - u_{i-1}}{2h} \tag{4}$$

Fourth order accuracy

$$u_{x,i} = u'(x_i) + O(h^4) \tag{5}$$

Connection to cubic splines

$$u_{x,i} = u_s'(x_i) \tag{6}$$

where $u_s(x)$ is the cubic spline approximation to $u(x)$.

Hermitian Derivative Operator

Definition

Suppose given $(u_i)_{i\in\mathbb{Z}}$. The hermitian derivative is $(u_{x,i})_{i\in\mathbb{Z}}$ given by

$$\frac{1}{6}u_{x,i-1} + \frac{2}{3}u_{x,i} + \frac{1}{6}u_{x,i+1} = \frac{u_{i+1} - u_{i-1}}{2h}, \quad i \in \mathbb{Z} \qquad (2)$$

Finite Difference form

Can be rewritten as

$$\sigma_x u_{x,i} = \delta_x u_i, \quad i \in \mathbb{Z} \qquad (3)$$

where σ_x, δ_x are

$$\sigma_x u_i = \frac{1}{6}u_{i-1} + \frac{2}{3}u_i + \frac{1}{6}u_{i+1}, \quad \delta_x u_i = \frac{u_{i+1} - u_{i-1}}{2h} \qquad (4)$$

Fourth order accuracy

$$u_{x,i} = u'(x_i) + O(h^4) \qquad (5)$$

Connection to cubic splines

$$u_{x,i} = u_s'(x_i) \qquad (6)$$

where $u_s(x)$ is the cubic spline approximation to $u(x)$.

Hermitian Derivative Operator

Definition

Suppose given $(u_i)_{i\in\mathbb{Z}}$. The hermitian derivative is $(u_{x,i})_{i\in\mathbb{Z}}$ given by

$$\frac{1}{6}u_{x,i-1} + \frac{2}{3}u_{x,i} + \frac{1}{6}u_{x,i+1} = \frac{u_{i+1} - u_{i-1}}{2h}, \quad i \in \mathbb{Z} \tag{2}$$

Finite Difference form

Can be rewritten as

$$\sigma_x u_{x,i} = \delta_x u_i, \quad i \in \mathbb{Z} \tag{3}$$

where σ_x, δ_x are

$$\sigma_x u_i = \frac{1}{6}u_{i-1} + \frac{2}{3}u_i + \frac{1}{6}u_{i+1}, \quad \delta_x u_i = \frac{u_{i+1} - u_{i-1}}{2h} \tag{4}$$

Fourth order accuracy

$$u_{x,i} = u'(x_i) + O(h^4) \tag{5}$$

Connection to cubic splines

$$u_{x,i} = u_s'(x_i) \tag{6}$$

where $u_s(x)$ is the cubic spline approximation to $u(x)$.

Hermitian Derivative Operator

Definition

Suppose given $(u_i)_{i\in\mathbb{Z}}$. The hermitian derivative is $(u_{x,i})_{i\in\mathbb{Z}}$ given by

$$\frac{1}{6}u_{x,i-1} + \frac{2}{3}u_{x,i} + \frac{1}{6}u_{x,i+1} = \frac{u_{i+1} - u_{i-1}}{2h}, \quad i \in \mathbb{Z} \tag{2}$$

Finite Difference form

Can be rewritten as

$$\sigma_x u_{x,i} = \delta_x u_i, \quad i \in \mathbb{Z} \tag{3}$$

where σ_x, δ_x are

$$\sigma_x u_i = \frac{1}{6}u_{i-1} + \frac{2}{3}u_i + \frac{1}{6}u_{i+1}, \quad \delta_x u_i = \frac{u_{i+1} - u_{i-1}}{2h} \tag{4}$$

Fourth order accuracy

$$u_{x,i} = u'(x_i) + O(h^4) \tag{5}$$

Connection to cubic splines

$$u_{x,i} = u_s'(x_i) \tag{6}$$

where $u_s(x)$ is the cubic spline approximation to $u(x)$.

Three-Point Biharmonic Operator

Definition

Suppose given $(u_i)_{i\in\mathbb{Z}}$ and $(u_{x,i})_{i\in\mathbb{Z}}$ the corresponding hermitian derivative. The Three-Point Biharmonic $(\delta_x^4 u_i)_{i\in\mathbb{Z}}$ is ($\delta_x^2 u_i = (u_{i+1} + u_{i-1} - 2u_i)/h^2$),

$$\delta_x^4 u_i = \frac{12}{h^2}\left(\delta_x u_{x,i} - \delta_x^2 u_i\right) \quad (7)$$

Fourth order accuracy

$$\delta_x^4 u_i = u^{(4)}(x_i) + O(h^4) \quad (8)$$

Connection to cubic splines

Denote by $u_s(x)$ the cubic spline interpolation of the data $(u_i)_{0\le i\le N}$ with endpoints derivatives $u_{x,0}, u_{x,N}$. For gridfunctions $(\mathfrak{u}_i)_{0\le i\le N}, (\mathfrak{v}_i)_{0\le i\le N}$ with $\mathfrak{u}_0 = \mathfrak{u}_N = \mathfrak{v}_0 = \mathfrak{v}_N = 0$,

$$(\delta_x^4 \mathfrak{u}, \mathfrak{v})_h = \int_0^1 u_s''(x) v_s''(x)\,dx \quad (9)$$

where $(\mathfrak{u}, \mathfrak{v})_h = h\sum_{i=1}^{N-1} \mathfrak{u}_i \mathfrak{v}_i$

Three-Point Biharmonic Operator

Definition

Suppose given $(u_i)_{i\in\mathbb{Z}}$ and $(u_{x,i})_{i\in\mathbb{Z}}$ the corresponding hermitian derivative. The Three-Point Biharmonic $(\delta_x^4 u_i)_{i\in\mathbb{Z}}$ is ($\delta_x^2 u_i = (u_{i+1} + u_{i-1} - 2u_i)/h^2$),

$$\delta_x^4 u_i = \frac{12}{h^2}\left(\delta_x u_{x,i} - \delta_x^2 u_i\right) \tag{7}$$

Fourth order accuracy

$$\delta_x^4 u_i = u^{(4)}(x_i) + O(h^4) \tag{8}$$

Connection to cubic splines

Denote by $u_s(x)$ the cubic spline interpolation of the data $(u_i)_{0\le i\le N}$ with endpoints derivatives $u_{x,0}, u_{x,N}$. For gridfunctions $(\mathfrak{u}_i)_{0\le i\le N}, (\mathfrak{v}_i)_{0\le i\le N}$ with $\mathfrak{u}_0 = \mathfrak{u}_N = \mathfrak{v}_0 = \mathfrak{v}_N = 0$,

$$(\delta_x^4 \mathfrak{u}, \mathfrak{v})_h = \int_0^1 u_s''(x) v_s''(x)\,dx \tag{9}$$

where $(\mathfrak{u}, \mathfrak{v})_h = h\sum_{i=1}^{N-1} \mathfrak{u}_i \mathfrak{v}_i$

Three-Point Biharmonic Operator

Definition

Suppose given $(u_i)_{i\in\mathbb{Z}}$ and $(u_{x,i})_{i\in\mathbb{Z}}$ the corresponding hermitian derivative. The Three-Point Biharmonic $(\delta_x^4 u_i)_{i\in\mathbb{Z}}$ is $(\delta_x^2 u_i = (u_{i+1} + u_{i-1} - 2u_i)/h^2)$,

$$\delta_x^4 u_i = \frac{12}{h^2}\left(\delta_x u_{x,i} - \delta_x^2 u_i\right) \qquad (7)$$

Fourth order accuracy

$$\delta_x^4 u_i = u^{(4)}(x_i) + O(h^4) \qquad (8)$$

Connection to cubic splines

Denote by $u_s(x)$ the cubic spline interpolation of the data $(u_i)_{0\le i\le N}$ with endpoints derivatives $u_{x,0}, u_{x,N}$. For gridfunctions $(\mathfrak{u}_i)_{0\le i\le N}, (\mathfrak{v}_i)_{0\le i\le N}$ with $\mathfrak{u}_0 = \mathfrak{u}_N = \mathfrak{v}_0 = \mathfrak{v}_N = 0$,

$$(\delta_x^4 \mathfrak{u}, \mathfrak{v})_h = \int_0^1 u_s''(x) v_s''(x)\,dx \qquad (9)$$

where $(\mathfrak{u}, \mathfrak{v})_h = h\sum_{i=1}^{N-1} \mathfrak{u}_i \mathfrak{v}_i$

A compact scheme for the biharmonic problem in 1D

One-dimensional biharmonic problem

Solve on $I = [0,1]$

$$\begin{cases} u^{(4)}(x) = f(x) \;\;,\;\; 0 < x < 1 \\ u(0) = u'(0) = u(1) = u'(1) = 0 \end{cases} \tag{10}$$

Compact scheme

The approximate problem is : find $u = [u_0, u_1, \cdots, u_{N-1}, u_N]$ solution of

$$\begin{cases} \delta_x^4 u_j = \frac{12}{h^2}\left(\delta_x u_{x,j} - \delta_x^2 u_j\right) = f(x_j) \;\;,\;\; 1 \le j \le N-1 \\ \frac{1}{6}u_{x,j-1} + \frac{2}{3}u_{x,j} + \frac{1}{6}u_{x,j+1} = \delta_x u_j \;\;,\;\; 1 \le j \le N-1 \\ u_0 = u_N = u_{x,0} = u_{x,N} = 0 \end{cases} \tag{11}$$

A compact scheme for the biharmonic problem in 1D

One-dimensional biharmonic problem

Solve on $I = [0, 1]$

$$\begin{cases} u^{(4)}(x) = f(x) \;,\; 0 < x < 1 \\ u(0) = u'(0) = u(1) = u'(1) = 0 \end{cases} \tag{10}$$

Compact scheme

The approximate problem is : find $\mathfrak{u} = [u_0, u_1, \cdots, u_{N-1}, u_N]$ solution of

$$\begin{cases} \delta_x^4 u_j = \frac{12}{h^2}\left(\delta_x u_{x,j} - \delta_x^2 u_j\right) = f(x_j) \;,\; 1 \leq j \leq N-1 \\ \frac{1}{6}u_{x,j-1} + \frac{2}{3}u_{x,j} + \frac{1}{6}u_{x,j+1} = \delta_x u_j \;,\; 1 \leq j \leq N-1 \\ u_0 = u_N = u_{x,0} = u_{x,N} = 0 \end{cases} \tag{11}$$

Properties of δ_x^4

Theorem

Let $\tilde{u}$ be the approximate solution of the biharmonic problem $u^{(4)}(x) = f(x)$ with Dirichlet B.C. . Let $u(x)$ be the exact solution and u^* its evaluation at grid points. The error $e = \tilde{u} - u^* = [u_1, \cdots, u_{N-1}]$ satifies

$$|e|_h \leq Ch^4 \tag{12}$$

where C depends only on f.

Proof

Not straightforward result, due to the boundary conditions ! Method of proof: careful analysis of the structure of the matrix of δ_x^4 on a bounded domain $[0, \cdots, N]$.

Accuracy

The pointwise truncation error on a bounded domain cannot be deduced from the fourth order acuracy in the "free" space. Here the pointwise truncation of δ_x^4 is 1 at $i = 1, 2, \cdots, N-1$.

Energy method

Energy methods (as in FEM) provide only a suboptimal error estimate (so far).

Properties of δ_x^4

Theorem

Let $\tilde{u}$ be the approximate solution of the biharmonic problem $u^{(4)}(x) = f(x)$ with Dirichlet B.C. . Let $u(x)$ be the exact solution and u^* its evaluation at grid points. The error $e = \tilde{u} - u^* = [u_1, \cdots, u_{N-1}]$ satifies

$$|e|_h \leq Ch^4 \quad (12)$$

where C depends only on f.

Proof

Not straightforward result, due to the boundary conditions ! Method of proof: careful analysis of the structure of the matrix of δ_x^4 on a bounded domain $[0, \cdots, N]$.

Accuracy

The pointwise truncation error on a bounded domain cannot be deduced from the fourth order acuracy in the "free" space. Here the pointwise truncation of δ_x^4 is 1 at $i = 1, 2, \cdots, N-1$.

Energy method

Energy methods (as in FEM) provide only a suboptimal error estimate (so far).

Properties of δ_x^4

Theorem

Let $\tilde{u}$ be the approximate solution of the biharmonic problem $u^{(4)}(x) = f(x)$ with Dirichlet B.C. . Let $u(x)$ be the exact solution and u^* its evaluation at grid points. The error $e = \tilde{u} - u^* = [u_1, \cdots, u_{N-1}]$ satifies

$$|e|_h \leq Ch^4 \tag{12}$$

where C depends only on f.

Proof

Not straightforward result, due to the boundary conditions ! Method of proof: careful analysis of the structure of the matrix of δ_x^4 on a bounded domain $[0, \cdots, N]$.

Accuracy

The pointwise truncation error on a bounded domain cannot be deduced from the fourth order acuracy in the “free” space. Here the pointwise truncation of δ_x^4 is 1 at $i = 1, 2, \cdots, N-1$.

Energy method

Energy methods (as in FEM) provide only a suboptimal error estimate (so far).

Properties of δ_x^4

Theorem

Let $\tilde{u}$ be the approximate solution of the biharmonic problem $u^{(4)}(x) = f(x)$ with Dirichlet B.C. . Let $u(x)$ be the exact solution and u^* its evaluation at grid points. The error $e = \tilde{u} - u^* = [u_1, \cdots, u_{N-1}]$ satifies

$$|e|_h \leq Ch^4 \tag{12}$$

where C depends only on f.

Proof

Not straightforward result, due to the boundary conditions ! Method of proof: careful analysis of the structure of the matrix of δ_x^4 on a bounded domain $[0, \cdots, N]$.

Accuracy

The pointwise truncation error on a bounded domain cannot be deduced from the fourth order acuracy in the “free” space. Here the pointwise truncation of δ_x^4 is 1 at $i = 1, 2, \cdots, N-1$.

Energy method

Energy methods (as in FEM) provide only a suboptimal error estimate (so far).

The nine-point Biharmonic Operator for the 2D bih. problem

A compact Biharmonic operator

Biharmonic operator:

$$\Delta^2\psi = \partial_x^4\psi + \partial_y^4\psi + 2\partial_x^2\partial_y^2\psi \qquad (13)$$

Approximation by:

$$\Delta_h^2\psi_{i,j} = \delta_x^4\psi_{i,j} + \delta_y^4\psi_{i,j} + 2\delta_x^2\delta_y^2\psi_{i,j} \qquad (14)$$

where the discrete gradient $\nabla_h\psi = \left(\psi_{x,i,j}, \psi_{y,x,y}\right)$ is defined by the hermitian relations

$$\begin{cases} \frac{1}{6}\psi_{x,i-1,j} + \frac{2}{3}\psi_{x,i,j} + \frac{1}{6}\psi_{x,i+1,j} = \delta_x\psi_{i,j} \ , \ 1 \leq i \leq N-1 \\ \frac{1}{6}\psi_{y,i,j-1} + \frac{2}{3}\psi_{y,i,j} + \frac{1}{6}\psi_{y,i,j+1} = \delta_y\psi_{i,j} \ , \ 1 \leq j \leq N-1 \end{cases} \qquad (15)$$

Stephenson Biharmonic

This operator is the same than the one introduced by J.W. Stephenson (*Jour. Comp. Phys.* 1984).

The nine-point Biharmonic Operator for the 2D bih. problem

A compact Biharmonic operator

Biharmonic operator:

$$\Delta^2\psi = \partial_x^4\psi + \partial_y^4\psi + 2\partial_x^2\partial_y^2\psi \tag{13}$$

Approximation by:

$$\Delta_h^2\psi_{i,j} = \delta_x^4\psi_{i,j} + \delta_y^4\psi_{i,j} + 2\delta_x^2\delta_y^2\psi_{i,j} \tag{14}$$

where the discrete gradient $\nabla_h\psi = \Big(\psi_{x,i,j}, \psi_{y,x,y}\Big)$ is defined by the hermitian relations

$$\left\{\begin{array}{ll} \frac{1}{6}\psi_{x,i-1,j} + \frac{2}{3}\psi_{x,i,j} + \frac{1}{6}\psi_{x,i+1,j} = \delta_x\psi_{i,j} & , \ 1 \le i \le N-1 \\ \frac{1}{6}\psi_{y,i,j-1} + \frac{2}{3}\psi_{y,i,j} + \frac{1}{6}\psi_{y,i,j+1} = \delta_y\psi_{i,j} & , \ 1 \le j \le N-1 \end{array}\right. \tag{15}$$

Stephenson Biharmonic

This operator is the same than the one introduced by J.W. Stephenson (*Jour. Comp. Phys.* 1984).

Stephenson scheme for the 2D Biharmonic Problem

Continuous Biharmonic problem

$$\begin{cases} \Delta^2\psi(x,y) = f(x,y), & (x,y) \in \Omega \\ \psi(x,y) = g_1(x,y), & (x,y) \in \partial\Omega \\ \frac{\partial\psi}{\partial n}(x,y) = g_2(x,y), & (x,y) \in \partial\Omega \end{cases} \tag{16}$$

Discrete Biharmonic problem in a square

Solve the system in $\psi_{i,j}, \quad 0 \le i,j \le N$

$$\Delta_h^2\psi_{i,j} = f^*(x_i,y_j), \quad 1 \le i,j \le N-1 \tag{17}$$

subject to the boundary conditions

$$\begin{cases} \psi_{i,j} = g_1^*(x_i,y_j), \; \{i=0,N, \quad 0 \le j \le N\} \quad \text{or} \quad \{j=0,N, \quad 0 \le i \le N\}, \\ \psi_{x,i,j} = -g_2^*(x_i,y_j), \quad i=0, \quad 0 \le j \le N, \\ \psi_{x,i,j} = g_2^*(x_i,y_j), \quad i=N, \quad 0 \le j \le N, \\ \psi_{y,i,j} = -g_2^*(x_i,y_j), \quad j=0, \quad 0 \le i \le N, \\ \psi_{y,i,j} = g_2^*(x_i,y_j), \quad j=N, \quad 0 \le i \le N. \end{cases} \tag{18}$$

Stephenson scheme for the 2D Biharmonic Problem

Continuous Biharmonic problem

$$\begin{cases} \Delta^2\psi(x,y) = f(x,y), & (x,y) \in \Omega \\ \psi(x,y) = g_1(x,y), & (x,y) \in \partial\Omega \\ \frac{\partial\psi}{\partial n}(x,y) = g_2(x,y), & (x,y) \in \partial\Omega \end{cases} \tag{16}$$

Discrete Biharmonic problem in a square

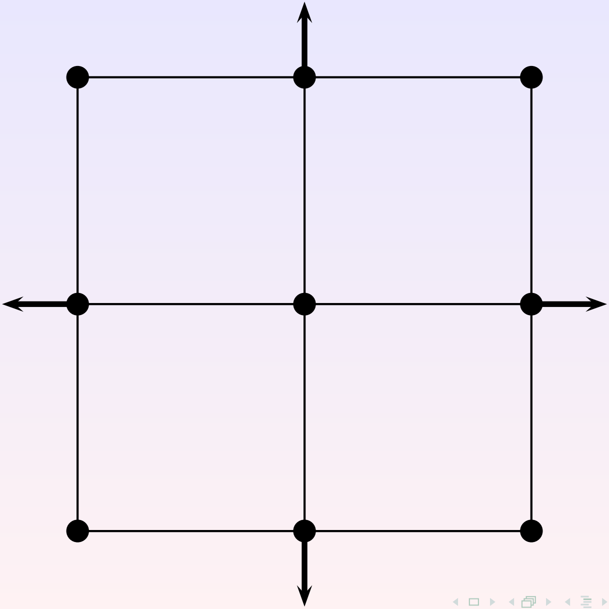
Solve the system in $\psi_{i,j}, \quad 0 \le i,j \le N$

$$\Delta_h^2 \psi_{i,j} = f^*(x_i, y_j), \quad 1 \le i,j \le N-1 \tag{17}$$

subject to the boundary conditions

$$\begin{cases} \psi_{i,j} = g_1^*(x_i,y_j), \ \{i = 0, N, \quad 0 \le j \le N\} \quad \text{or} \quad \{j = 0, N, \quad 0 \le i \le N\}, \\ \psi_{x,i,j} = -g_2^*(x_i,y_j), \quad i = 0, \quad 0 \le j \le N, \\ \psi_{x,i,j} = g_2^*(x_i,y_j), \quad i = N, \quad 0 \le j \le N, \\ \psi_{y,i,j} = -g_2^*(x_i,y_j), \quad j = 0, \quad 0 \le i \le N, \\ \psi_{y,i,j} = g_2^*(x_i,y_j), \quad j = N, \quad 0 \le i \le N. \end{cases} \tag{18}$$

Stencil of the nine-point Bih. operator



Properties of the Stephenson scheme for the 2D Bih. Problem

No artificial BC on the vorticity $\Delta\psi$

Only the natural BC on ψ are required by the scheme. In the Dirichlet case, it is ψ, $\frac{\partial \psi}{\partial n}$.

Second order accuracy

The operator Δ_h^2 is second order accurate. The one-dimensional operators $\delta_x^4\psi$, $\delta_y^4\psi$ are 4th order accurate (in the “free” setting). The second order accuracy is due only to the mixed term $\delta_x^2\delta_y^2\psi$.

Properties of the Stephenson scheme for the 2D Bih. Problem

No artificial BC on the vorticity $\Delta\psi$

Only the natural BC on ψ are required by the scheme. In the Dirichlet case, it is ψ, $\frac{\partial \psi}{\partial n}$.

Second order accuracy

The operator Δ_h^2 is second order accurate. The one-dimensional operators $\delta_x^4\psi$, $\delta_y^4\psi$ are 4th order accurate (in the “free” setting). The second order accuracy is due only to the mixed term $\delta_x^2\delta_y^2\psi$.

Matrix operator of δ_x^2 and δ_x^4

Matrix operators

One has $-\delta_x^2 = T/h^2$ with

$$T = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{bmatrix} \in \mathbb{M}_{N-1}(\mathbb{R}) \quad (19)$$

The symmetric positive definite matrix P is deduced from T by

$$P = 6I - T, \quad (20)$$

The nine-point Biharmonic

$$\Delta_h^2 = \frac{1}{h^4}\left[6P^{-1}T^2 \otimes I + 6I \otimes P^{-1}T^2 + 2T \otimes T\right] \quad (21)$$

$$+ \frac{36}{h^4}\left[v_1, v_2\right]\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} \otimes I_{N-1} + \frac{36}{h^4} I_{N-1} \otimes \left[v_1, v_2\right]\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}.$$

$$\begin{cases} v_1 = (\alpha - \beta)^{1/2} P^{-1}\left(\frac{\sqrt{2}}{2}e_1 - \frac{\sqrt{2}}{2}e_{N-1}\right) \in \mathbb{R}^{N-1} \\ v_2 = (\alpha + \beta)^{1/2} P^{-1}\left(\frac{\sqrt{2}}{2}e_1 + \frac{\sqrt{2}}{2}e_{N-1}\right) \in \mathbb{R}^{N-1} \end{cases} \quad (22)$$

Matrix operator of δ_x^2 and δ_x^4

Matrix operators

One has $-\delta_x^2 = T/h^2$ with

$$T = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{bmatrix} \in \mathbb{M}_{N-1}(\mathbb{R}) \quad (19)$$

The symmetric positive definite matrix P is deduced from T by

$$P = 6I - T, \quad (20)$$

The nine-point Biharmonic

$$\begin{aligned} \Delta_h^2 &= \frac{1}{h^4}\left[6P^{-1}T^2 \otimes I + 6I \otimes P^{-1}T^2 + 2T \otimes T\right] \quad (21) \\ &+ \frac{36}{h^4}\left[v_1, v_2\right]\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} \otimes I_{N-1} + \frac{36}{h^4} I_{N-1} \otimes \left[v_1, v_2\right]\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}. \end{aligned}$$

$$\begin{cases} v_1 = (\alpha - \beta)^{1/2} P^{-1}\left(\frac{\sqrt{2}}{2}e_1 - \frac{\sqrt{2}}{2}e_{N-1}\right) \in \mathbb{R}^{N-1} \\ v_2 = (\alpha + \beta)^{1/2} P^{-1}\left(\frac{\sqrt{2}}{2}e_1 + \frac{\sqrt{2}}{2}e_{N-1}\right) \in \mathbb{R}^{N-1} \end{cases} \quad (22)$$

Fast solver

Shermann-Morrison formula

The matrix of Δ_h^2 is a low-rank perturbation (due to the BC) of a diagonal operator (in a spectral basis), which represents the biharmonic in the "free space":

$$\mathcal{A} = \mathcal{B} + \frac{36}{h^4}\mathcal{R}\mathcal{R}^T, \tag{23}$$

The Sherman-Morrison formula gives

$$\bar{\mathcal{A}}^{-1} = \bar{\mathcal{B}}^{-1} - 36\mathcal{B}^{-1}\mathcal{R}\left[I_{4(N-1)} + 36\mathcal{R}^T\mathcal{B}^{-1}\mathcal{R}\right]^{-1}\mathcal{R}^T\bar{\mathcal{B}}^{-1}. \tag{24}$$

Fast resolution procedure

A fast solver ($N^2\ \ln_2(N)$) is deduced in 8 steps. The key steps are:

1. Using the FFT to compute $BU = F$ (system in $\mathbb{R}^{(N-1)^2}$).
2. Using the PCG to solve

$$\left(I_{4(N-1)} + 36R^TB^{-1}R\right)V = G, \quad \text{(system in } \mathbb{R}^{4(N-1)}\text{)}. \tag{25}$$

Fast solver

Shermann-Morrison formula

The matrix of Δ_h^2 is a low-rank perturbation (due to the BC) of a diagonal operator (in a spectral basis), which represents the biharmonic in the "free space":

$$\mathcal{A} = \mathcal{B} + \frac{36}{h^4}\mathcal{R}\mathcal{R}^T, \tag{23}$$

The Sherman-Morrison formula gives

$$\bar{\mathcal{A}}^{-1} = \bar{\mathcal{B}}^{-1} - 36\mathcal{B}^{-1}\mathcal{R}\left[I_{4(N-1)} + 36\mathcal{R}^T\mathcal{B}^{-1}\mathcal{R}\right]^{-1}\mathcal{R}^T\bar{\mathcal{B}}^{-1}. \tag{24}$$

Fast resolution procedure

A fast solver ($N^2 \ln_2(N)$) is deduced in 8 steps. The key steps are:

- Using the FFT to compute $BU = F$ (system in $\mathbb{R}^{(N-1)^2}$).
- Using the PCG to solve

$$\left(I_{4(N-1)} + 36\mathcal{R}^T\mathcal{B}^{-1}\mathcal{R}\right)V = G, \quad \text{(system in } \mathbb{R}^{4(N-1)}\text{)}. \tag{25}$$

Fast solver

Shermann-Morrison formula

The matrix of Δ_h^2 is a low-rank perturbation (due to the BC) of a diagonal operator (in a spectral basis), which represents the biharmonic in the "free space":

$$\mathcal{A} = \mathcal{B} + \frac{36}{h^4}\mathcal{R}\mathcal{R}^T, \quad (23)$$

The Sherman-Morrison formula gives

$$\bar{\mathcal{A}}^{-1} = \bar{\mathcal{B}}^{-1} - 36\mathcal{B}^{-1}\mathcal{R}\left[I_{4(N-1)} + 36\mathcal{R}^T\mathcal{B}^{-1}\mathcal{R}\right]^{-1}\mathcal{R}^T\bar{\mathcal{B}}^{-1}. \quad (24)$$

Fast resolution procedure

A fast solver ($N^2 \ln_2(N)$) is deduced in 8 steps. The key steps are:

- Using the FFT to compute $BU = F$ (system in $\mathbb{R}^{(N-1)^2}$).
- Using the PCG to solve

$$\left(I_{4(N-1)} + 36\mathcal{R}^T\mathcal{B}^{-1}\mathcal{R}\right)V = G, \quad \text{(system in } \mathbb{R}^{4(N-1)}\text{)}. \quad (25)$$

A fourth order biharmonic operator

Fourth order Biharmonic

It is possible to modify the mixed term in the Stephenson operator to obtain a 4th order accurate scheme. Simply replace $\delta_x^2 \delta_y^2 u$ by

$$\widetilde{\delta_x^2 \delta_y^2} \psi_{i,j} = 3\delta_x^2 \delta_y^2 \psi_{i,j} - \delta_x^2 \delta_y \psi_{y,i,j} - \delta_y^2 \delta_x \psi_{x,i,j} = \partial_x^2 \partial_y^2 \psi_{i,j} + O(h^4). \tag{26}$$

Fast solver for the fourth order Biharmonic

The fast solver follows the same principle than for the second order Biharmonic.

A fourth order biharmonic operator

Fourth order Biharmonic

It is possible to modify the mixed term in the Stephenson operator to obtain a 4th order accurate scheme. Simply replace $\delta_x^2 \delta_y^2 u$ by

$$\widetilde{\delta_x^2 \delta_y^2} \psi_{i,j} = 3\delta_x^2 \delta_y^2 \psi_{i,j} - \delta_x^2 \delta_y \psi_{y,i,j} - \delta_y^2 \delta_x \psi_{x,i,j} = \partial_x^2 \partial_y^2 \psi_{i,j} + O(h^4). \tag{26}$$

Fast solver for the fourth order Biharmonic

The fast solver follows the same principle than for the second order Biharmonic.

Computing efficiency

N	N=128	N=256	N=512	N=1024	N=2048
CPU$_{tot}$	0.11s	0.45s	1.84s	7.91s	34.63s
CPU$_{\infty}$	0.093s	0.39s	1.47s	6.46s	27.72s
CPU$_{tot}$ $/(N^2\,\mathrm{Log}(N))$	1.37(-6)	1.24(-6)	1.16(-6)	1.09(-6)	1.07(-6)

Table: *Indicative CPU time on a Laptop*

Fourth order accuracy for $\psi, \nabla\psi, \Delta\psi$

N	$\|\psi - \psi_h\|_{\infty,h}$	$\|\psi_x - \psi_{x,h}\|_{\infty,h}$	$\|\psi - \psi_{y,h}\|_{\infty,h}$	$\|\Delta\psi - \tilde{\Delta}_h \psi_h\|_{\infty,h}$
$N = 16$	3.42(-5)	1.00(-4)	1.00(-4)	3.99(-4)
conv. rate	4.04	4.01	4.01	4.00
$N = 32$	2.08(-6)	6.21(-6)	6.21(-6)	2.48(-5)
conv. rate	4.01	4.00	4.00	4.00
$N = 64$	1.29(-7)	3.87(-7)	3.87(-7)	1.55(-6)
conv. rate	4.00	4.00	4.00	4.00
$N = 128$	8.06(-9)	2.41(-8)	2.41(-8)	9.68(-8)
conv. rate	3.99	3.99	3.99	3.83
$N = 256$	5.04(-10)	1.51(-9)	1.51(-9)	6.77(-9)
conv. rate	3.74	4.02	4.02	-0.22
$N = 512$	3.76(-11)	9.27(-11)	9.07(-11)	7.90(-9)
conv. rate	-0.13	0.19	0.19	0.59
$N = 1024$	4.12(-11)	8.09(-11)	8.09(-11)	5.22(-8)

Table: *Error and convergence rate for Test Case 1 with the fourth order scheme*

Second order scheme for the Navier-Stokes equation

Navier-Stokes equation in streamfunction

$$\partial_t \Delta\psi + (\nabla^\perp \psi)\cdot\nabla(\Delta\psi) - \nu\Delta^2\psi = 0 \;\;,\;\; x\in\Omega \;\;,\;\; t>0 \tag{27}$$

+ Dirichlet B.C on ψ.

Approximation in space (method of lines)

$\psi(x_i, y_j, t) \simeq \tilde\psi_{i,j}(t)$, solution of

$$\partial_t \Delta_h \tilde\psi_{i,j} - \tilde\psi_{y,i,j}\Delta_h\tilde\psi_{x,i,j} + \tilde\psi_{x,i,j}\Delta_h\tilde\psi_{y,i,j} - \nu\Delta_h^2\tilde\psi_{i,j} = 0 \;\;,\;\; x\in\Omega \;\;,\;\; t>0 \tag{28}$$

+ Dirichlet B.C on $\tilde\psi_{i,j}$, $\tilde\psi_{x,i,j}$, $\tilde\psi_{y,i,j}$.

Fuly centered second order scheme

The operator in space are just translated on the discrete grid using:

- Second order Laplacian, second order Biharmonic Five-point Laplacian:

$$\Delta\psi(x_i, y_j) \simeq \Delta_h\tilde\psi_{i,j}, \quad \Delta^2\psi(x_i, y_j) \simeq \Delta_h^2\tilde\psi_{i,j} \tag{29}$$

- Second order convective term

$$(\nabla^\perp\psi(x_i, y_j))\cdot\nabla(\Delta\psi(x_i, y_j)) \simeq -\tilde\psi_{y,i,j}\Delta_h\tilde\psi_{x,i,j} + \tilde\psi_{x,i,j}\Delta_h\tilde\psi_{y,i,j} \tag{30}$$

Second order scheme for the Navier-Stokes equation

Navier-Stokes equation in streamfunction

$$\partial_t \Delta\psi + (\nabla^{\perp}\psi)\cdot\nabla(\Delta\psi) - \nu\Delta^2\psi = 0 \ , \ \ x \in \Omega \ , \ \ t > 0 \qquad (27)$$

+ Dirichlet B.C on ψ.

Approximation in space (method of lines)

$\psi(x_i, y_j, t) \simeq \tilde{\psi}_{i,j}(t)$, solution of

$$\partial_t \Delta_h \tilde{\psi}_{i,j} - \tilde{\psi}_{y,i,j}\Delta_h\tilde{\psi}_{x,i,j} + \tilde{\psi}_{x,i,j}\Delta_h\tilde{\psi}_{y,i,j} - \nu\Delta_h^2\tilde{\psi}_{i,j} = 0 \ , \ \ x \in \Omega \ , \ \ t > 0 \qquad (28)$$

+ Dirichlet B.C on $\tilde{\psi}_{i,j}$, $\tilde{\psi}_{x,i,j}$, $\tilde{\psi}_{y,i,j}$.

Fuly centered second order scheme

The operator in space are just translated on the discrete grid using:

- Second order Laplacian, second order Biharmonic Five-point Laplacian:

$$\Delta\psi(x_i, y_j) \simeq \Delta_h\tilde{\psi}_{i,j}, \quad \Delta^2\psi(x_i, y_j) \simeq \Delta_h^2\tilde{\psi}_{i,j} \qquad (29)$$

- Second order convective term

$$(\nabla^{\perp}\psi(x_i, y_j))\cdot\nabla(\Delta\psi(x_i, y_j)) \simeq -\tilde{\psi}_{y,i,j}\Delta_h\tilde{\psi}_{x,i,j} + \tilde{\psi}_{x,i,j}\Delta_h\tilde{\psi}_{y,i,j} \qquad (30)$$

Second order scheme for the Navier-Stokes equation

Navier-Stokes equation in streamfunction

$$\partial_t \Delta\psi + (\nabla^{\perp}\psi)\cdot\nabla(\Delta\psi) - \nu\Delta^2\psi = 0 \ , \quad x \in \Omega \ , \quad t > 0 \qquad (27)$$

+ Dirichlet B.C on ψ.

Approximation in space (method of lines)

$\psi(x_i, y_j, t) \simeq \tilde{\psi}_{i,j}(t)$, solution of

$$\partial_t \Delta_h \tilde{\psi}_{i,j} - \tilde{\psi}_{y,i,j}\Delta_h \tilde{\psi}_{x,i,j} + \tilde{\psi}_{x,i,j}\Delta_h\tilde{\psi}_{y,i,j} - \nu\Delta_h^2\tilde{\psi}_{i,j} = 0 \ , \quad x \in \Omega \ , \quad t > 0 \qquad (28)$$

+ Dirichlet B.C on $\tilde{\psi}_{i,j}$, $\tilde{\psi}_{x,i,j}$, $\tilde{\psi}_{y,i,j}$.

Fuly centered second order scheme

The operator in space are just translated on the discrete grid using:

- Second order Laplacian, second order Biharmonic Five-point Laplacian:

$$\Delta\psi(x_i, y_j) \simeq \Delta_h\tilde{\psi}_{i,j}, \quad \Delta^2\psi(x_i, y_j) \simeq \Delta_h^2\tilde{\psi}_{i,j} \qquad (29)$$

- Second order convective term

$$(\nabla^{\perp}\psi(x_i, y_j))\cdot\nabla(\Delta\psi(x_i, y_j)) \simeq -\tilde{\psi}_{y,i,j}\Delta_h\tilde{\psi}_{x,i,j} + \tilde{\psi}_{x,i,j}\Delta_h\tilde{\psi}_{y,i,j} \qquad (30)$$

Second order scheme for the Navier-Stokes equation

Navier-Stokes equation in streamfunction

$$\partial_t \Delta\psi + (\nabla^\perp \psi)\cdot\nabla(\Delta\psi) - \nu\Delta^2\psi = 0 \ , \quad x\in\Omega \ , \quad t>0 \qquad (27)$$

+ Dirichlet B.C on ψ.

Approximation in space (method of lines)

$\psi(x_i, y_j, t) \simeq \tilde{\psi}_{i,j}(t)$, solution of

$$\partial_t \Delta_h \tilde{\psi}_{i,j} - \tilde{\psi}_{y,i,j}\Delta_h\tilde{\psi}_{x,i,j} + \tilde{\psi}_{x,i,j}\Delta_h\tilde{\psi}_{y,i,j} - \nu\Delta_h^2\tilde{\psi}_{i,j} = 0 \ , \quad x\in\Omega \ , \quad t>0 \qquad (28)$$

+ Dirichlet B.C on $\tilde{\psi}_{i,j}$, $\tilde{\psi}_{x,i,j}$, $\tilde{\psi}_{y,i,j}$.

Fuly centered second order scheme

The operator in space are just translated on the discrete grid using:

- Second order Laplacian, second order Biharmonic Five-point Laplacian:

$$\Delta\psi(x_i, y_j) \simeq \Delta_h\tilde{\psi}_{i,j}, \quad \Delta^2\psi(x_i, y_j) \simeq \Delta_h^2\tilde{\psi}_{i,j} \qquad (29)$$

- Second order convective term

$$(\nabla^\perp\psi(x_i, y_j))\cdot\nabla(\Delta\psi(x_i, y_j)) \simeq -\tilde{\psi}_{y,i,j}\Delta_h\tilde{\psi}_{x,i,j} + \tilde{\psi}_{x,i,j}\Delta_h\tilde{\psi}_{y,i,j} \qquad (30)$$

Numerical analysis

Theorem

Let $T > 0$. Then there exist constants $C, h_0 > 0$, depending possibly on T, ν and on the exact solution ψ, such that, for all $0 \leq t \leq T$,

$$|\delta_x^+(\psi(t) - \tilde{\psi}(t))|_h^2 + |\delta_y^+(\psi(t) - \tilde{\psi}(t))|_h^2 \leq Ch^3 \quad , \quad 0 < h \leq h_0 \tag{31}$$

where $\psi(t) = \psi_{i,j}(t)$ is the pointwise interpolated exact solution and $\tilde{\psi}_{i,j}(t)$ is the solution of the semidiscrete scheme.

Properties

- Second order centered approximation (no upwinding).
- No need of boundary conditions on the vorticity and no uncontrolled pressure modes.

Numerical analysis

Theorem

Let $T > 0$. Then there exist constants $C, h_0 > 0$, depending possibly on T, ν and on the exact solution ψ, such that, for all $0 \le t \le T$,

$$|\delta_x^+(\psi(t) - \tilde{\psi}(t))|_h^2 + |\delta_y^+(\psi(t) - \tilde{\psi}(t))|_h^2 \le Ch^3 \quad , \quad 0 < h \le h_0 \qquad (31)$$

where $\psi(t) = \psi_{i,j}(t)$ is the pointwise interpolated exact solution and $\tilde{\psi}_{i,j}(t)$ is the solution of the semidiscrete scheme.

Properties

- Second order centered approximation (no upwinding).
- No need of boundary conditions on the vorticity and no uncontrolled pressure modes.

Fourth order scheme for the Navier-Stokes equation

Centered fourth order scheme

The operators in space are just translated from the continuous ones on the discrete grid using:

- Fourth order Laplacian, fourth order Biharmonic

$$\begin{cases} \Delta\psi(x_i, y_j) \simeq \Delta_h \psi_{i,j} - \frac{h^2}{12}(\delta_x^4 \psi_{i,j} + \delta_y^4 \psi_{i,j}) \\ \Delta^2\psi(x_i, y_j) \simeq \Delta_h^2 \psi - \delta_x^4 \left(I - \frac{h^2}{6}\delta_y^2\right) \psi_{i,j} + \delta_y^4 \left(I - \frac{h^2}{6}\delta_x^2\right) \psi_{i,j} + 2\delta_x^2 \delta_y^2 \psi_{i,j} \end{cases} \tag{32}$$

- Fourth order convective term

$$\begin{aligned} (\nabla^\perp \psi(x_i, y_j)) \cdot \nabla(\Delta\psi(x_i, y_j)) \quad &\simeq \quad -\psi_{y,i,j}\Delta_h \psi_{x,i,j} + \psi_{x,i,j}\Delta_h \psi_{y,i,j} \\ &- \quad \frac{h^2}{12}\Big(- \delta_x \left(\psi_{y,i,j}(\delta_x^4 \psi_{i,j} + \delta_y^4 \psi_{i,j})\right) \\ &+ \quad \delta_y \left(\psi_{x,i,j}(\delta_x^4 \psi_{i,j} + \delta_y^4 \psi_{i,j})\right) \Big) \end{aligned}$$

Centered fourth order scheme

The operators in space are just translated from the continuous ones on the discrete grid using:

- Fourth order Laplacian, fourth order Biharmonic

$$\begin{cases} \Delta\psi(x_i, y_j) \simeq \Delta_h \psi_{i,j} - \frac{h^2}{12}(\delta_x^4 \psi_{i,j} + \delta_y^4 \psi_{i,j}) \\ \Delta^2\psi(x_i, y_j) \simeq \Delta_h^2 \psi - \delta_x^4 \left(I - \frac{h^2}{6}\delta_y^2\right)\psi_{i,j} + \delta_y^4 \left(I - \frac{h^2}{6}\delta_x^2\right)\psi_{i,j} + 2\delta_x^2\delta_y^2\psi_{i,j} \end{cases} \tag{32}$$

- Fourth order convective term

$$\begin{aligned} (\nabla^{\perp}\psi(x_i, y_j)) \cdot \nabla(\Delta\psi(x_i, y_j)) \quad &\simeq \quad -\psi_{y,i,j}\Delta_h\psi_{x,i,j} + \psi_{x,i,j}\Delta_h\psi_{y,i,j} \\ &- \quad \frac{h^2}{12}\Big(-\delta_x\left(\psi_{y,i,j}(\delta_x^4\psi_{i,j} + \delta_y^4\psi_{i,j})\right) \\ &+ \quad \delta_y\left(\psi_{x,i,j}(\delta_x^4\psi_{i,j} + \delta_y^4\psi_{i,j})\right)\Big) \end{aligned}$$

Fourth order scheme for the Navier-Stokes equation

Centered fourth order scheme

The operators in space are just translated from the continuous ones on the discrete grid using:

- Fourth order Laplacian, fourth order Biharmonic

$$\begin{cases} \Delta\psi(x_i, y_j) \simeq \Delta_h \psi_{i,j} - \frac{h^2}{12}(\delta_x^4 \psi_{i,j} + \delta_y^4 \psi_{i,j}) \\ \Delta^2\psi(x_i, y_j) \simeq \Delta_h^2 \psi - \delta_x^4 \left(I - \frac{h^2}{6}\delta_y^2\right)\psi_{i,j} + \delta_y^4 \left(I - \frac{h^2}{6}\delta_x^2\right)\psi_{i,j} + 2\delta_x^2\delta_y^2\psi_{i,j} \end{cases} \quad (32)$$

- Fourth order convective term

$$\begin{aligned} (\nabla^{\perp}\psi(x_i, y_j)) \cdot \nabla(\Delta\psi(x_i, y_j)) &\simeq -\psi_{y,i,j}\Delta_h \psi_{x,i,j} + \psi_{x,i,j}\Delta_h \psi_{y,i,j} \\ &- \frac{h^2}{12}\Big(-\delta_x \left(\psi_{y,i,j}(\delta_x^4\psi_{i,j} + \delta_y^4\psi_{i,j})\right) \\ &+ \delta_y \left(\psi_{x,i,j}(\delta_x^4\psi_{i,j} + \delta_y^4\psi_{i,j})\right) \Big) \end{aligned}$$

High order IMEX time-scheme (Spalart-Moser-Rogers)

Algorithm: 3 biharmonic solving by time-step

$$\begin{cases} U = \tilde{\Delta}_h \psi \\ D = \nu \tilde{\Delta}_h^2(\psi) \\ C = \tilde{C}_h(\psi), \end{cases} \tag{33}$$

The scheme is

$$\begin{cases} U^1 = \tilde{\Delta}_h \psi^n \\ U^2 = U^1 + \Delta t \left(\gamma_1(-C_h^1) + \alpha_1 D_h^1 + \beta_1 D_h^2\right) + \frac{8}{15}\Delta t F^{n+4/15} \\ U^3 = U^2 + \Delta t \left(\gamma_2(-C_h^2) + \zeta_1(-C_h^1) + \alpha_2 D_h^2 + \beta_2 D_h^3\right) + \Delta t \left(\frac{2}{3} F^{n+1/3} - \frac{8}{15} F^{n+4/15}\right) \\ U^4 = U^3 + \Delta t \left(\gamma_3(-C_h^3) + \zeta_2(-C_h^2) + \alpha_3 D_h^3 + \beta_3 D_h^4\right) + \Delta t \left(\frac{1}{6} F^n + \frac{2}{3} F^{n+1/2} + \frac{1}{6} F^{n+1} - \frac{2}{3} F^{n+1/3}\right). \end{cases} \tag{34}$$

The values of the parameters are

$$\begin{cases} \alpha_1 = \frac{29}{96} \quad \alpha_2 = \frac{-3}{40} \quad \alpha_3 = \frac{1}{6} \\ \beta_1 = \frac{37}{160} \quad \beta_2 = \frac{5}{24} \quad \beta_3 = \frac{1}{6} \\ \gamma_1 = \frac{8}{15} \quad \gamma_2 = \frac{5}{12} \quad \gamma_3 = \frac{3}{4} \\ \zeta_1 = \frac{-17}{60} \quad \zeta_2 = \frac{-5}{12}. \end{cases} \tag{35}$$

Cost of one time-step

Three biharmonic problems of the form $(\Delta - \alpha\Delta^2)\psi = f$ to solve at each time step

High order IMEX time-scheme (Spalart-Moser-Rogers)

Algorithm: 3 biharmonic solving by time-step

$$\begin{cases} U = \tilde{\Delta}_h \psi \\ D = \nu \tilde{\Delta}_h^2(\psi) \\ C = \tilde{C}_h(\psi), \end{cases} \tag{33}$$

The scheme is

$$\begin{cases} U^1 = \tilde{\Delta}_h \psi^n \\ U^2 = U^1 + \Delta t \left(\gamma_1(-C_h^1) + \alpha_1 D_h^1 + \beta_1 D_h^2\right) + \frac{8}{15}\Delta t F^{n+4/15} \\ U^3 = U^2 + \Delta t \left(\gamma_2(-C_h^2) + \zeta_1(-C_h^1) + \alpha_2 D_h^2 + \beta_2 D_h^3\right) + \Delta t \left(\frac{2}{3} F^{n+1/3} - \frac{8}{15} F^{n+4/15}\right) \\ U^4 = U^3 + \Delta t \left(\gamma_3(-C_h^3) + \zeta_2(-C_h^2) + \alpha_3 D_h^3 + \beta_3 D_h^4\right) + \Delta t \left(\frac{1}{6} F^n + \frac{2}{3} F^{n+1/2} + \frac{1}{6} F^{n+1} - \frac{2}{3} F^{n+1/3}\right). \end{cases} \tag{34}$$

The values of the parameters are

$$\begin{cases} \alpha_1 = \frac{29}{96} \quad \alpha_2 = \frac{-3}{40} \quad \alpha_3 = \frac{1}{6} \\ \beta_1 = \frac{37}{160} \quad \beta_2 = \frac{5}{24} \quad \beta_3 = \frac{1}{6} \\ \gamma_1 = \frac{8}{15} \quad \gamma_2 = \frac{5}{12} \quad \gamma_3 = \frac{3}{4} \\ \zeta_1 = \frac{-17}{60} \quad \zeta_2 = \frac{-5}{12}. \end{cases} \tag{35}$$

Cost of one time-step

Three biharmonic problems of the form $(\Delta - \alpha\Delta^2)\psi = f$ to solve at each time step

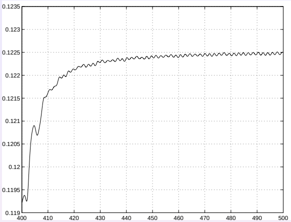
Assessing the fourth order accuracy

$e =$ absolute error for ψ, $e_r =$, relative error for ψ_x, $e_x =$ absolute error for ψ_x.

mesh		9 × 9	Rate	17 × 17	Rate	33 × 33	Rate	65 × 65
$t = 0.25$	e	5.0867(-3)	4.06	3.0525(-4)	4.02	1.8835(-5)	4.00	1.1734(-6)
	e_r	9.4936(-3)		5.7441(-4)		3.5460(-5)		2.2092(-6)
	e_x	2.6390(-3)	3.89	1.7837(-4)	3.93	1.1670(-5)	3.98	7.3752(-7)
$t = 0.5$	e	3.2224(-3)	4.00	2.0085(-4)	4.00	1.2541(-5)	4.00	7.8361(-7)
	e_r	7.7407(-3)		4.8536(-4)		3.0317(-5)		1.8944(-6)
	e_x	3.2285(-3)	4.02	1.9896(-4)	4.00	1.2436(-5)	4.00	7.7745(-7)
$t = 0.75$	e	2.4887(-3)	4.00	1.5508(-4)	4.00	9.6887(-6)	4.00	6.0551(-7)
	e_r	7.6730(-3)		4.8119(-4)		3.0075(-5)		1.8796(-6)
	e_x	2.5516(-3)	4.02	1.5723(-4)	4.00	9.8187(-6)	4.00	6.1364(-7)
$t = 1$	e	1.9376(-3)	4.00	1.2074(-4)	4.00	7.5434(-6)	4.00	4.7145(-7)
	e_r	7.6796(-3)		4.8103(-4)		3.0066(-5)		1.8791(-6)
	e_x	1.9885(-3)	4.02	1.2255(-4)	4.00	7.6526(-6)	4.00	4.7826(-7)

Table 1: Compact scheme for Navier-Stokes with exact solution: $\psi = (1 - x^2)^3(1 - y^2)^3 e^{-t}$ on $[-1, 1] \times [-, 1]$. We represent e,: the l_2 error for the streamfunction and e_x the max error in the U^x velocity $= -\partial_y \psi$. $\Delta t = Ch^2$.

Max $|\psi|$ behaviour at $\mathrm{Re} = 10000$



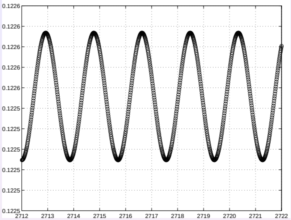
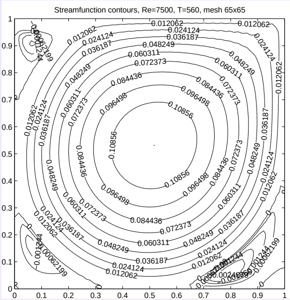
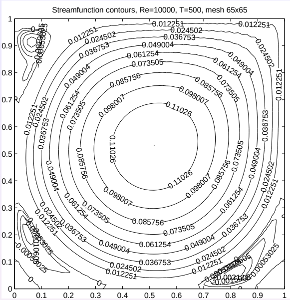


Figure: Driven Cavity for $\mathrm{Re} = 10000$: Max streamfunction. Computations are done with $N = 65$, with $\Delta t = 1/90$.

Isolines, $Re = 7500$, $Re = 10000$



(a) Re=7500



(b) Re=10000

Figure: Driven Cavity for $Re = 7500, 10000$: Streamfunction Contours with the fourth-order scheme

Velocity in the middle of the cavity, Re $= 7500$, Re $= 10000$

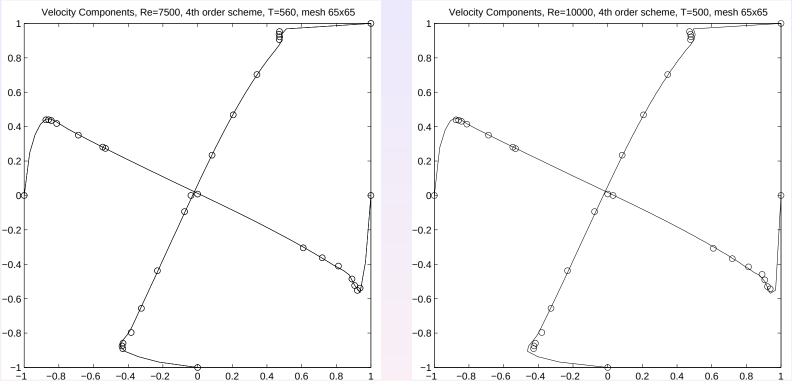


Figure: Velocity components for the driven cavity problem. Left: $Re = 7500$, fourth-order scheme with $N = 65$ (solid line), Ghia-Ghia-Shin. with $N = 257$ (circles). Right: $Re = 10000$ fourth-order scheme with $N = 65$ (solid Line), Ghia-Ghia-Shin with $N = 257$ (circles).

Computing efficiency for NS (driven cavity)

$N = 65$, Re $= 1000$ 8000 it., $\Delta t = 1/60$	$N = 129$, Re $= 1000$ 12000 it., $\Delta t = 1/60$	$N = 256$, Re $= 5000$ 50000 it., $\Delta t = 1/180$
4 min (0.03 sec/it.)	23min30sec. (0.11sec/it.)	7h 50min.(0.56sec/it.)

Table: *Indicative CPU time for the driven cavity on a Laptop*

Implementation so far

Outline

- Fourth order scheme with fast solver in $O(N^2 \ln_2(N))$. Fortran90 code.
- Driven cavity computations up to $\mathrm{Re} = 10000$, beyond the first Hopf bifurcation.
- Numerical analysis
- Derivation and first implementation of the 3D NS equations in streamfunction formulation in a cube
- Design and tests of a cartesian embedded biharmonic scheme for irregular geometries
- Application to other models involving biharmonic equations (e.g. image processing).

In progress

Outline

- Spectral analysis of fourth order problems. Application to the Stokes modes in a square/cube.
- Still enhance the fast solver (also in 3D)
- Other applications to fourth order problems solving: HJ (Hamilton-Jacobi), KS (Kuramoto-Sivashinsky), MEMS (Micro-Electro-Mechanical Systems).
- Driven cavity in a cube.
- Irregular geometries on cartesian grids using embedded/immersed boundaries seem tractable.