Comparison of the Bergman and Szegö kernels

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based on joint work with Boyong Chen

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Comparing Bergman and Szegö

Stefan Bergman
(5/5/1895-6/6/1977)

Gábor Szegö
(1/20/1895-8/7/1985)

Motivations

Problem (Stein, 72): "What are the relations between K and S?"

K: The Bergman kernel, the reproducing kernel for the Bergman space $A^2(\Omega)$ of L^2 holomorphic functions on Ω. Let $\{b_j\}$ be an orthonormal basis for $A^2(\Omega)$. Then

$$K(z,w) = \sum b_j(z)\overline{b_j(w)}.$$

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$$\|f\|^2_{H^2} := \limsup_{\varepsilon\to 0^+} \int_{b\Omega_\varepsilon} |f|^2\, dS < \infty,$$

where $\Omega_\varepsilon = \{z \in \Omega; \delta(z) = \varepsilon\}$, δ: the Euclidean distance to $b\Omega$.

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where $\Omega_\varepsilon = \{z \in \Omega; \delta(z) = \varepsilon\}$, δ: the Euclidean distance to $b\Omega$.

"The relation of K and S is known also only in very special circumstances."

Backgrounds

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- The Ball in $\mathbb{C}^n$:

$$K(z,w) = \frac{n!}{\pi^n}\frac{1}{(1-z\bar{w})^{n+1}}; S(z,w) = \frac{(n-1)!}{2\pi^n}\frac{1}{(1-z\bar{w})^n}$$

In particular,

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- $b\Omega$ is smooth, strictly pseudoconvex:

$$S(z,z)/K(z,z) \sim \delta(z)/n$$

(Hörmander; Fefferman; Boutet de Monvel-Sjöstrand)

- $b\Omega$ is C^∞, pseudoconvex in $\mathbb{C}^2$ or convex in $\mathbb{C}^n$ and of finite type:

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- Relating mapping properties of the Bergman and Szegö projections. (Boas-Straube; Nagel et al; Bonami-Charpentier; Cumenge; Ligocka; Koenig...)

Goal: Study boundary behavior of S/K on diagonal

Main theorem

$\Omega \subset\subset \mathbb{C}^n$, $b\Omega$: C^2-smooth, pseudoconvex.

- Upper estimate: For any $\varepsilon \in (0,1)$,

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- When Ω has a defining function psh on $b\Omega$ or Ω is pseudoconvex of finite type: $\forall \varepsilon \in (0,1)$:

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- Ω is convex:

$$\frac{S(z,z)}{K(z,z)} \approx \delta(z).$$

D-F exponent

- Diederich-Fornæss exponent ε: $\exists \phi \in psh(\Omega)$,

$$-\phi(z) \approx \delta^{\varepsilon}.$$

(Diederich-Fornæss; Kerzman-Rosay; Demailly; Harrington)

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- When Ω is pseudoconvex of finite type or has a psh defining function, the Diederich-Fornæss index, the sup of the D-F exponents, is 1. (Catlin; Sibony; Fornaess-Herbig).

- δ-regular: $\exists$ bounded continuous $\varphi \in$ *psh*(Ω) and a defining function ρ of Ω such that

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    - Case II: finite type. By Catlin, $\exists$ bounded continuous λ, $\partial\overline{\partial}\lambda \gtrsim \partial\overline{\partial}|z|^2/\delta^{\varepsilon}$. Take $\phi = C(\lambda - (-\rho)^{\eta})$, $\eta << \varepsilon$.

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Basic properties

- $$K_\Omega(z,z) = \sup\{|f(z)|^2 \mid f \in A^2(\Omega), \|f\|_\Omega \le 1\}$$

  and

  $$S_\Omega(z,z) = \sup\{|f(z)|^2 \mid f \in H^2(\Omega), \|f\|_{b\Omega} \le 1\}$$

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- Localization property: U a neighborhood of $z^0 \in b\Omega$:

  $$K_{\Omega\cap U}(z,z) \lesssim K_\Omega(z,z) \le K_{\Omega\cap U}(z,z)$$

  and

  $$S_\Omega(z,z) \lesssim S_{\Omega\cap U}(z,z)$$

  for z near z^0.

- For any harmonic function f on Ω, $1 < p < \infty$,

$$\limsup_{\varepsilon \to 0^+} \int_{b\Omega_\varepsilon} |f|^p \, dS = \limsup_{r \to 1^-} (1-r) \int_\Omega |f(z)|^p \delta^{-r}(z) \, dV.$$

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- Given $z \in \Omega$ and $f \in A^2(\Omega)$, find $g \in H(\Omega)$, $g(z) = f(z)$,

$$(1-r)\int_\Omega |g|^2 \delta^{-r} \lesssim \frac{1}{\delta(z)} \int_\Omega |f|^2 \Rightarrow \frac{S(z,z)}{K(z,z)} \gtrsim \delta(z)$$

Hörmander's estimates

- The $\overline{\partial}$-Problem: $\Omega \subset\subset \mathbb{C}^n$. Given $(0,1)$-form $v = \sum_{j=1}^n v_j d\bar{z}_j$. Find u such that

$$\overline{\partial} u = \sum_{j=1}^n \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j = v, \qquad (\overline{\partial}\text{-equation})$$

provided $\overline{\partial} v = 0$.

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- Hörmander (65): Ω is pseudoconvex. $\psi \in psh(\Omega)$. Suppose $\partial\overline{\partial}\psi \geq c(z)\partial\overline{\partial}|z|^2$ for a positive continuous function $c(z)$. Then the $\overline{\partial}$-equation has a solution satisfying

$$\int_\Omega |u|^2 e^{-\psi}\, dV \leq \int_\Omega \frac{|v|^2}{c(z)} e^{-\psi}\, dV < \infty.$$

Demailly's estimates

- Demailly(82):

$$\int |u|^2 e^{-\psi} \leq \int |v|^2_{\partial\bar\partial\psi} e^{-\psi}.$$

where

$$|v|^2_{\partial\bar\partial\psi} = \sup\{|\langle v, X\rangle|; \quad |X|_{\partial\bar\partial\psi} \leq 1\}$$

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- If $u \perp \mathcal{N}(\bar{\partial})$ in $L^2(\Omega, e^{-\psi})$, then

$$\int_\Omega |u|^2 e^{-\psi} \leq \int_\Omega |\bar{\partial} u|^2_{\partial\bar{\partial}\psi} e^{-\psi}.$$

Berndtsson's Estimates

- Berndtsson (01): $\rho \in C^2(\Omega)$, $\rho < 0$. Suppose $\exists \psi \in psh(\Omega) \cap C^2(\Omega)$ such that

$$\Theta = (-\rho)\partial\overline{\partial}\psi + \partial\overline{\partial}\rho > 0.$$

If $u \perp \mathcal{N}(\overline{\partial})$ in $L^2(\Omega, e^{-\psi})$, then $\forall r \in (0,1)$,

$$(1-r)\int |u|^2(-\rho)^{-r}e^{-\psi} \leq \frac{1}{r}\int |v|^2_{\Theta}(-\rho)^{1-r}e^{-\psi}$$

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- Demailly $\Rightarrow$ Berndtsson: Let

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Then $ue^{\phi} \perp \mathcal{N}(\overline{\partial})$ in $L^2(\Omega, e^{-\varphi})$. Applying Demailly,

$$\int_\Omega |u|^2 e^{\phi-\psi} \leq \int_\Omega |\overline{\partial}u + u\overline{\partial}\phi|^2_{\partial\overline{\partial}\varphi} e^{\phi-\psi}.$$

- Berndtsson then follows from

$$\partial\overline{\partial}\varphi \geq \frac{r}{-\rho} + \frac{1}{r}\partial\psi \wedge \overline{\partial}\psi$$

and Cauchy-Schwarz:

$$|\overline{\partial}u + u\overline{\partial}\phi|^2_{\partial\overline{\partial}\varphi} \leq r|u|^2 + \frac{1}{r}|\overline{\partial}u|^2_{\Theta}(-\rho)$$

Pluricomplex Green function

$\Omega \subset\subset \mathbb{C}^n$. Pluricomplex Green function:

$$g(z,w) = \sup\{u(z) \mid u \in psh(\Omega), u < 0, \limsup_{z \to w}(u(z) - \log|z-w|) < \infty\}.$$

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- Blocki (05): Ω has D-F exponent $\varepsilon > 0$ and $\delta = \delta(w)$:

$$\{g(\cdot, w) < -1\} \subset \{\delta|\log\delta|^{-\frac{1}{\varepsilon}} \lesssim \delta(\cdot) \lesssim \delta|\log\delta|^{\frac{n}{\varepsilon}}\}$$

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When Ω is convex:

$$\{g(\cdot, w) < -1\} \subset \{\frac{1}{C}\delta \leq \delta(\cdot) \leq C\delta\}$$

Upper estimates

$b\Omega$ pseudoconvex, C^2-smooth, $\forall \varepsilon \in (0,1)$,

$$\frac{S(z,z)}{K(z,z)} \lesssim \delta(z)|\log \delta(z)|^{n/\varepsilon}$$

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- Step 1 (Chen/Herbot; 99): Ω bounded, pseudoconvex

$$K(z,z) \gtrsim K_{\{g(\cdot,z)<-1\}}(z,z)$$

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Proof: Given f holomorphic on $\Omega_z = \{g(\cdot,z) < -1\}$. Solve $\overline{\partial}u = v$. Applying Demailly with weight

$\psi = 2ng(\cdot,z) - \log(-g(\cdot,z)+1)$; $v = \overline{\partial}\chi(-\log(-g(\cdot,z)))f$

where χ is cut-off function $= 1$ on $(-\infty,-1)$, and $= 0$ on $(0,\infty)$. Then let $g = \chi(-\log(-g(\cdot,z)))f - u$.

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- Step 2: Applying Blocki. Write $\delta = \delta(z)$. For any $f \in H^2(\Omega)$.

$$\int_{\Omega_z} |f|^2 \leq \int_0^{\delta|\log\delta|^{\frac{n}{\varepsilon}}} dt \int_{\{\delta=t\}} |f|^2 \lesssim \|f\|^2_{b\Omega}\delta|\log\delta|^{\frac{n}{\varepsilon}}.$$

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- Using the fact that for any $z^0 \in b\Omega$, $\forall \varepsilon \in (0, 1)$, $\exists$ defining function r of Ω and a neighborhood U of z^0 such that $\varphi_2 = -(-r)^{\varepsilon}$ is psh on $\Omega \cap U$ (Diederich-Fornæss).

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- Consider

$$\widetilde{\Omega} = \{\varphi_1 = -(-r)^\varepsilon + M\chi(|z - z_0|^2/m^2) < 0\}$$

where χ is positive , increasing, and convex when $t > 1$. M large, m small.

$$\tilde{\delta} \lesssim -\varphi_1 \lesssim \tilde{\delta}^\varepsilon.$$

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- Cannot directly applying Blocki. Nonetheless, we have

$$\{g_{\widetilde{\Omega}}(\cdot, z) \leq -1\} \subset \{\tilde{\delta}(\cdot) \lesssim \tilde{\delta} |\log \tilde{\delta}|^{n/\varepsilon}\}.$$

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$$v = \overline{\partial}\chi(-\log(-g(\cdot, z)))f$$

where χ is cut-off function $= 1$ on $(-\infty, -1)$, and $= 0$ on $(0, \infty)$.

- Notice that

$$\operatorname{Supp}\bar\partial\chi(*) \subset \{-e \le g(\cdot,w) \le -1\} \subset \{C\delta|\delta|^{-1/\varepsilon} \le \delta(\cdot)\}$$

and

$$\partial\bar\partial\psi \ge \partial\log(-g(\cdot,w)+1)\wedge\bar\partial\log(-g(\cdot,w)+1)$$

Hence $|\bar\partial\chi|_{\partial\bar\partial\psi} \lesssim 1$.

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  Hence $u(w) = 0$.
- Let $g = \chi(-\log(-g(\cdot,z)))f - u$. Then g is holomorphic on Ω, $g(w) = f(w)$, and $g = -u$ near $b\Omega$.

- Notice that

$$\operatorname{Supp} \bar\partial\chi(*) \subset \{-e \le g(\cdot, w) \le -1\} \subset \{C\delta|\delta|^{-1/\varepsilon} \le \delta(\cdot)\}$$

  and

$$\partial\bar\partial\psi \ge \partial \log(-g(\cdot, w) + 1) \wedge \bar\partial \log(-g(\cdot, w) + 1)$$

  Hence $|\bar\partial\chi|_{\partial\bar\partial\psi} \lesssim 1$.
- By Demailly:

$$\int_\Omega |u|^2 e^{-\psi} \le \int_\Omega |v|^2_{\partial\bar\partial\psi} e^{-\psi} < \infty.$$

  Hence $u(w) = 0$.
- Let $g = \chi(-\log(-g(\cdot, z)))f - u$. Then g is holomorphic on Ω, $g(w) = f(w)$, and $g = -u$ near $b\Omega$. Furthermore,

$$\|g\|^2_{b\Omega} = \lim_{r\to 1^-} (1 - r) \int_\Omega |g|^2 \delta^{-r} \lesssim \frac{|\delta(z)|^{1/\varepsilon}}{\delta(z)} \|f\|^2.$$

- Notice that

$$\operatorname{Supp}\overline{\partial}\chi(*) \subset \{-e \le g(\cdot, w) \le -1\} \subset \{C\delta|\delta|^{-1/\varepsilon} \le \delta(\cdot)\}$$

  and

$$\partial\overline{\partial}\psi \ge \partial \log(-g(\cdot, w)+1) \wedge \overline{\partial} \log(-g(\cdot, w)+1)$$

  Hence $|\overline{\partial}\chi|_{\partial\overline{\partial}\psi} \lesssim 1$.
- By Demailly:

$$\int_\Omega |u|^2 e^{-\psi} \le \int_\Omega |v|^2_{\partial\overline{\partial}\psi} e^{-\psi} < \infty.$$

  Hence $u(w) = 0$.
- Let $g = \chi(-\log(-g(\cdot, z)))f - u$. Then g is holomorphic on Ω, $g(w) = f(w)$, and $g = -u$ near $b\Omega$. Furthermore,

$$\|g\|^2_{b\Omega} = \lim_{r\to 1^-} (1-r) \int_\Omega |g|^2 \delta^{-r} \lesssim \frac{|\delta(z)|^{1/\varepsilon}}{\delta(z)} \|f\|^2.$$

- Make precise: convolute with the Friderichs' mollifiers.

Thank You!