A new class of splitting methods for incompressible flow using direction splitting

Jean-Luc Guermond and Peter Minev

Texas A&M University and University of Alberta

November 21-26, 2010, BIRS

Outline

1. Navier-Stokes equations
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3. New splitting scheme
4. Higher order version
5. Parallel implementation
6. Numerical results
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INCOMPRESSIBLE NEWTONIAN FLUIDS

- Navier-Stokes equations:

$$\frac{D\boldsymbol{u}}{Dt} = \frac{1}{Re}\nabla^2\boldsymbol{u} - \nabla p + \boldsymbol{f} \quad \text{in}\,\Omega$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in}\,\Omega$$

$$\boldsymbol{u}|_{\partial\Omega} = 0$$

CHORIN-TEMAM

$$\begin{cases} \dfrac{\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n}{\Delta t} - \nu\Delta\tilde{\mathbf{u}}^{n+1} = \boldsymbol{f} \quad \text{in } \Omega\times[0,T], \quad \mathbf{u}^{n+1}|_{\partial\Omega} = 0 \\ \Delta t\nabla \mathrm{p} + \mathbf{u}^{n+1} - \tilde{\mathbf{u}}^{n+1} = 0, \quad \nabla\cdot\mathbf{u}^{n+1} = 0 \quad \text{in } \Omega\times[0,T], \\ \partial_n \mathrm{p}|_{\partial\Omega} = 0 \quad \text{in } [0,\mathrm{T}], \quad \text{ and } \mathbf{u}|_{t=0} = \mathbf{u}_0,\ \mathrm{p}|_{t=0} = \mathrm{p}_0 \quad \text{in } \Omega, \end{cases}$$

$$\begin{cases} \partial_t\mathbf{u}_\epsilon - \nu\Delta\mathbf{u}_\epsilon + \nabla\mathrm{p}_\epsilon = \boldsymbol{f} \quad \text{in } \Omega\times[0,T], \\ -\epsilon\Delta\mathrm{p}_\epsilon + \nabla\cdot\mathbf{u}_\epsilon = 0 \quad \text{in } \Omega\times[0,T], \\ \mathbf{u}_\epsilon|_{\partial\Omega} = 0,\ \partial_n\mathrm{p}_\epsilon|_{\partial\Omega} = 0 \quad \text{in } [0,\mathrm{T}], \quad \text{ and } \mathbf{u}_\epsilon|_{t=0} = \mathbf{u}_0,\ \mathrm{p}_\epsilon|_{t=0} = \mathrm{p}_0 \quad \text{in } \Omega, \end{cases}$$

New perturbation

$$\begin{cases} \partial_t \mathbf{u}_\epsilon - \nu \Delta \mathbf{u}_\epsilon + \nabla \mathrm{p}_\epsilon = \boldsymbol{f} \quad \text{in } \Omega \times [0, T], \\ \Delta t A \mathrm{p}_\epsilon + \nabla \cdot \mathbf{u}_\epsilon = 0 \quad \text{in } \Omega \times [0, T], \\ \mathbf{u}_\epsilon|_{\partial\Omega} = 0, \mathrm{p}_\epsilon \in D(A), \quad \text{in [0,T]}, \quad \text{and } \mathbf{u}_\epsilon|_{t=0} = \mathbf{u}_0, \ \mathrm{p}_\epsilon|_{t=0} = p_0 \quad \text{in } \Omega, \end{cases}$$

a is symmetric, and $\|\nabla q\|^2_{\boldsymbol{L}^2} \leq a(q, q), \quad \forall q \in D(A).$

Theorem

$$\|\mathbf{u} - \mathbf{u}_\epsilon\|_{L^2((0,T);\boldsymbol{L}^2\Omega)} \leq c\,\Delta t,$$

$$\|\mathbf{u} - \mathbf{u}_\epsilon\|_{L^2((0,T);\boldsymbol{H}^1(\Omega))} + \|\mathrm{p} - \mathrm{p}_\epsilon\|_{L^2((0,T);\boldsymbol{L}^2(\Omega))} \leq c\,\Delta t^{\frac{1}{2}}.$$

New direction splitting scheme

$$p^{*,n+\frac{1}{2}} = p^{n-\frac{1}{2}}.$$

$$\frac{\boldsymbol{u}^{n+\frac{1}{2}} - \boldsymbol{u}^{n}}{\frac{1}{2}\Delta t} - \nu\left(\partial_{xx}\boldsymbol{u}^{n+\frac{1}{2}} + \partial_{yy}\boldsymbol{u}^{n}\right) + \nabla p^{*,n+\frac{1}{2}} = \boldsymbol{f}^{n+\frac{1}{2}}; \quad \boldsymbol{u}^{n+\frac{1}{2}}|_{x=0,1} = 0,$$

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n+\frac{1}{2}}}{\frac{1}{2}\Delta t} - \nu\left(\partial_{xx}\boldsymbol{u}^{n+\frac{1}{2}} + \partial_{yy}\boldsymbol{u}^{n+1}\right) + \nabla p^{*,n+\frac{1}{2}} = \boldsymbol{f}^{n+\frac{1}{2}}; \quad \boldsymbol{u}^{n+1}|_{y=0,1} = 0.$$

$$\begin{cases} A := (1-\partial_{xx})(1-\partial_{yy}) \\ D(A) := \{p, (1-\partial_{yy})p, Ap \in L^2(\Omega), p|_{y=0,1} = 0, \partial_x((1-\partial_{yy})p)|_{x=0,1} = 0\} \end{cases}$$

$$\psi - \partial_{xx}\psi = -\frac{\nabla\cdot\boldsymbol{u}^{n+1}}{\Delta t}, \quad \partial_x\psi|_{x=0,1} = 0;$$

$$p^{n+\frac{1}{2}} - \partial_{yy}p^{n+\frac{1}{2}} = \psi, \qquad \partial_y p^{n+\frac{1}{2}}|_{y=0,1} = 0.$$

STABILITY RESULT

LEMMA

Let $f \in L^2(\Omega)$. Let ψ and p solve

$$\psi - \partial_{xx}\psi = f, \qquad \partial_x \psi|_{x=0,1} = 0;$$
$$p - \partial_{yy} p = \psi, \qquad \partial_y p|_{y=0,1} = 0,$$

then $Ap = f$ and the bilinear form $a(p,q) := \int_\Omega qAp \,\mathrm{d}\boldsymbol{x}$ is symmetric and H^1 coercive.

THEOREM

The solution to the new scheme, with $p^{-\frac{1}{2}} = 0$, satisfies the following stability estimate for all $T > 0$:

$$\|\boldsymbol{u}\|^2_{\ell^\infty(0,T;\boldsymbol{L}^2)} + 2\nu \|\nabla \boldsymbol{u}\|^2_{\ell^2(0,T;\boldsymbol{L}^2)} + \Delta t^2 \|p\|_{\ell^2(-\frac{\Delta t}{2}, T-\frac{\Delta t}{2}, D(A))} \le \|\boldsymbol{u}^0\|^2_{\boldsymbol{L}^2}.$$

HIGHER ORDER VERSION

$$
\begin{cases}
\partial_t \mathbf{u}_\epsilon - \nu \Delta \mathbf{u}_\epsilon + \nabla \mathrm{p}_\epsilon = \boldsymbol{f} \quad \text{in } \Omega \times [0,T], & \mathbf{u}_\epsilon|_{\partial\Omega\times[0,T]} = 0, \quad \mathbf{u}_\epsilon|_{t=0} = \boldsymbol{u}_0 \\
\Delta t A \phi_\epsilon + \nabla\cdot\mathbf{u}_\epsilon = 0 \quad \text{in } \Omega \times [0,T], & \partial_n \phi_\epsilon|_{\partial\Omega\times[0,T]} = 0 \quad \phi_\epsilon \in D(A), \\
\Delta t \partial_t \mathrm{p}_\epsilon = \phi_\epsilon - \chi \nu \nabla\cdot\mathbf{u}_\epsilon & \mathrm{p}_\epsilon|_{t=0} = p_0,
\end{cases}
$$

CONJECTURE

$$\|\mathbf{u} - \mathbf{u}_\epsilon\|_{L^2((0,T);\boldsymbol{L}^2\Omega)} \leq c\,\Delta t^2,$$

$$\|\mathbf{u} - \mathbf{u}_\epsilon\|_{L^2((0,T);\boldsymbol{H}^1(\Omega))} + \|\mathrm{p} - \mathrm{p}_\epsilon\|_{L^2((0,T);\boldsymbol{L}^2(\Omega))} \leq c\,\Delta t, \quad \textit{if } \chi = 0.$$

$$\|\mathbf{u} - \mathbf{u}_\epsilon\|_{L^2((0,T);\boldsymbol{H}^1(\Omega))} + \|\mathrm{p} - \mathrm{p}_\epsilon\|_{L^2((0,T);\boldsymbol{L}^2(\Omega))} \leq c\,\Delta t^{\frac{3}{2}}, \quad \textit{if } \chi \in (0,1].$$

Higher order version

$$p^{*,n+\frac{1}{2}} = 2p^{n-\frac{1}{2}} - p^{n-\frac{3}{2}}$$

$$\begin{aligned}
&\frac{\boldsymbol{\xi}^{n+1} - \boldsymbol{u}^n}{\Delta t} - \nu\Delta\boldsymbol{u}^n + \nabla p^{*,n+\frac{1}{2}} = \boldsymbol{f}(t^{n+\frac{1}{2}}), \quad && \boldsymbol{u}^n|_{\partial\Omega} = 0,\\
&\frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\xi}^{n+1}}{\Delta t} - \frac{\nu}{2}\partial_{xx}\left(\boldsymbol{\eta}^{n+1} - \boldsymbol{u}^n\right) = 0, && \boldsymbol{\eta}^{n+1}|_{x=0,1} = 0,\\
&\frac{\boldsymbol{u}^{n+1} - \boldsymbol{\eta}^{n+1}}{\Delta t} - \frac{\nu}{2}\partial_{yy}\left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n\right) = 0, && \boldsymbol{u}^{n+1}|_{y=0,1} = 0.
\end{aligned}$$

$$\begin{aligned}
\tilde{\phi}^{n+\frac{1}{2}} - \partial_{xx}\tilde{\phi}^{n+\frac{1}{2}} &= -\frac{\nabla\cdot\boldsymbol{u}^{n+1}}{\Delta t}; \quad \partial_x\tilde{\phi}^{n+\frac{1}{2}}|_{x=0,1} = 0,\\
\phi^{n+\frac{1}{2}} - \partial_{xx}\tilde{\phi}_p^{n+\frac{1}{2}} - \partial_{yy}\phi^{n+\frac{1}{2}} &= -\frac{\nabla\cdot\boldsymbol{u}^{n+1}}{\Delta t}; \quad \partial_y\phi^{n+\frac{1}{2}}|_{y=0,1} = 0.
\end{aligned}$$

$$p^{n+\frac{1}{2}} = p^{n-\frac{1}{2}} + \phi^{n+\frac{1}{2}} - \chi\nu\nabla\cdot(\tfrac{1}{2}(\boldsymbol{u}^{n+1} + \boldsymbol{u}^n))$$

Higher order version

Theorem

$$\|\boldsymbol{u}\|^2_{\ell^\infty(0,T;\boldsymbol{L}^2)} + 2\nu\|\nabla\overline{\boldsymbol{u}}\|^2_{\ell^2(0,T;\boldsymbol{L}^2)} + \Delta t^2\|p\|_{\ell^\infty(-\frac{\Delta t}{2},T-\frac{\Delta t}{2},D(A))}$$
$$\leq \|\boldsymbol{u}^0\|^2_{\boldsymbol{L}^2} + \Delta t^2\|p^{-\frac{1}{2}}\|^2_A + \frac{1}{2}\Delta t\nu\|\nabla\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2}. \quad (4.1)$$

Parallel implementation

- Comparison to FFT.
- Parallel efficiency.

# procs	$2.7\,10^4$ nds/proc	$2.16\,10^5$ nds/proc	10^6 nds/proc
$1\times1\times1$	0.056s	0.41s	2.0s
$8\times8\times8$	0.077s	0.54s	2.3s
$8\times8\times16$	0.094s	0.55s	2.34s

Table: Weak scalability: CPU time (in second) per time step for fully split Douglas scheme + explicit nonlinear terms.

ANALYTIC SOLUTION

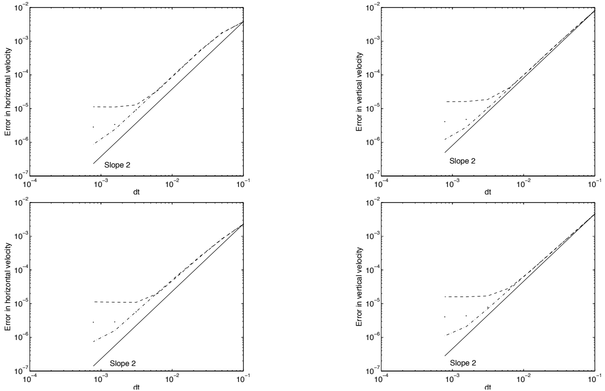


FIGURE: L^2-norm of the error on horizontal (left column) and vertical (right column) components of the velocity at $T = 2$ on grids of 40×40 (dashed line), 80×80 (dotted line) and 160×160 (dash-dotted line). Standard scheme (top) and rotational scheme (bottom).

Analytic solution

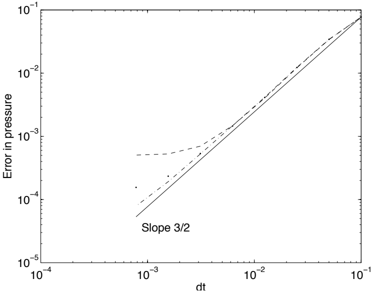


Figure: L^2-norm of the error on the pressure at $T = 2$ on grids of 40×40 (dashed line), 80×80 (dotted line) and 160×160 (dash-dotted line). Standard scheme (left column) and rotational scheme (right column).

ANALYTIC SOLUTION

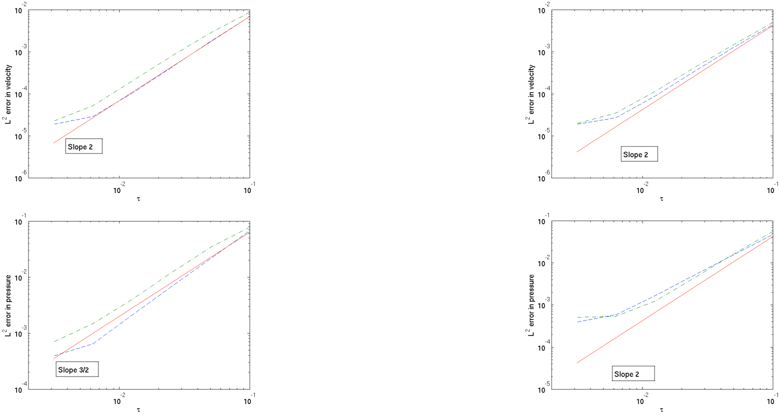


FIGURE: $\boldsymbol{L}^2$-norm of the error on the velocity (top) and pressure (bottom) at $T = 2$ on a uniform grid of 40×40 (dashed line); Left: standard schemes. Right: schemes in a rotational form with $\gamma = 0.5$.

2D CAVITY

$Re = 1000$, vertical component			
x	BS	BP	Present
1.0000	1.00000	1.0000000	1.0000000
0.9688	0.58031	0.5808359	0.5808318
0.9531	0.47239	0.4723329	0.4723260
0.7344	0.18861	0.1886747	0.1886680
0.5000	-0.06205	-0.0620561	-0.0620535
0.2813	-0.28040	-0.2803696	-0.2803632
0.1016	-0.30029	-0.3004561	-0.3004504
0.0625	-0.20227	-0.2023300	-0.2023277
0.0000	0.00000	0.0000000	0.0000000

3D CAVITY

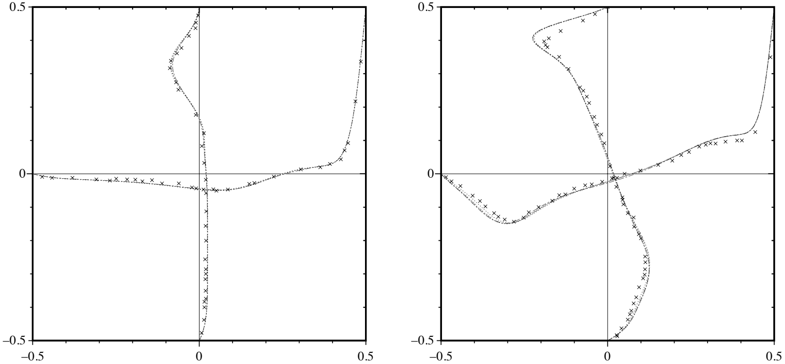


FIGURE: Center line profiles in the plane z=0.5 for a lid-driven cavity of size 1x1x2 at t=4, 8, 12.

3D CAVITY

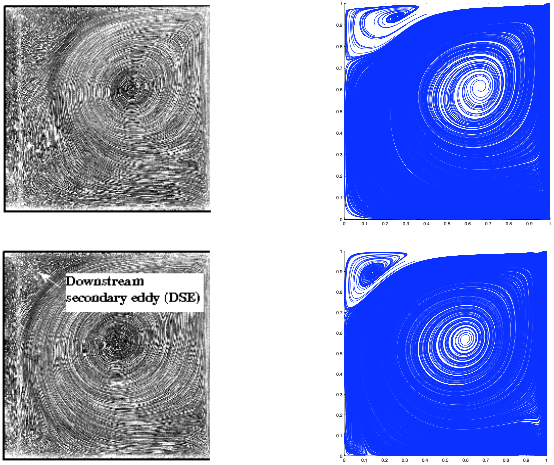
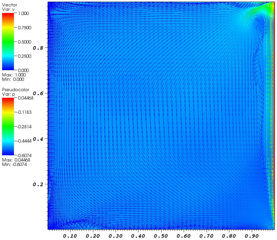
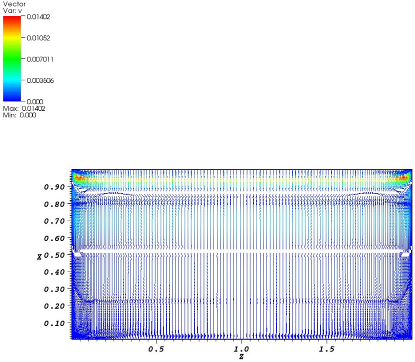


FIGURE: Streamlines at t= 8, 12. Left: experiment, right: simulation

3D Cavity


Vector
Var: v
1.000
0.7500
0.5000
0.2500
0.000
Max: 1.000
Min: 0.000
Pseudocolor
Var: p
0.04468
-0.1183
-0.2814
-0.4444
-0.6074
Max: 0.04468
Min: -0.6074
0.8
0.6
0.4
0.2
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90

3D Cavity


Vector
Var: v
0.01402
0.01052
0.007011
0.003506
0.000
Max: 0.01402
Min: 0.000
0.90
0.80
0.70
0.60
x 0.50
0.40
0.30
0.20
0.10
0.5
1.0
1.5
z

FUTURE DIRECTIONS

- Complete error analysis.
- Complex geometries: with Ph. Angot (U. of Provence).
- Adaptivity (for time dependent problems): in progress.
- Computing of extremely large problems: planning under way.
- Free-boundary and fluid-structure interaction problems: soon to come.
- Parallelization on larger machines ($O(10^4) - O(10^5)$ procs): subject to approval by a "higher authority".

ACKNOWLEDGEMENTS

- NSERC Discovery grant.
- KAUST and ISC funding.

- **Thank you.**