A nonlinear filtering technique for fluid-structure interaction problems

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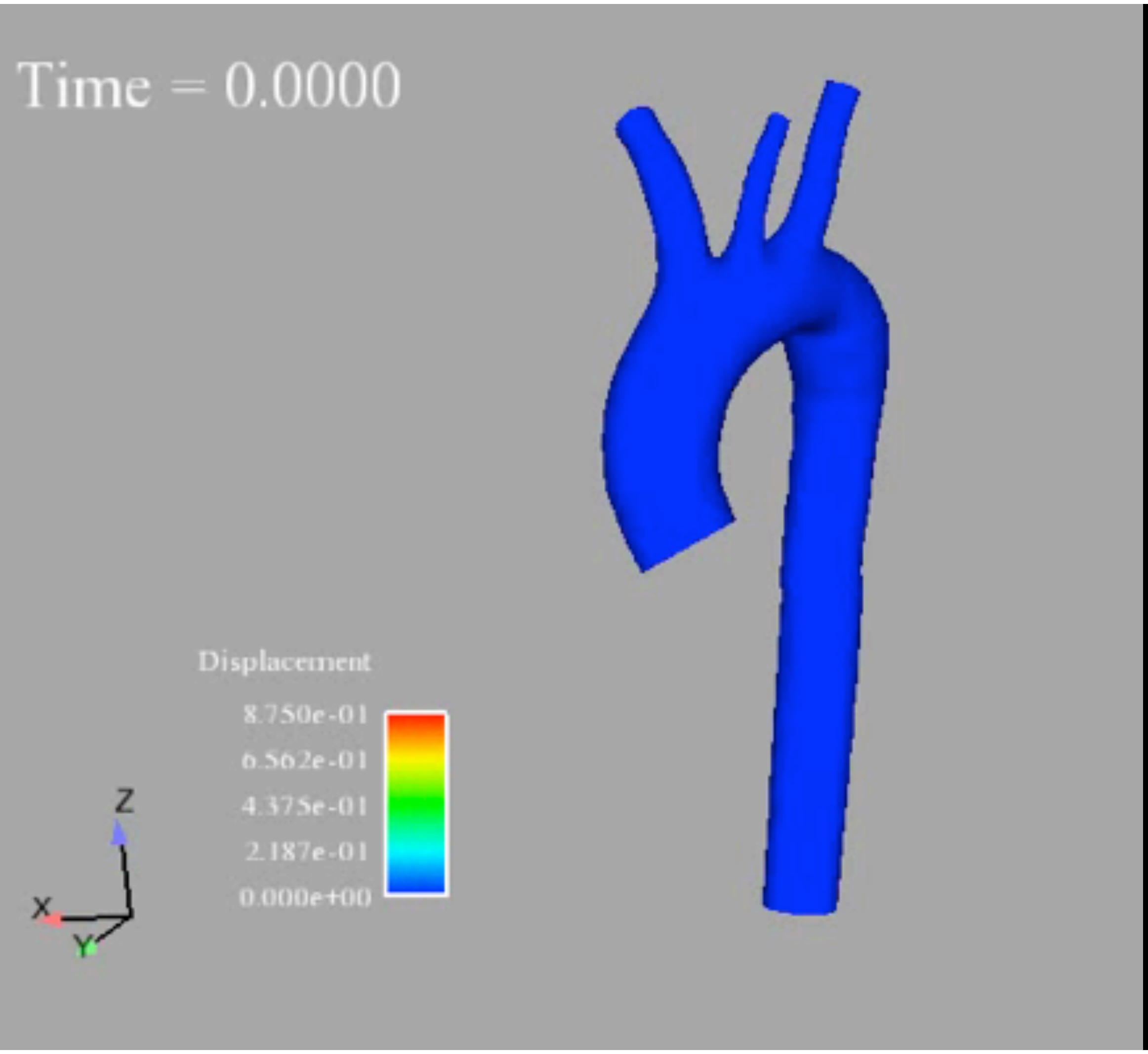
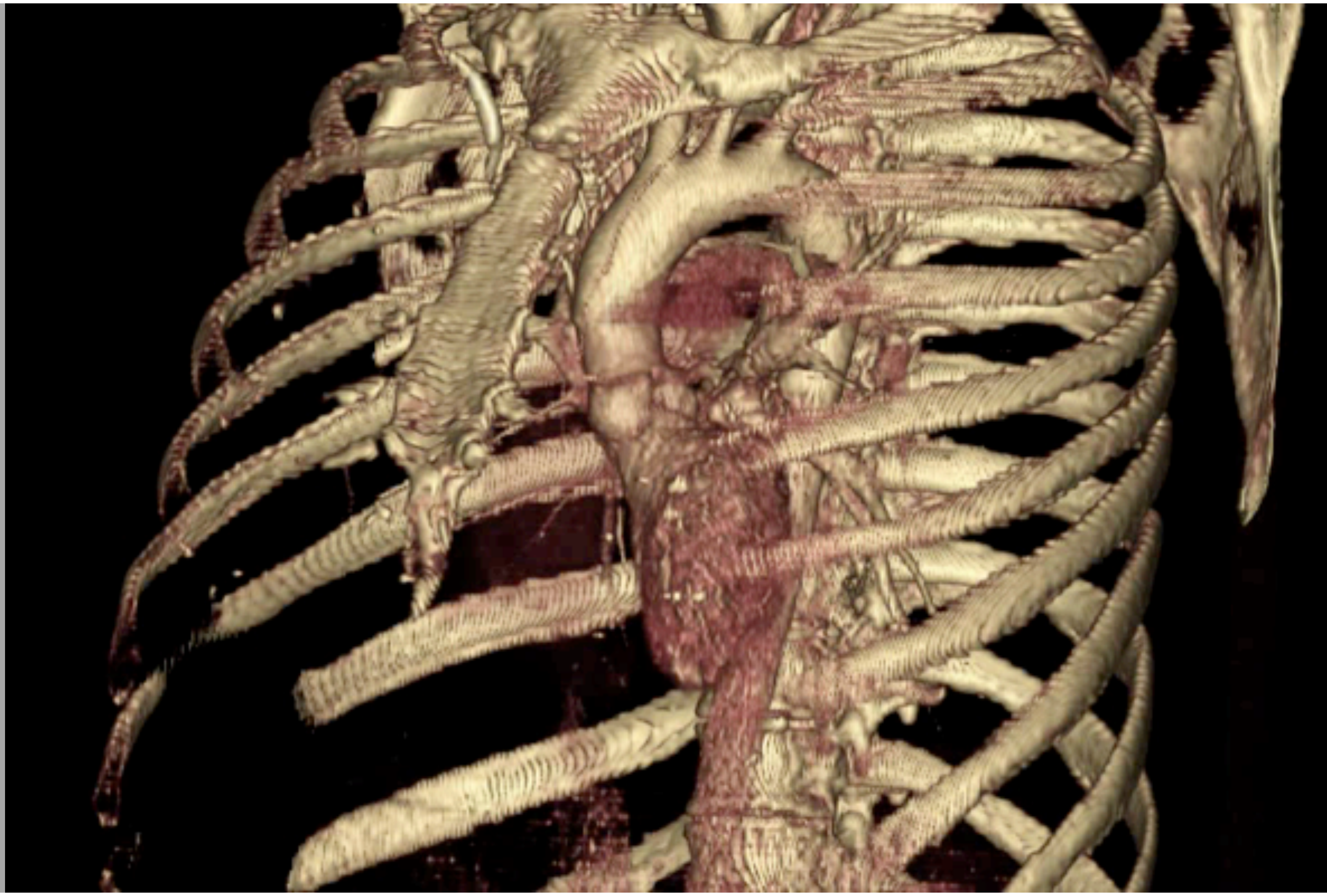
Medical Data Assimilation





INRIA *CVBRL, Stanford*

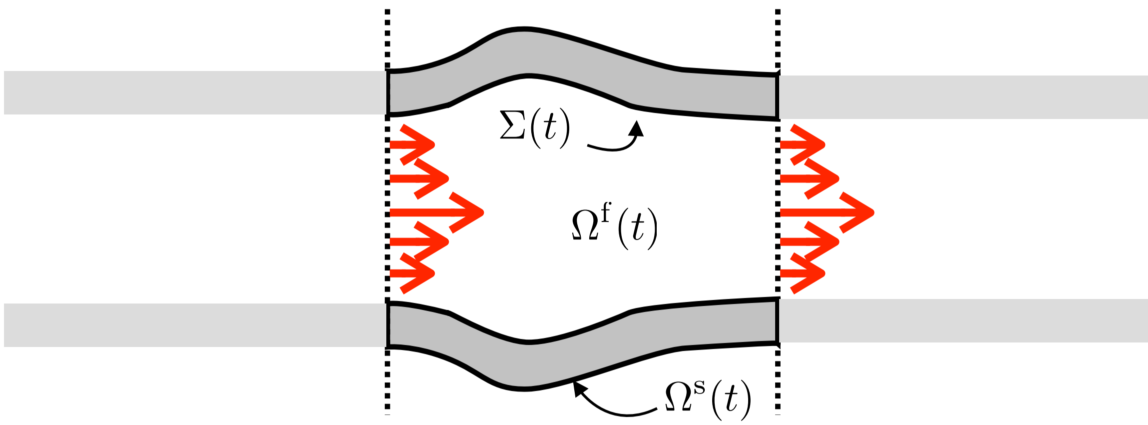
Data assimilation

- Reduce model uncertainties using observations
- Access to "hidden" quantities
- Smooth the data

Outline

- Fluid-Solid Interaction
  - Equations & algorithms
- Data assimilation in a nutshell
  - Kalman filters (linear and nonlinear cases)
  - Luenberger filter
- Some preliminary results

Fluid-Structure Interaction (FSI)



- **Fluid equations:** Navier-Stokes (ALE)

$$\rho^{\mathrm{f}}\left(\frac{\partial \boldsymbol{u}}{\partial t}_{|\widehat{\boldsymbol{x}}} + (\boldsymbol{u}-\boldsymbol{w})\cdot\boldsymbol{\nabla}\boldsymbol{u}\right) - 2\mu\,\mathbf{div}\,\boldsymbol{\epsilon}(\boldsymbol{u}) + \boldsymbol{\nabla} p = \boldsymbol{0}, \quad \text{in} \quad \Omega^{\mathrm{f}}(t)$$

$$\operatorname{div}\boldsymbol{u} = 0, \quad \text{in} \quad \Omega^{\mathrm{f}}(t)$$

- **Solid equations:** nonlinear elasticity

$$\rho^{\mathrm{s}}\frac{\partial^2 \boldsymbol{d}}{\partial t^2} - \mathbf{div}\big(\boldsymbol{F}(\boldsymbol{d})\boldsymbol{S}(\boldsymbol{d})\big) = \boldsymbol{0}, \qquad \text{in} \quad \widehat{\Omega}^{\mathrm{s}}$$

- **Coupling conditions:**

$$\boldsymbol{d}^{\mathrm{f}} = \mathrm{Ext}(\boldsymbol{d}_{|\widehat{\Sigma}}), \quad \boldsymbol{w} = \frac{\partial \boldsymbol{d}^{\mathrm{f}}}{\partial t} \quad \text{in} \quad \widehat{\Omega}^{\mathrm{f}}, \quad \Omega^{\mathrm{f}}(t) = (I + \boldsymbol{d}^{\mathrm{f}})(\widehat{\Omega}^{\mathrm{f}}), \quad \text{(geometry)}$$

$$\boldsymbol{u} = \frac{\partial \boldsymbol{d}}{\partial t}, \quad \text{on} \quad \Sigma(t), \quad \text{(velocity)}$$

$$\boldsymbol{F}(\boldsymbol{d})\boldsymbol{S}(\boldsymbol{d})\widehat{\boldsymbol{n}} = J(\boldsymbol{d}^{\mathrm{f}})\boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{F}(\boldsymbol{d}^{\mathrm{f}})^{-\mathrm{T}}\widehat{\boldsymbol{n}}, \quad \text{on} \quad \widehat{\Sigma}, \quad \text{(stress)}$$

Fluid-Structure Interaction (FSI)

Semi-implicit coupling schemes:

- Step 1: advection / diffusion / ALE

$$\rho_f \frac{\tilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^n}{\delta t} + \rho_f(\tilde{\boldsymbol{u}}^n - \boldsymbol{w}) \cdot \boldsymbol{\nabla} \tilde{\boldsymbol{u}}^{n+1} - \mu \triangle \tilde{\boldsymbol{u}}^{n+1} = 0$$

Explicit coupling with the structure for efficiency

- Step 2 : projection

$$\begin{cases} \rho_f \dfrac{\boldsymbol{u}^{n+1} - \tilde{\boldsymbol{u}}^{n+1}}{\delta t} + \boldsymbol{\nabla} p^{n+1} & = & 0 \\ \operatorname{div} \boldsymbol{u}^{n+1} & = & 0 \end{cases}$$

Implicit coupling with the structure for stability

Causin-JFG-Nobile, 05
Fernández-JFG-Grandmont, 07

Benchmark: pressure wave in a straight tube

COUPLING	ALGORITHM	CPU time	
Implicit	FP-Aitken	24.86	← 2001
	quasi-Newton	6.05	← 2003
	Newton	4.77	← 2003
Semi-Implicit	Newton	1	← 2007

Other non fully implicit schemes based on different ideas :

Guidoboni-Glowinski-Cavallini-Canic 09, Burman-Fernández 09

Data assimilation in a nutshell

- FSI dynamical system: $\left\{ \begin{array}{rcl} B\dot{X} & = & A(X, \theta) + R \\ X(0) & = & X_0 \end{array} \right.$

- Time discretization: $X^{n+1} = F^{n+1}(X^n, \theta)$

- **State variable**: X=$[\boldsymbol{u}, p, \boldsymbol{d}^f, \boldsymbol{d}, \boldsymbol{v}]$

- **Parameters**: θ = [Young modulus, viscosity, boundary conditions, $\ldots$]

- **Uncertainties** on the initial condition X_0 and the parameters θ

- Partial **observations** of X: $Z = H(X)$

Data assimilation in a nutshell

- Uncertainties: $\zeta = [\zeta_X, \zeta_\theta]$

$$\left\{ \begin{array}{rcl} X_0 & = & \hat{X}_0 + \zeta_X \\ \theta & = & \hat{\theta} + \zeta_\theta \end{array} \right.$$

- Minimize

$$J(\zeta) = \frac{1}{2}\int_0^T \|Z - H(\hat{X})\|_W^2 \, dt + \frac{1}{2}\|\zeta\|_P^2$$

where $\hat{X} = \hat{X}(\zeta)$ is solution to the problem.

- **Variational approach:**
  - Optimization algorithms
  - Usually based on gradient (adjoint equations)
- **Filtering approach:**
  - Sequential correction of the state and the parameters
  - Large full matrices (Kalman)

Data assimilation in a nutshell

Static linear case: least square approach

- Assume there is no dynamics, but there is a guess $\hat{X}_-$
- The error on the guess $e_- = X - \hat{X}_-$ has a covariance P_-
- The observation error $e_H = Z - HX$ has a covariance W
- We look for $\hat{X}_+$ that accounts for $\hat{X}_-$ and an observation $Z = HX$
- A natural idea is to minimize:

$$J(\hat{X}) = \frac{1}{2}(\hat{X} - \hat{X}_-)^T P_-^{-1}(\hat{X} - \hat{X}_-) + \frac{1}{2}(Z - H\hat{X})^T W^{-1}(Z - H\hat{X})$$

- Solution:

$$\hat{X}_+ = \hat{X}_- + K(Z - H\hat{X}_-)$$

Gain (Kalman matrix)

$$K = P_- H^T (W + H P_- H^T)^{-1}$$

Innovation

Data assimilation in a nutshell

Dynamical linear case: Kalman filter

- Linear dynamical system with state uncertainty ζ_X:

$$\begin{cases} \dot{X} &= FX \\ X(0) &= X_0 + \zeta_X \end{cases}$$

- Time discretization:

$$X^{n+1} = F^{n+1} X^n$$

- Assume $\hat{X}_+^n$ is known with a covariance P_+^n
  - Prediction: $$\hat{X}_-^{n+1} = F^{n+1} \hat{X}_+^n$$
    $$P_-^{n+1} = F^{n+1} P_+^n F^{n+1}$$
  - Correction: $$\hat{X}_+^{n+1} = \hat{X}_-^{n+1} + K^{n+1}(Z - H\hat{X}_-^{n+1})$$
    $$P_+^{n+1} = (I - KH)P_-^{n+1}$$

Data assimilation in a nutshell

Extension to nonlinear problems

- Nonlinear dynamical system: $X^{n+1} = F^{n+1}(X^n)$
- First natural idea: Extended Kalman Filter (EKF)
  - Nonlinear prediction: $\hat{X}_-^{n+1} = F^{n+1}(\hat{X}_+^n)$
  - Gain & propagation with tangent op.: $P_-^{n+1} = \nabla F^{n+1} \, P_+^n \, \nabla F^{n+1}$
- Two drawbacks of EKF:
  - Need to compute ∇F^n
  - Nonlinear prediction step may be unaccurate:

Let $\hat{X} = \mathbb{E}(X)$ and $P_X = \mathrm{Cov}(X - \hat{X}, X - \hat{X})$

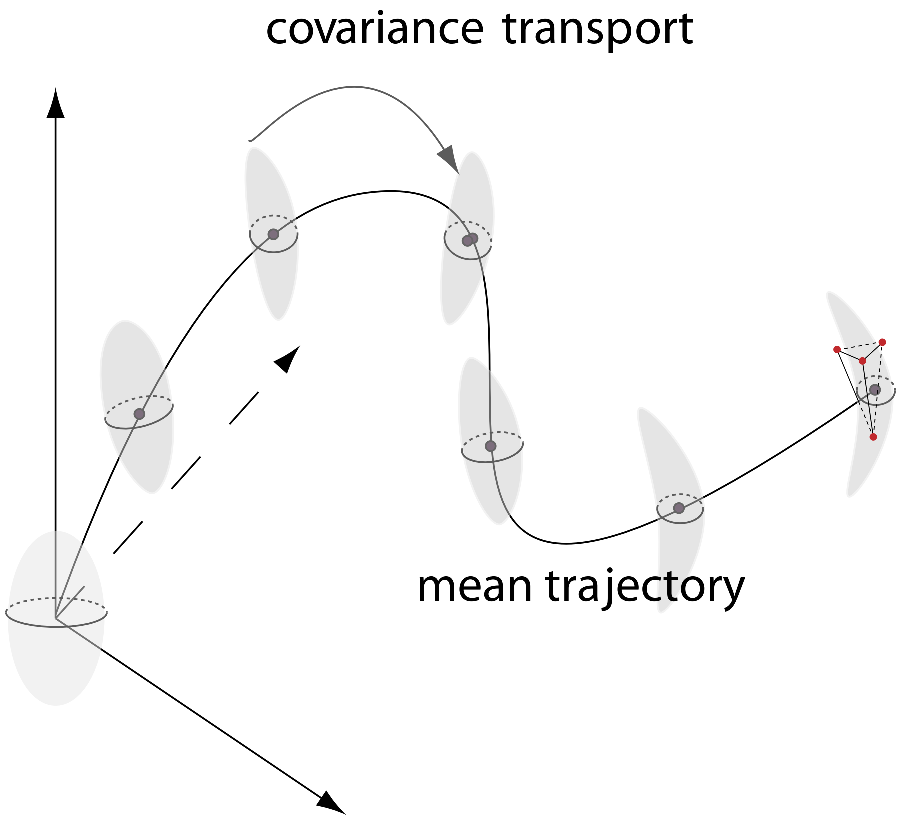
$$F(X) = F(\hat{X}) + \nabla F(X - \hat{X}) + \frac{1}{2}(X - \hat{X})^T \nabla^2 F(X - \hat{X}) + \dots$$

Hence: $\mathbb{E}(F(X)) = F(\hat{X}) + \frac{1}{2} \nabla^2 F : P_X + \dots$

Data assimilation in a nutshell

Extension to nonlinear problems

- **Unscented Kalman Filter (UKF)**
  (Julier-Uhlmann-97)
  - Nonlinear propagation of a few "particles"
  - Empirical mean and covariance :
    no tangent operator !



- Rationale in 1D:
  - Let $\hat{X} = \mathbb{E}(X)$ and $\sigma^2 = \mathrm{Var}(X - \hat{X})$
  - Let be 3 "particles": $\hat{X}_1 = \hat{X}, \quad \hat{X}_2 = \hat{X} + \frac{\sigma}{\sqrt{2}}, \quad \hat{X}_3 = \hat{X} - \frac{\sigma}{\sqrt{2}}$
  - By construction: $\sum_{i=1}^{3} \frac{1}{3} F(\hat{X}_i) = F(\hat{X}) + \frac{\sigma^2}{2} F''(\hat{X}) + \ldots$

  (which is an approximation of $\mathbb{E}(F(X))$ **better** than $F(\hat{X})$)

- **In N dimensions** : needs *2N+1* particles and a **Cholesky** factorisation

Data assimilation in a nutshell

Extension to parameters estimation

- Introduce an pseudo-dynamics for θ:

$$\left\{ \begin{array}{rcl} \dot{X} & = & F(X,\theta) \\ \dot{\theta} & = & 0 \end{array} \right. \quad \text{with} \quad \left\{ \begin{array}{rcl} X(0) & = & X_0 + \zeta_X \\ \theta(0) & = & \theta_0 + \zeta_\theta \end{array} \right.$$

- Let $dim(X) = N$, $dim(\theta) = p$, and $dim(Z) = m$
- **Major concern:** K is $(N+p) \times m$ and full !

Untractable for large systems (PDE) !

Strategy : reduced filtering

- Kalman filtering (UKF) is only used for the **parameters** θ $(p \ll N)$
- A much cheaper filter (Luenberger) is used for the **state** X

Automatic control : *Zhang-02*
Oceanography: *Pham-Verron-Roubeaud-97*
Elasticity: *Moireau-Chapelle-09*

Data assimilation in a nutshell

Luenberger filters

- Observer *(Luenberger, 1971)*:

$$B\frac{d\hat{X}}{dt} = A\hat{X} + R + K(Z - H\hat{X})$$

- Linear stability of the error dynamics $e_X = X - \hat{X}$

$$B\frac{\mathrm{d}e_X}{\mathrm{d}t} = (A - KH)e_X$$

- Eigenmodes (λ_k, Φ_k):

$$(A - KH)\Phi_k = \lambda_k B\Phi_k$$

- Devise K to reduce $Re(\lambda_k) < 0$

Data assimilation in solid mechanics

Luenberger filters

- Elastodynamics equations $X = [\boldsymbol{d}, \boldsymbol{v}]$
- Velocity filtering: *Direct Velocity Feedback* (**DVF**)

(Moireau-Chapelle-Le Tallec-08)

$$\begin{cases} M_s \dfrac{\mathrm{d}\hat{\boldsymbol{v}}}{\mathrm{d}t} + K_s \hat{\boldsymbol{d}} &= R + \gamma_v H^T M_H (Z - H\hat{\boldsymbol{v}}) \\ \dfrac{\mathrm{d}\hat{\boldsymbol{d}}}{\mathrm{d}t} &= \hat{\boldsymbol{v}} \end{cases}$$

- Equation of the error: $e_{\boldsymbol{v}} = \boldsymbol{v} - \hat{\boldsymbol{v}}$, $e_{\boldsymbol{d}} = \boldsymbol{d} - \hat{\boldsymbol{d}}$

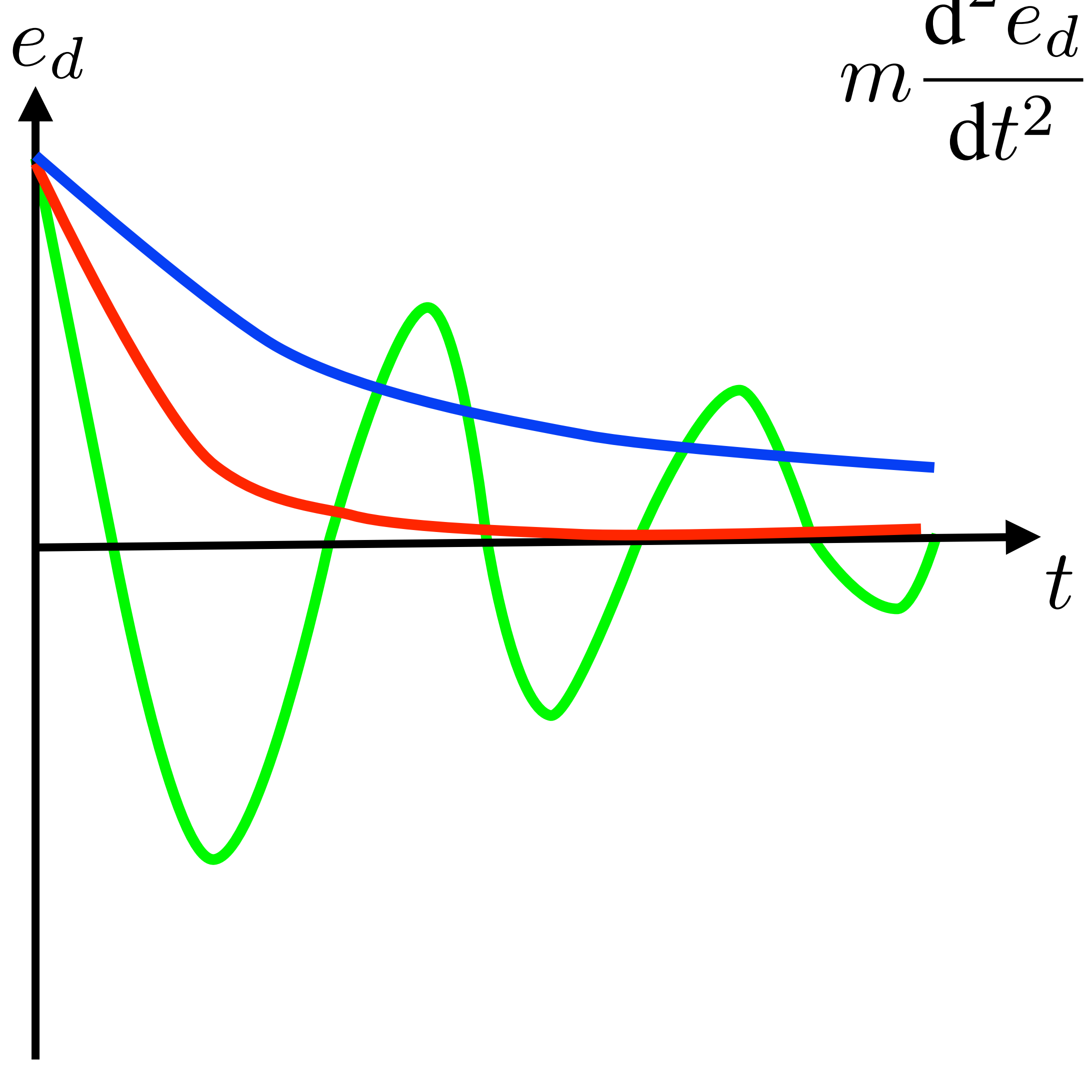
$$M_s \frac{\mathrm{d}e_{\boldsymbol{v}}}{\mathrm{d}t} + K_s e_{\boldsymbol{d}} = -\gamma_v H^T M_H H e_{\boldsymbol{v}}$$

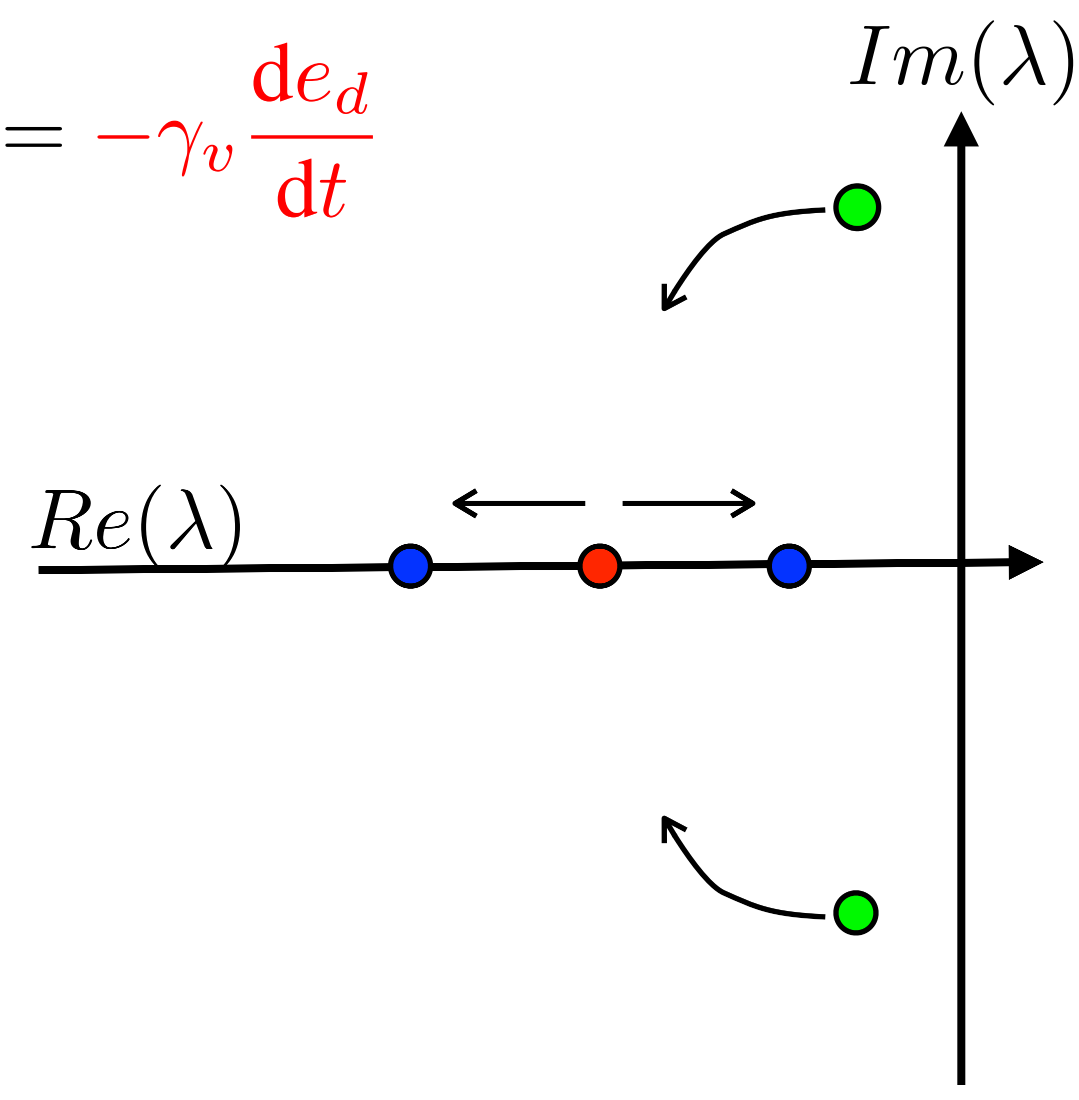
- Energy equation of the error: $e_{\boldsymbol{v}} = \boldsymbol{v} - \hat{\boldsymbol{v}}$, $e_{\boldsymbol{d}} = \boldsymbol{d} - \hat{\boldsymbol{d}}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(M_s e_{\boldsymbol{v}}, e_{\boldsymbol{v}}) + (K_s e_{\boldsymbol{d}}, e_{\boldsymbol{d}})\right] = -\gamma_v (M_H H e_{\boldsymbol{v}}, H e_{\boldsymbol{v}})$$

- A trivial example: oscillator in 1D

$$m\frac{\mathrm{d}^2 e_d}{\mathrm{d}t^2} + k e_d = -\gamma_v \frac{\mathrm{d}e_d}{\mathrm{d}t}$$





- Let $\omega_0 = \sqrt{\frac{k}{m}}$ and $\beta = \frac{\gamma_v}{2m}$

  - If $\beta < \omega_0$: underdamped
  - If $\beta = \omega_0$: critically damped
  - If $\beta > \omega_0$: overdamped

Data assimilation in a nutshell

Luenberger filters

- In practice, it is more convenient to work with displacement
- Displacement filtering: *Schur Displacement Feedback* **(SDF)**
  (Moireau-Chapelle-Le Tallec, 2009)

$$\left\{\begin{array}{rcl} M_s \dfrac{\mathrm{d}\hat{\boldsymbol{v}}}{\mathrm{d}t} + K_s \hat{\boldsymbol{d}} & = & R \\ K_\mu \dfrac{\mathrm{d}\hat{\boldsymbol{d}}}{\mathrm{d}t} & = & K_\mu \hat{\boldsymbol{v}} + \gamma H^T M_H (Z - H(\hat{\boldsymbol{d}})) \end{array}\right.$$

with $K_\mu = K_s + \mu H^T M_\Gamma H$.

- **Remarks:**
  - velocity is no longer the derivative of displacement
  - be careful in the Fluid-Structure algorithm !

Data assimilation in a nutshell

Joint state-parameter estimation

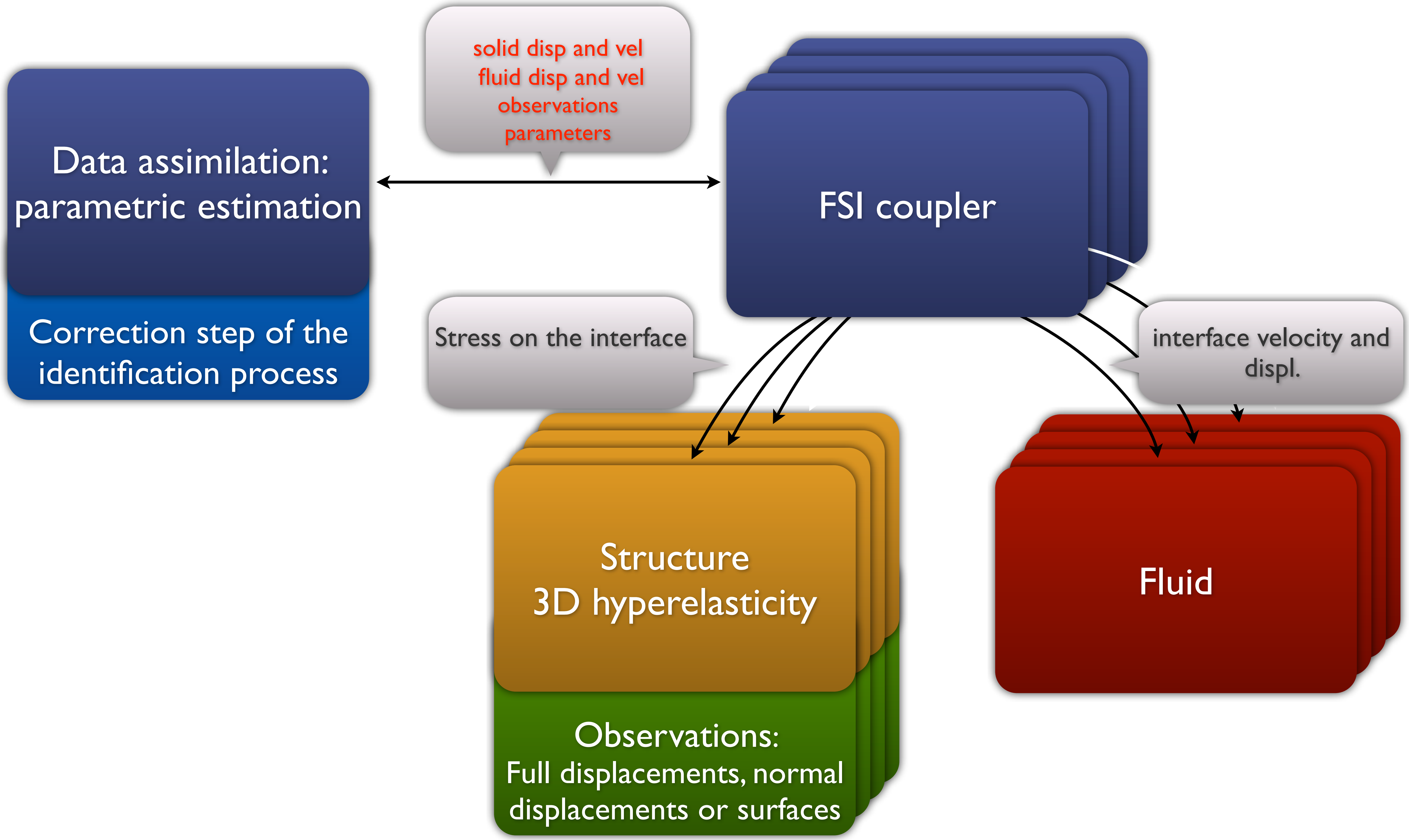
Luenberger filter
for the state

Summary

- Prediction: $$\begin{cases} \hat{X}_-^{n+1} = \hat{F}^{n+1}(\hat{X}_+^n, \theta_+^n, Z^{n+1}, K) \\ \hat{\theta}_-^{n+1} = \hat{\theta}_+^n \end{cases}$$
- Correction: $$\begin{cases} \hat{X}_+^{n+1} = \hat{X}_-^{n+1} + \hat{K}_X^{n+1}(Z^{n+1} - H(\hat{X}_-^{n+1})) \\ \hat{\theta}_+^{n+1} = \hat{\theta}_-^{n+1} + \hat{K}_\theta^{n+1}(Z^{n+1} - H(\hat{X}_-^{n+1})) \end{cases}$$

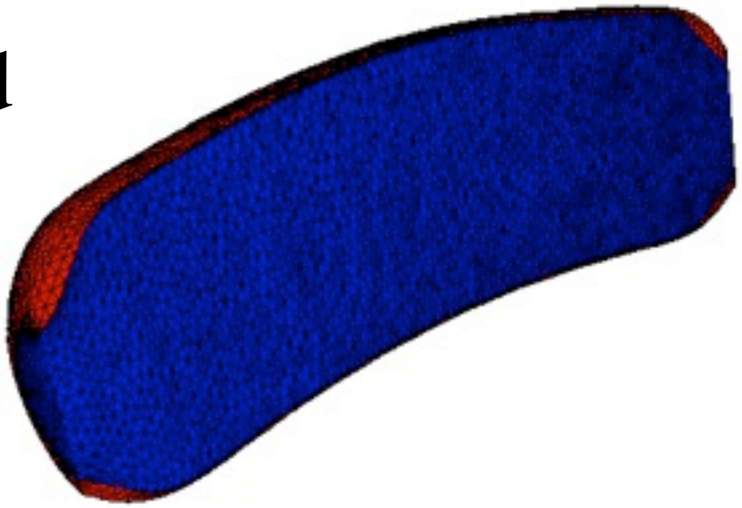
Kalman-like filter for
for the parameters

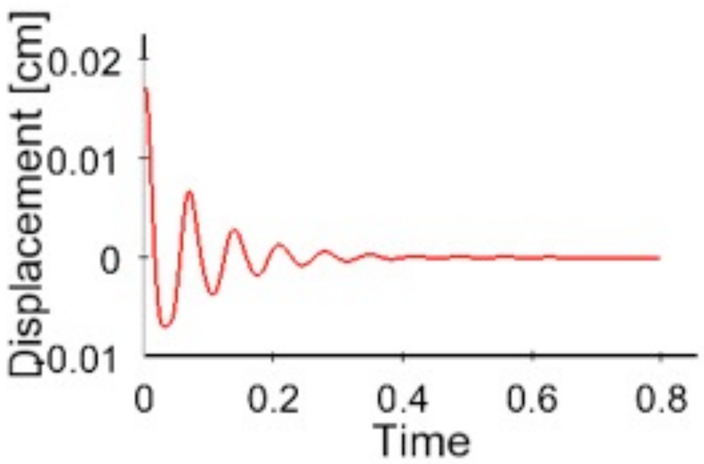
Implementation

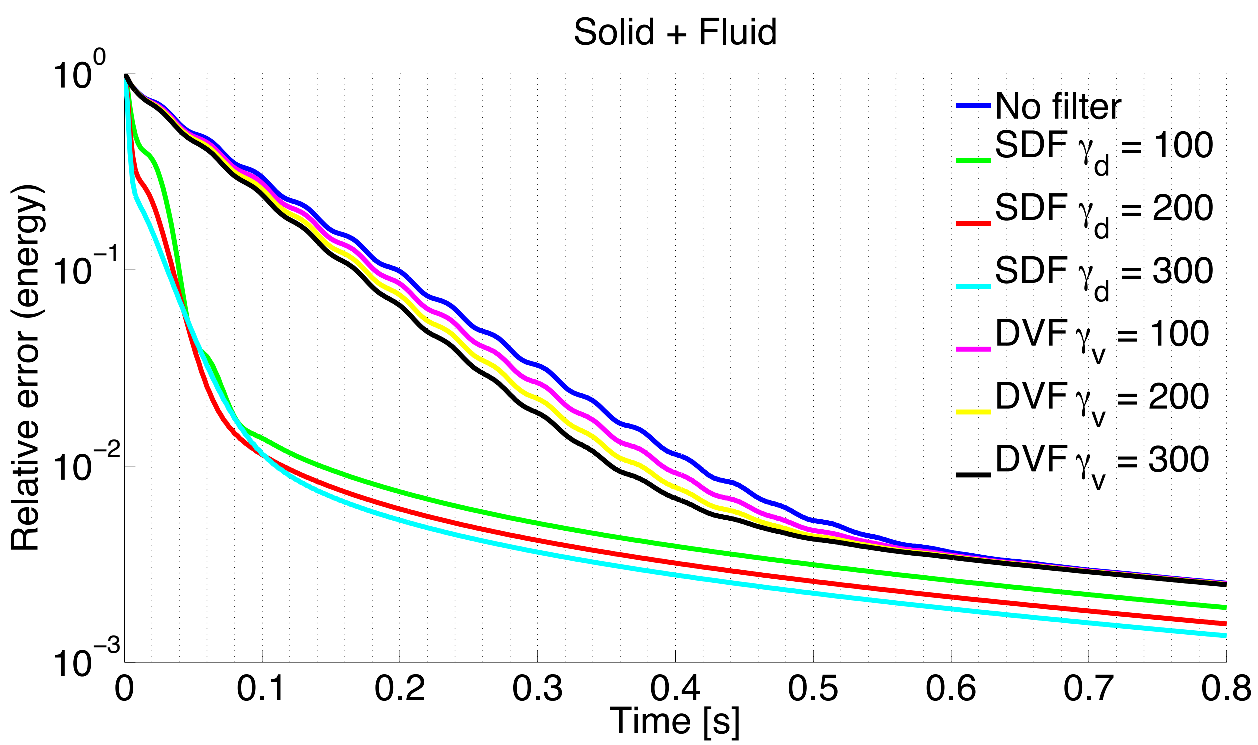


Example 1: Displacement *vs* Velocity in FSI

- Fluid at rest
- Initial perturbation in the solid
- Stabilization to equilibrium







Example 1: Displacement *vs* Velocity in FSI

Analysis of a simplified model

- Potential fluid:

$$\left\{\begin{aligned} \rho^{\mathrm{f}}\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} p &= 0, \text{ in } \Omega^{\mathrm{f}}\\ \operatorname{div} \boldsymbol{u} &= 0, \text{ in } \Omega^{\mathrm{f}}\\ \boldsymbol{u}\cdot\boldsymbol{n} &= \dot{\boldsymbol{d}}, \text{ on } \Sigma \end{aligned}\right. \quad \underset{\text{div}}{\Longrightarrow} \left\{\begin{aligned} &-\Delta p = 0, \text{ in } \Omega^{\mathrm{f}}\\ &\frac{\partial p}{\partial \boldsymbol{n}} = -\rho^{\mathrm{f}}\frac{\partial \boldsymbol{u}}{\partial t}\cdot\boldsymbol{n} = -\rho^{\mathrm{f}}\ddot{\boldsymbol{d}}\cdot\boldsymbol{n}, \text{ on } \Sigma \end{aligned}\right.$$

- Let $\mathcal{M}_{\mathrm{A}}$ be the "Neumann-to-Dirichlet" operator: $p_{|\Sigma} = -\rho^{\mathrm{f}}\mathcal{M}_{\mathrm{A}}\ddot{\boldsymbol{d}}\cdot\boldsymbol{n}$

- Linear elasticity:

$$\left\{\begin{aligned} &\rho^{\mathrm{s}}\ddot{\boldsymbol{d}} - \operatorname{div}\sigma(\boldsymbol{d}) = 0, \text{ in } \Omega^{\mathrm{s}}\\ &\sigma(\boldsymbol{d})\cdot\boldsymbol{n} = p|_{\Sigma}\boldsymbol{n} = -\rho^{f}\mathcal{M}_{A}\ddot{\boldsymbol{d}}\cdot\boldsymbol{n}\,\boldsymbol{n}, \text{ on } \Sigma \end{aligned}\right.$$

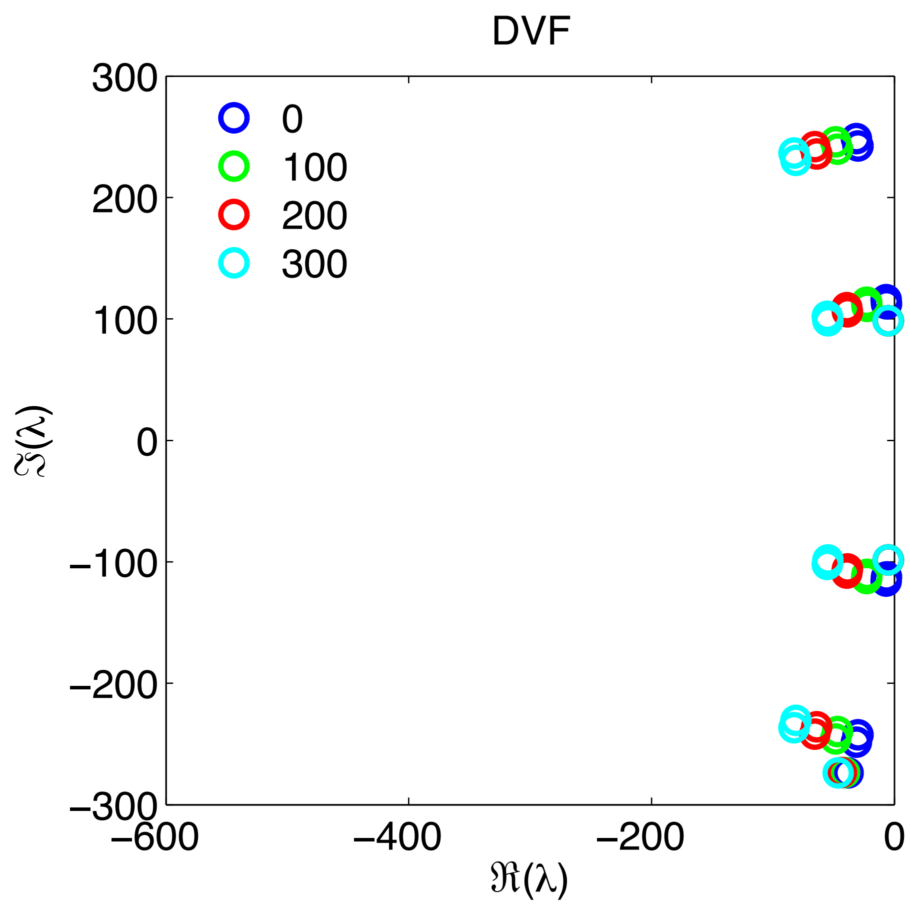
Example 1: Displacement *vs* Velocity in FSI

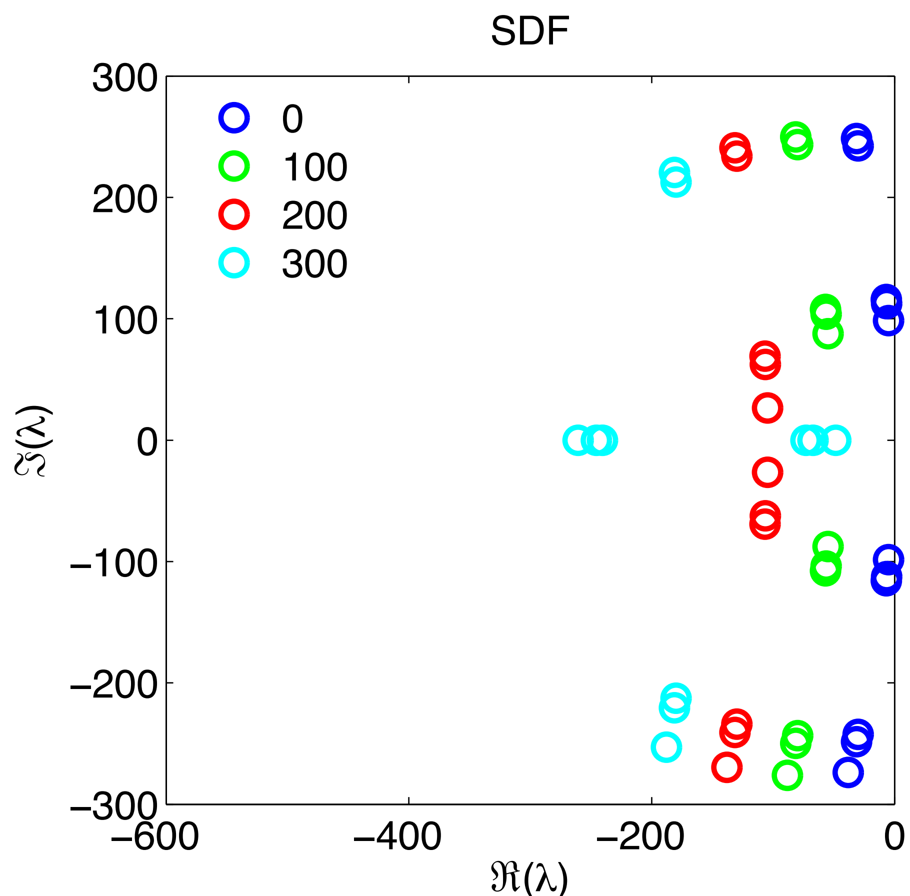
- Simplified FSI problem, with SDF or DVF

Added mass (FSI)

$$\left\{\begin{array}{rcl} (M_s + M_A)\dfrac{\mathrm{d}\hat{\boldsymbol{v}}}{\mathrm{d}t} + K_s\hat{\boldsymbol{d}} & = & R + \gamma_v H_v^T M_\Gamma (Z_v - H_v(\hat{\boldsymbol{v}})) \\ K_\mu \dfrac{\mathrm{d}\hat{\boldsymbol{d}}}{\mathrm{d}t} & = & K_\mu \hat{\boldsymbol{v}} + \gamma_d H_d^T M_\Gamma (Z_d - H_d(\hat{\boldsymbol{d}})) \end{array}\right.$$

- Evolution of λ for increasing γ:





N.B.: viscoelastic structure

Example 1: Displacement *vs* Velocity in FSI

Sensitivity

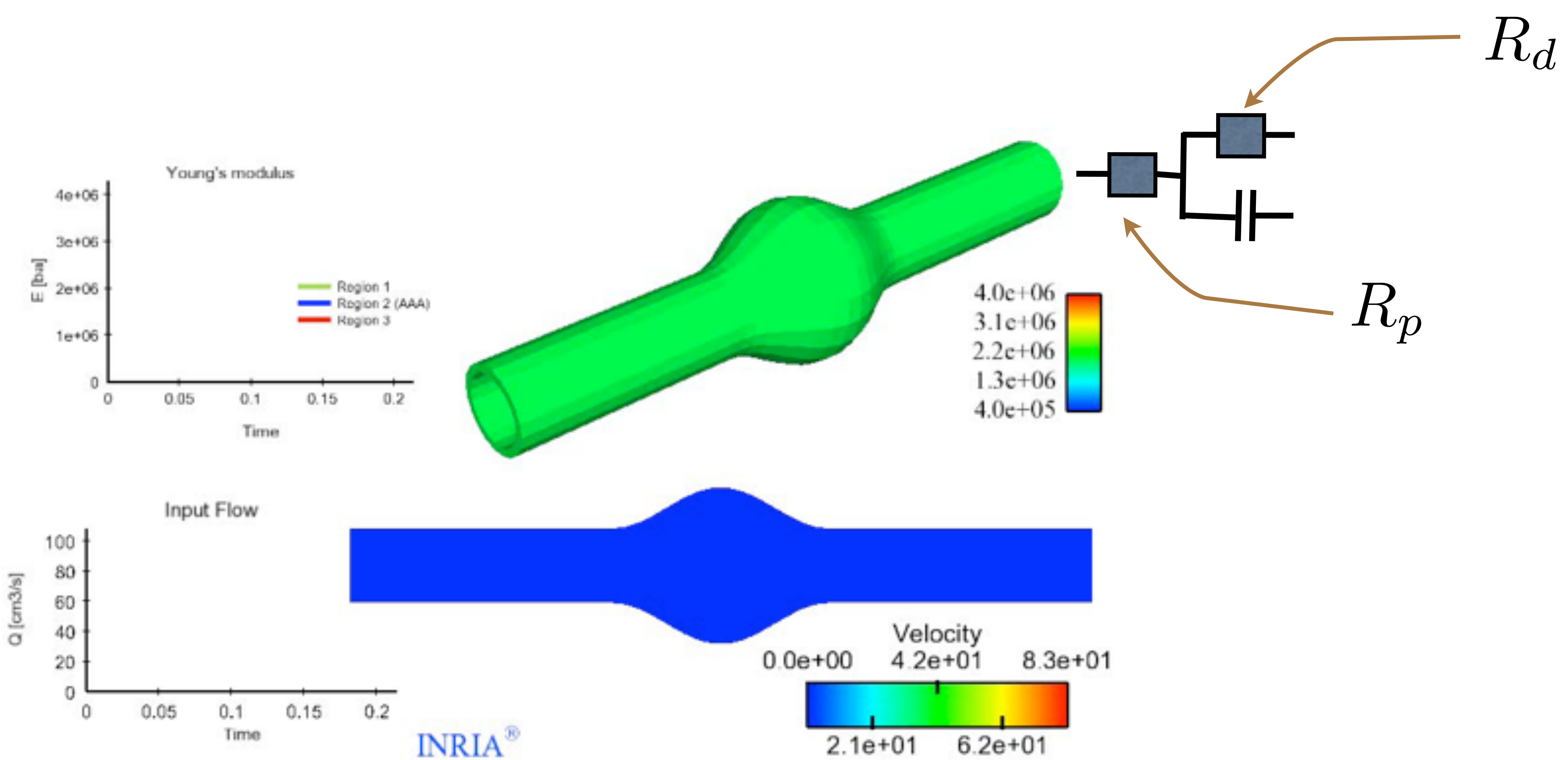
- Let $(\lambda(\gamma), \Phi(\gamma))$ an eigenmode. Assuming full observation:
  - Velocity filter: $\left.\dfrac{\partial \lambda}{\partial \gamma_v}\right|_{\gamma_v=0} = -\dfrac{1 - \Phi^T M_A \Phi}{2}$
  - Displacement filter: $\left.\dfrac{\partial \lambda}{\partial \gamma_d}\right|_{\gamma_d=0} = -\dfrac{1}{2}$

Remark: In our experiment $\Phi^T M_A \Phi$ is close to 1

Example 2: Compliance estimation

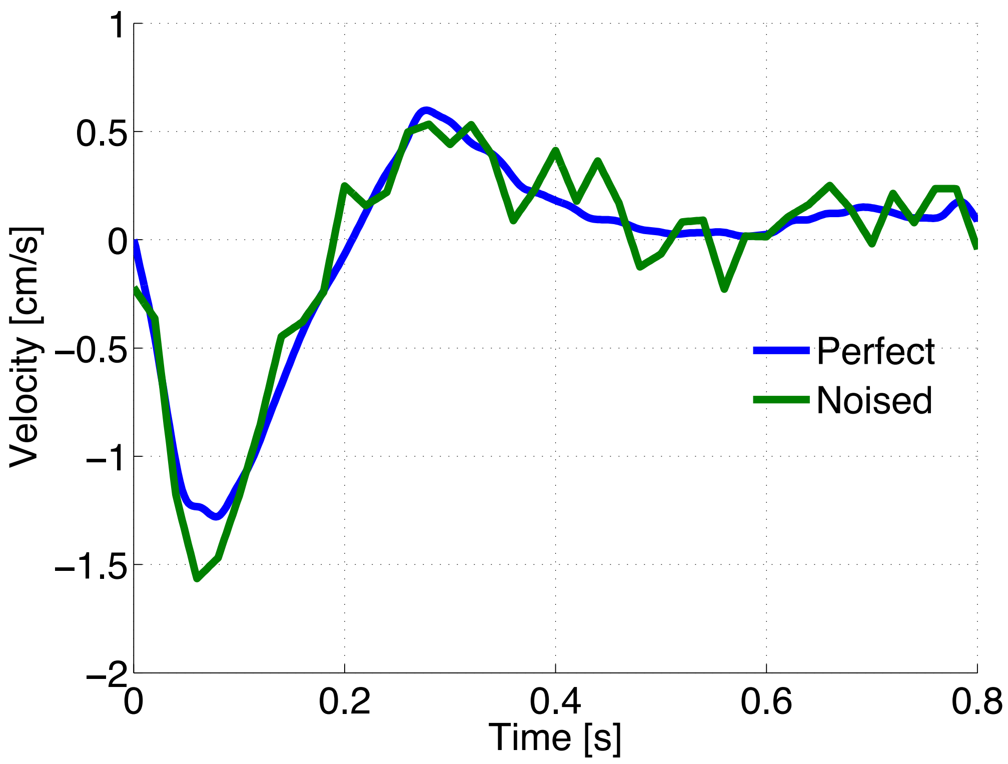
Parameter estimation

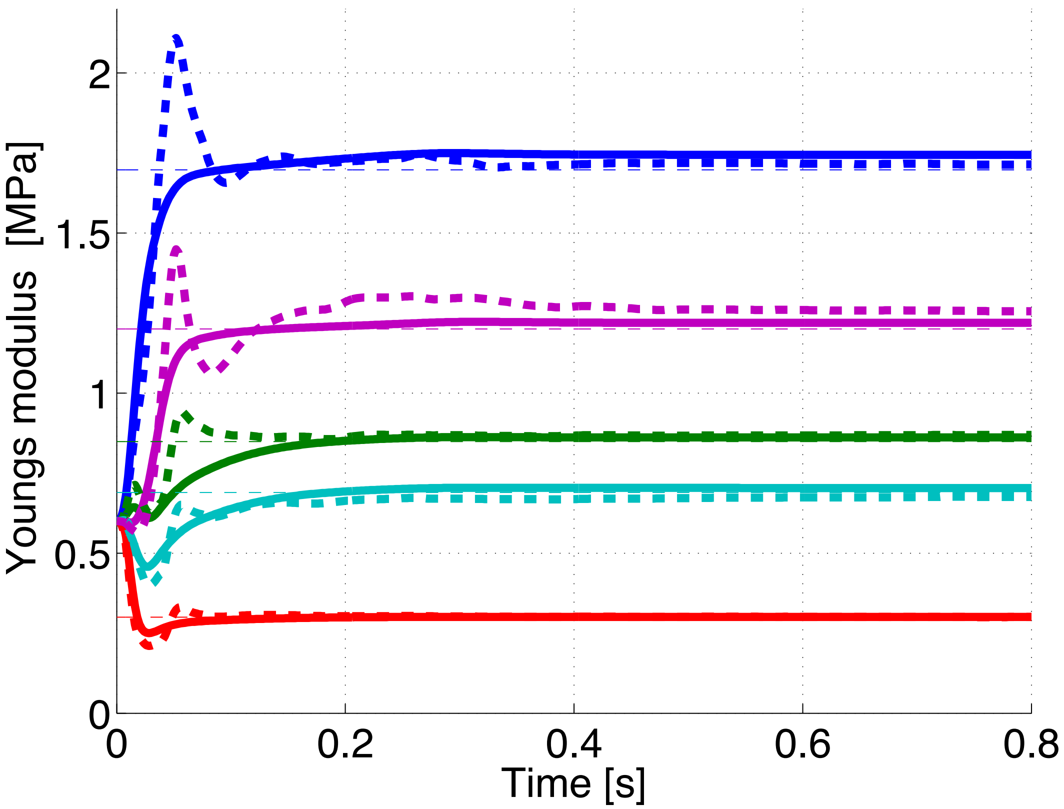
- Parameter estimation: Young modulus E in 3 regions
- Synthetic data with $E_1 = 0.5, E_2 = 2, E_3 = 4MPa$
- Initial guess: $E = 2MPa$ in the three regions
- Observations: wall velocity



Simulation : C.Bertoglio

- Similar experiment with 5 regions
- With noise (10%) and resampling:

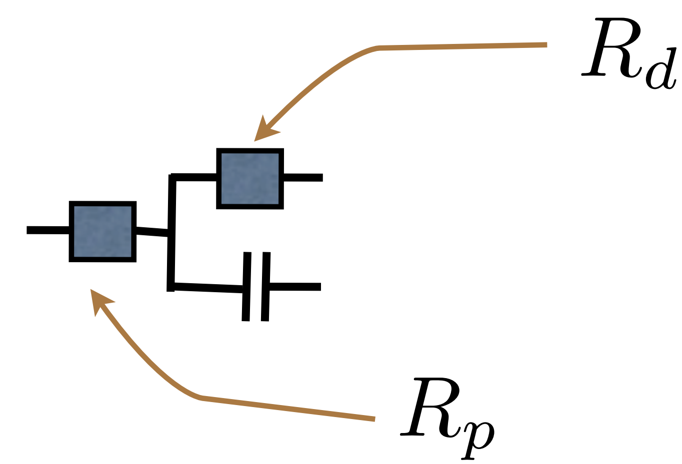


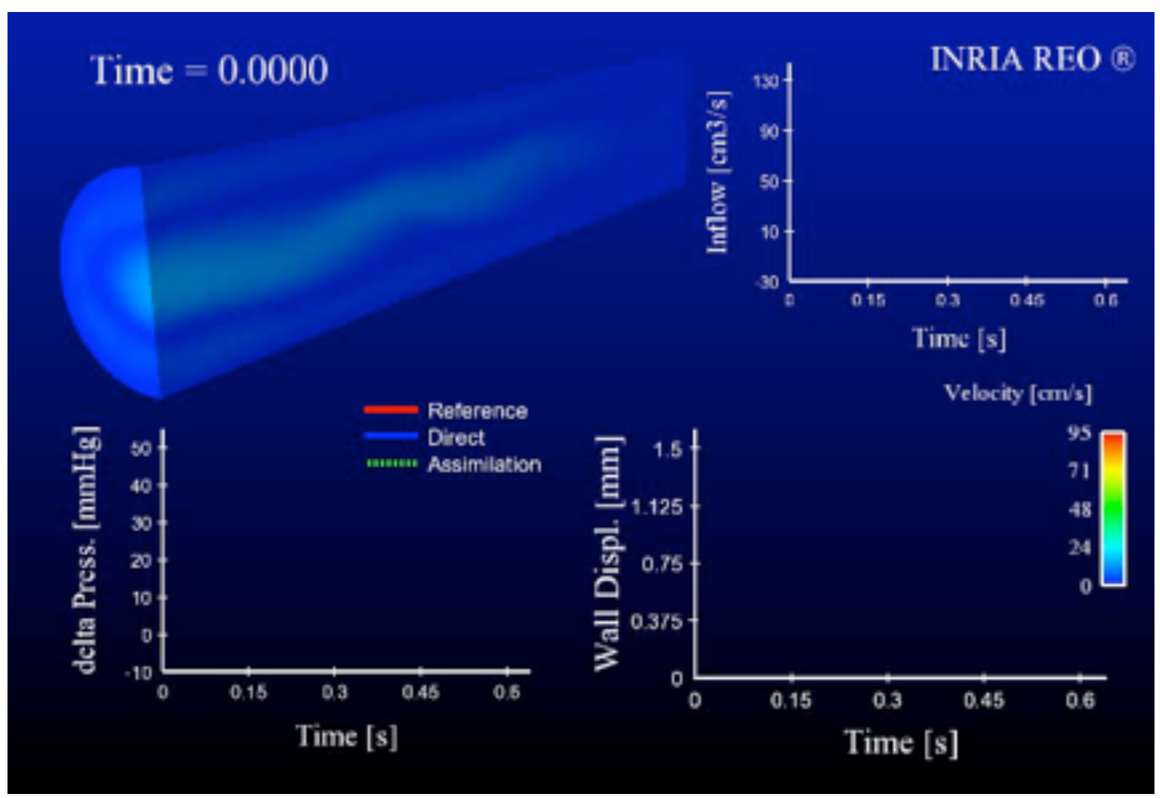


Example 2: Compliance estimation

State estimation

- Computer model : $E = 3\,\mathrm{MPa},\ R_p = 800,\ R_d = 1.2\,10^4$
- Patient with "hypertension" : $E = 5\,\mathrm{MPa},\ R_p = 900,\ R_d = 1.5\,10^4$
- 1st attempt : Observation = wall velocity
- State estimation only





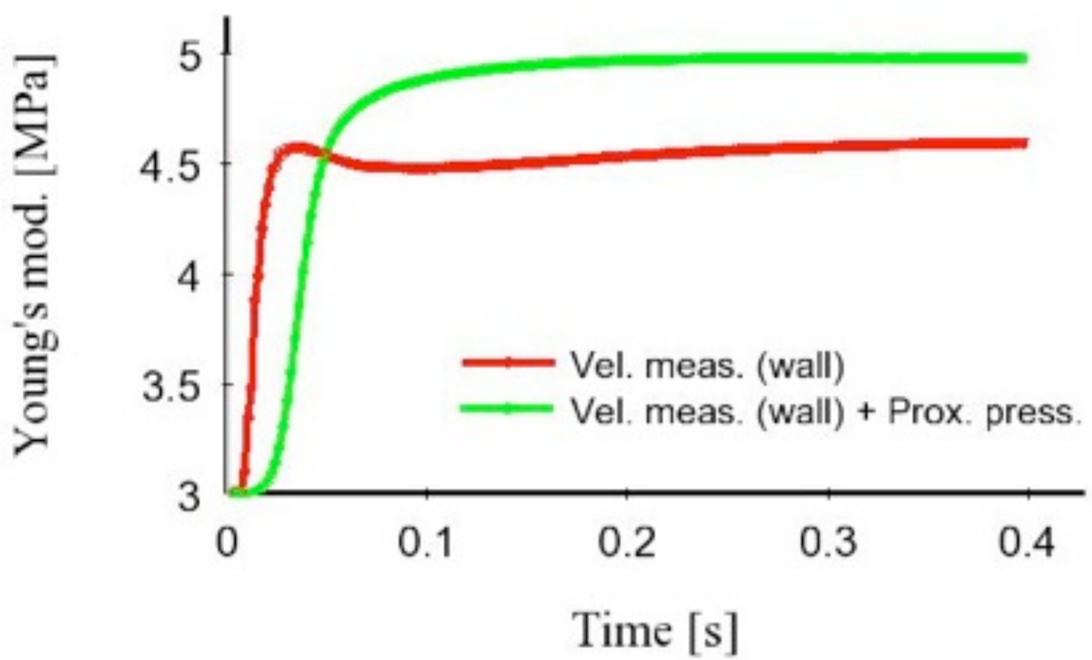
- **Reference** : "real" patient
- **Direct** : computer model
- **Assimilation** : state estimation

Simulation : C.Bertoglio

Example 2: Compliance estimation

Parameter & State estimation

- State **and parameter** estimation (Young modulus E)
- Patient with "hypertension": $E = 5\,\mathrm{MPa},\;\; R_p = 900,\;\; R_d = 1.5\,10^4$
- Observation : wall velocity
  - ★ Young modulus underestimated $E \approx 4.5$ instead of $5\,\mathrm{MPa}$
- Observation : wall velocity **and outlet blood pressure**
  - ★ Young modulus correctly estimated



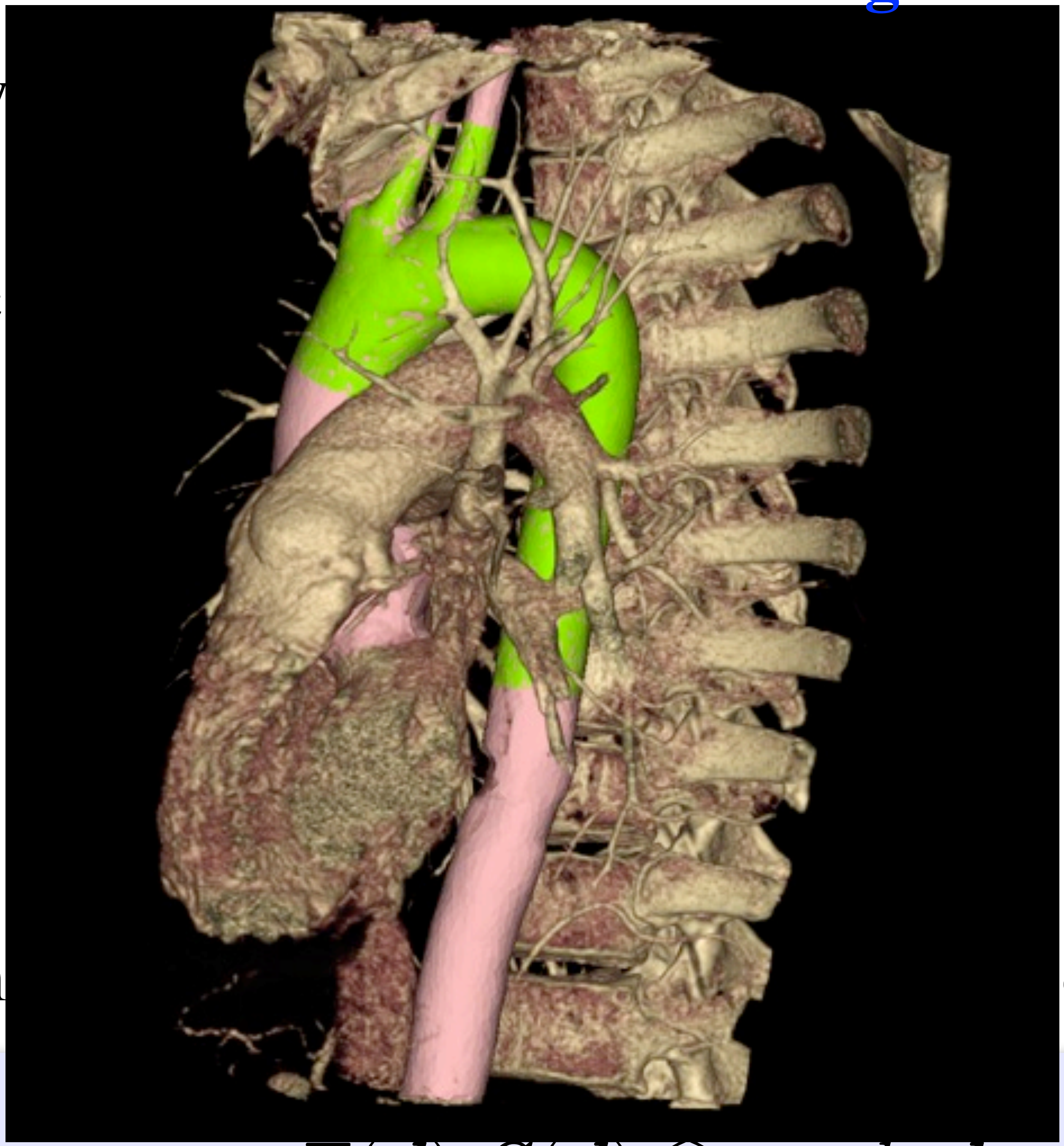
Simulation : C.Bertoglio

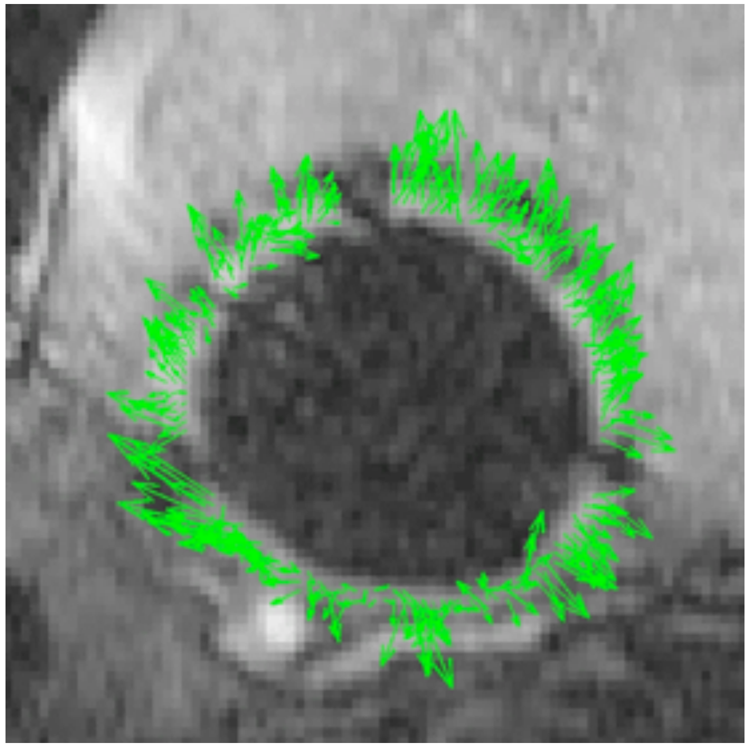
Example 3: External tissue support

Modeling

- Many
but not

- Typic
vessel :





Abdominal aorta
Courtesy of C. Taylor, Stanford

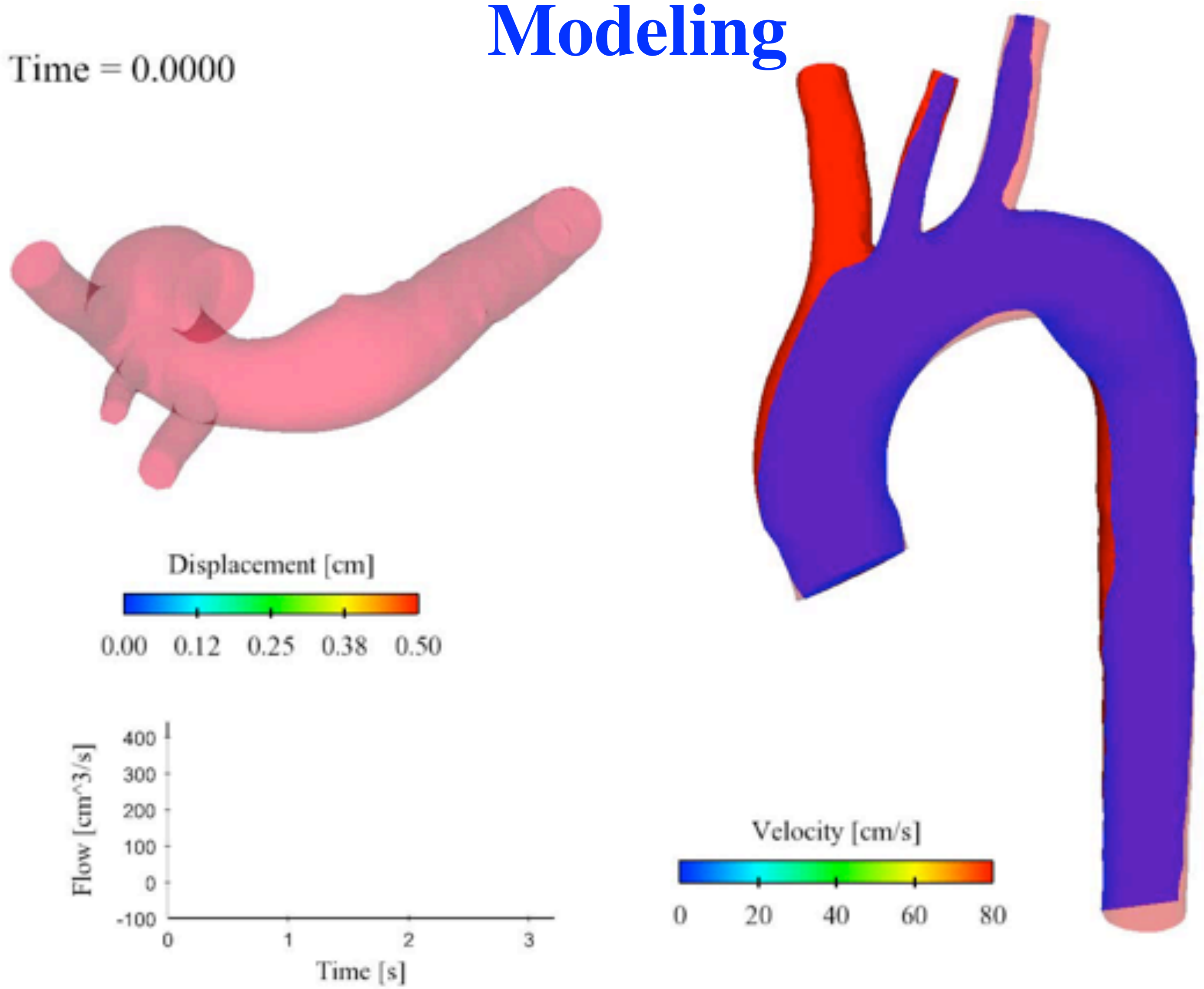
- A sim external tissues :

$$\boldsymbol{F}(\boldsymbol{d})\ \boldsymbol{S}(\boldsymbol{d})\ \widehat{\boldsymbol{n}} = -k_s \boldsymbol{d} - c_s \frac{\partial \boldsymbol{d}}{\partial t}$$

Moireau, Xiao, Astorino, et al. (2010)

Example 3: External tissue support

Modeling

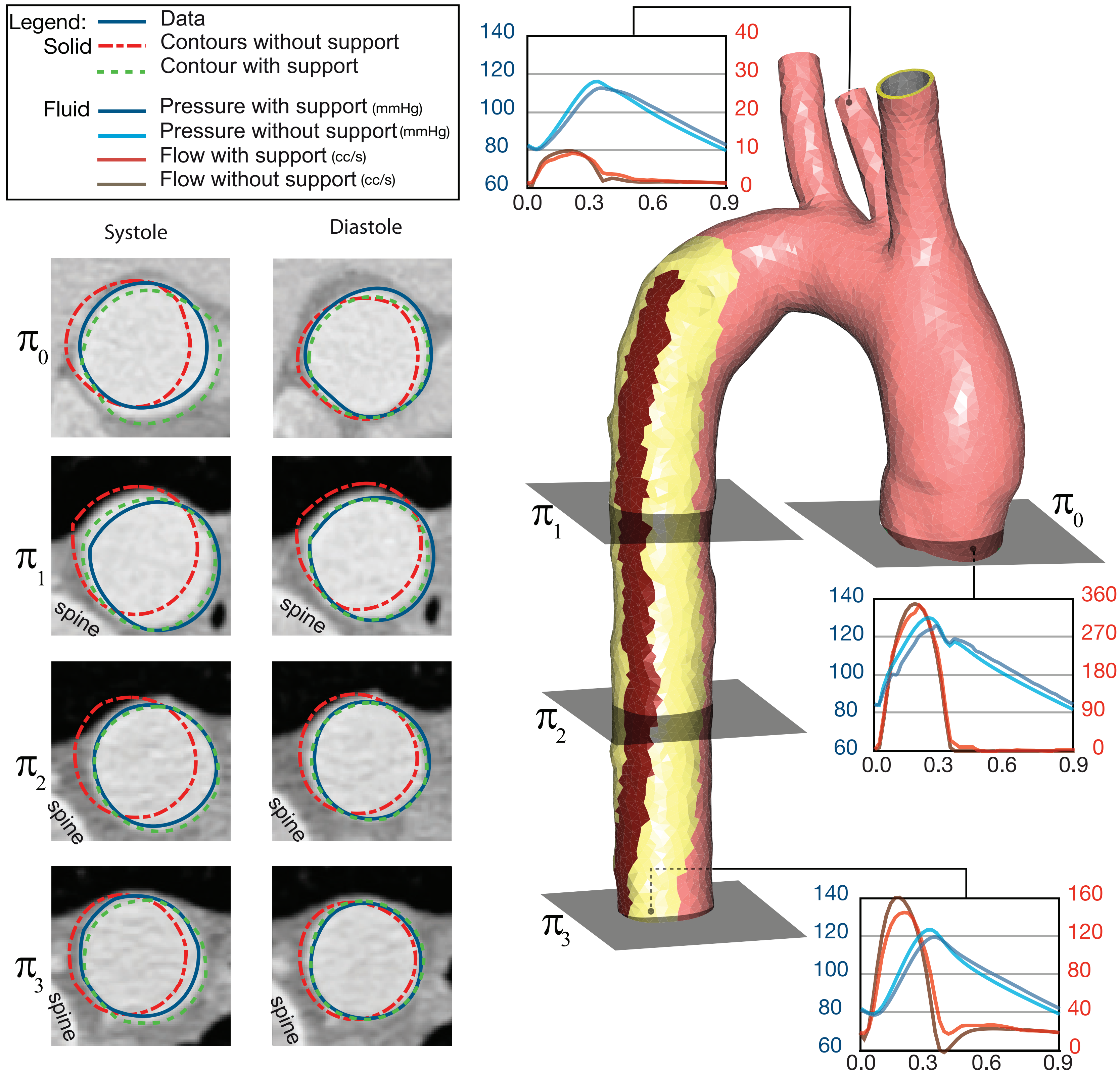


$$\boldsymbol{F}(\boldsymbol{d})\ \boldsymbol{S}(\boldsymbol{d})\ \widehat{\boldsymbol{n}} = -k_s \boldsymbol{d} - c_s \frac{\partial \boldsymbol{d}}{\partial t}$$

with heterogeneous coefficients

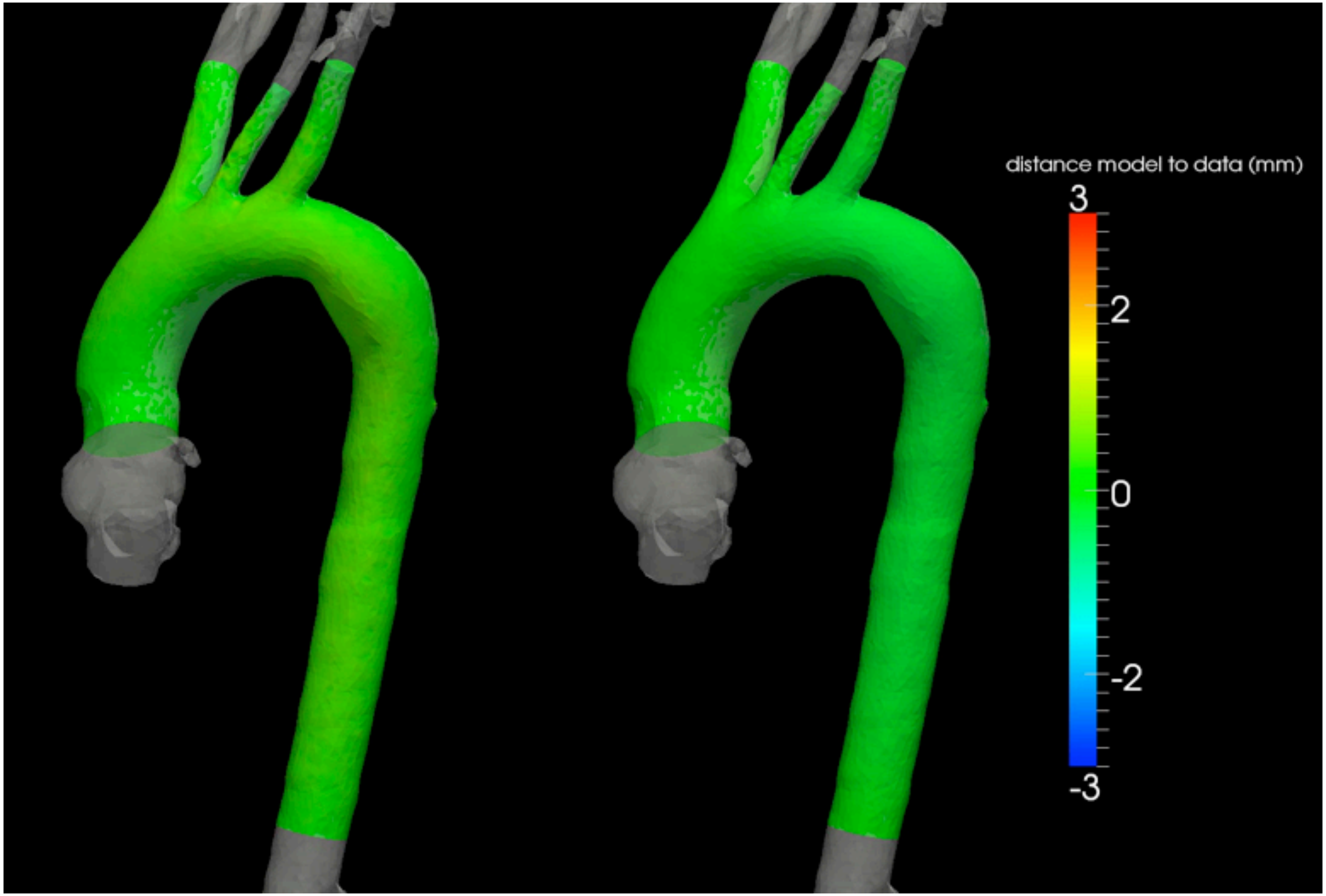
Example 3: External tissue support

Modeling



Example 3: External tissue support

State estimation



Data assimilation (state only)

Direct similation

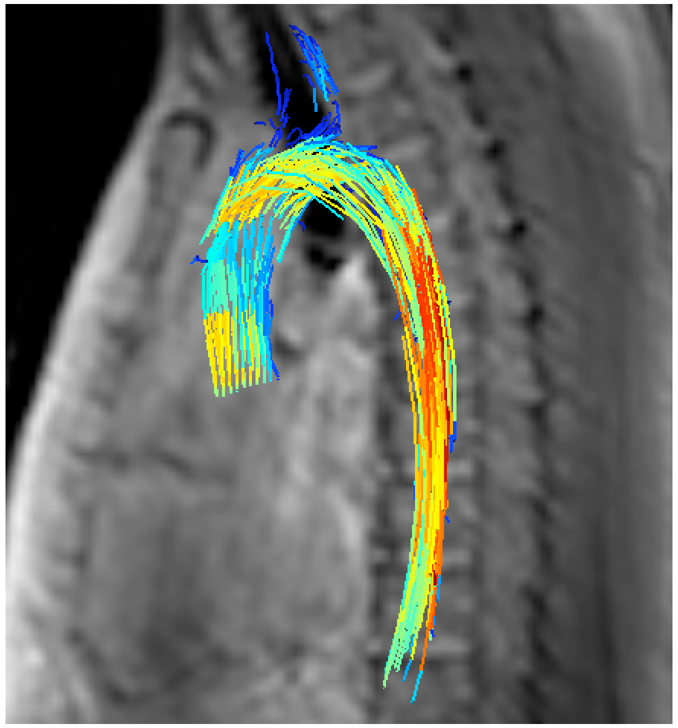
Simulation: N. Xiao

Collaboration INRIA/ Stanford

Conclusion

- Fluid-structure in arteries
  - ★ Tremendous progress over the last few years
  - ★ Important modeling issues (pre-stress, external tissues, ...)
- Grand challenge: medical data assimilation
  - ★ Our approach: filtering techniques for parameter and state

- Work in progress:
  - ★ Real data for aortic coarctation
  - ★ Introduce fluid observations: flow rate, pressure, velocity field.



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