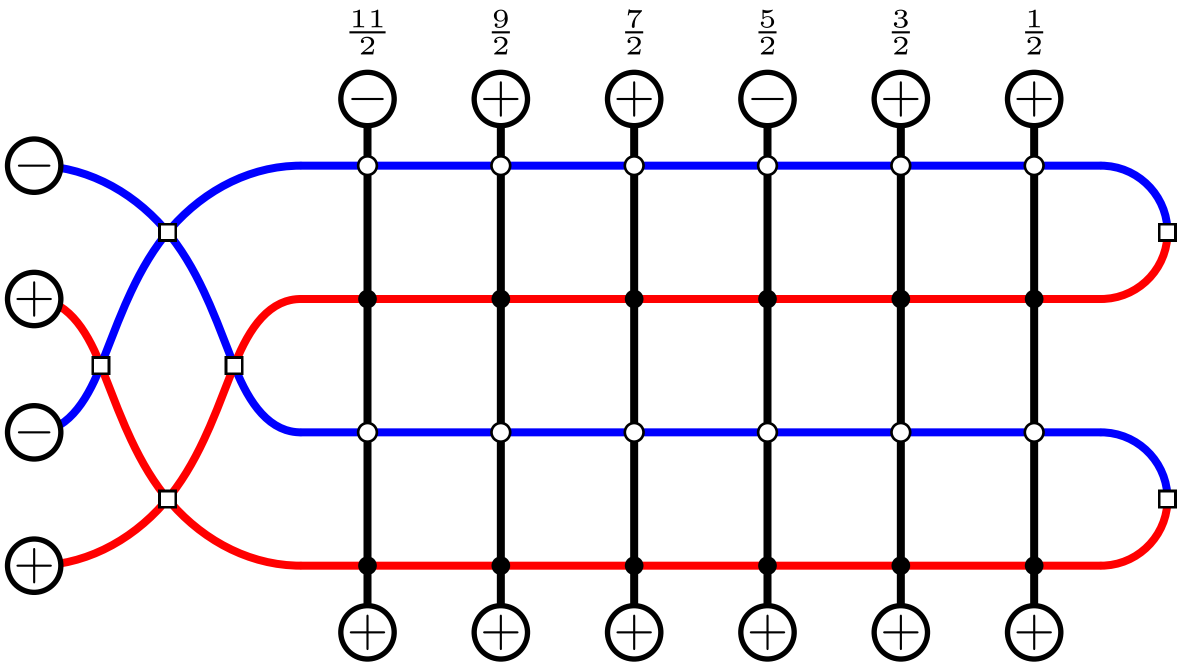
Modeling p-adic Whittaker Functions



Whittaker functions

- F – a locally compact field
- G – a split reductive group over F
- B – positive Borel subgroup
- T – maximal torus
- U – unipotent radical of $B = TU$
- ψ – nondegenerate character of U

Example: $G = \mathrm{GL}_n$

$$U = \left\{ \begin{pmatrix} 1 & u_{12} & \cdots & u_{n-1,n} \\ & 1 & & u_{n-2,n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \right\}$$

$\psi(u) = \psi_0(u_{12} + u_{23} + \cdots)$

$\psi_0 \colon F \longrightarrow \mathbb{C}$ a nontriv char.

Theorem. (Gelfand-Graev, Shalika, Piatetski-Shapiro) *The representation* $\mathrm{Ind}_U^G(\psi)$ *is multiplicity-free.*

A **Whittaker model** of an irreducible representation (π, V) is a space of functions $\mathcal{W}_\pi$ on G that satisfy

$$W(ug) = \psi(u) W(g), \qquad u \in U,$$

that is closed under right translation:

$$W \in \mathcal{W}_\pi \quad \Rightarrow \quad \rho(g) W \in W_\pi, \qquad\qquad \rho(g) W(x) = W(xg)$$

and such that $\mathcal{W}_\pi \cong V$ as G-modules. The content of the theorem is that **the Whittaker model (if it exists) is unique**.

Principal Series representations

Let χ be a character of T_F. Extend χ to B_F (Borel subgroup) and induce:

$$V(\chi) = \{f : G_F \longrightarrow \mathbb{C} \mid f(bg) = (\delta^{1/2}\chi)(b) f(g)$$

$(\delta = \text{modular character of } B_F)$

G_F acts by right translation: $\pi(g) f(x) = f(xg)$

- $V(\chi)$ is **usually irreducible**.
- If $w \in W$ (Weyl group) $V(\chi)$ and $V({}^w\chi)$ are **isomorphic** (if irreducible).

Suppose F is nonarchimedean, $\mathfrak{o} =$ integers in F. Let $K = G(\mathfrak{o})$, max'l compact.

- Given any representation (π, V), let $V^K =$ space of K-fixed vectors.

Proposition 1. *If* (π, V) *is irreducible* $\dim(V^K) \leqslant 1$.

The irreducible representation is **spherical** if $\dim(V^K) = 1$.
If χ is a character of T_F, χ is **unramified** if $\chi(T_{\mathfrak{o}}) = 1$.

Proposition. *If χ is unramified $V(\chi)$ is spherical.*

(If $V(\chi)$ is reducible, it has a unique spherical quotient.)

The L-group

Given a group G there is a group $\hat{G}$ whose root data are dual to G.

- G – a split reductive group
- T – maximal split torus in G
- Φ – root system of G
- P – weight lattice in G
- $\hat{T}$ – maximal split torus in $\hat{G}$
- $\hat{G}$ – the L-group
- $\hat{G}$ – root system of $\hat{G}$
- $P^\vee$ – the weight lattice in $\hat{G}$

Example:	
$G = \mathrm{GL}_n$	$P = \mathbb{Z}^n$
$\hat{G} = \mathrm{GL}_n$	$P^\vee = \mathbb{Z}^n$
Example:	
$G = \mathrm{Sp}_{2r}$	$P = \mathbb{Z}^r$
$\hat{G} = \mathrm{SO}_{2r+1}$	$P^\vee = \mathbb{Z}^r$

Assume that the ground field F is nonarchimedean local.

- $\hat{T}(\mathbb{C}) \cong$ group of characters of $T_F/T_{\mathfrak{o}}$ $\quad \boldsymbol{z} \in \hat{T}(\mathbb{C}) \longleftrightarrow \chi_{\boldsymbol{z}} \in X(T_F/T_{\mathfrak{o}})$
- $T_F/T_{\mathfrak{o}} \cong$ coweight lattice $P^\vee$ $\quad \lambda \in P^\vee \longleftrightarrow t_{\lambda^\vee} \in T_F$

  (if G is of adjoint type, otherwise $\subseteq P^\vee$.)
- Dominant $\lambda \in P^\vee$ parametrize irreducible characters of $\hat{G}(\mathbb{C})$ $\quad \lambda \in P^\vee$ (dominant) $\longleftrightarrow \xi_\lambda$, irreducible char of $\hat{G}(\mathbb{C})$

If $\boldsymbol{z} \in \hat{T}(\mathbb{C})$, we may consider the induced representation $V(\chi_{\boldsymbol{z}})$.

Duality

Recap: (Semisimple) (spherical)

Conjugacy classes of $\hat{G}(\mathbb{C})$ correspond to **irreps of $G(\mathbf{F})$**

(Semisimple) (finite-dimensional)

Conjugacy classes of $G(F)$ correspond to **irreps of $\hat{G}(\mathbb{C})$**

(Not bijectively: $t_{\lambda^\vee}$ is only determined up to multiplication by a unit.)

- $\boldsymbol{z} \in \hat{T}(\mathbb{C})$ $\longleftrightarrow$ $\chi_{\boldsymbol{z}} \in X(T_F/T_{\mathfrak{o}})$
  L-group torus element
- $\boldsymbol{z} \in \hat{T}(\mathbb{C})$ $\longleftrightarrow$ $V(\chi_{\boldsymbol{z}}) = \mathrm{Ind}(\delta^{1/2}\chi_{\boldsymbol{z}})$
  (If irreducible – usually)
- $\boldsymbol{z}' = {}^w\boldsymbol{z}$ $(w \in W)$ $\Longrightarrow$ $V(\chi_{\boldsymbol{z}}) \cong V(\chi_{\boldsymbol{z}'})$
- $\lambda \in P^\vee$ $\longleftrightarrow$ $t_{\lambda^\vee} \in T_F/T_{\mathfrak{o}}$
  (Coweight)
- $\lambda \in P^\vee$ $\longleftrightarrow$ ξ_λ irr char of $\hat{G}(\mathbb{C})$
  (Dominant weight)

L-group elements index unramified chars of T_F and by induction, irreps of G_F. Conjugate $\boldsymbol{z}$ index isomorphic $V(\chi)$. Elements of $T_F/T_{\mathfrak{o}}$ are indexed by coweights; dominant coweights index irreps of $\hat{G}(\mathbb{C})$. Each conjugacy class contains a unique coset $t_{\lambda^\vee}$ mod $T_{\mathfrak{o}}$ with $\lambda^\vee$ dominant.

Casselman-Shalika Formula

Let $\boldsymbol{z} \in \hat{T}(\mathbb{C})$. Let $W_{\boldsymbol{z}}^{\circ}$ be the spherical vector in the Whittaker model of $V(\chi_{\boldsymbol{z}})$. Langlands conjectured that the values of $W_{\boldsymbol{z}}^{\circ}$ are the values of irreducible characters of $\hat{G}$. This was proved by Shintani, S. Kato, Casselman and Shalika and is referred to as the **Casselman-Shalika formula**.

Theorem. *We have*

$$W_{\boldsymbol{z}}^{\circ}(t_{\lambda^\vee}) = \begin{cases} \text{const} \times \delta^{1/2}(t_{\lambda^\vee})\chi_\lambda(\boldsymbol{z}) & \text{if } \lambda^\vee \text{ is dominant,} \\ 0 & \text{otherwise} \end{cases}$$

In a natural normalization the constant is $\prod_{\alpha \in \Phi^+} (1 - q^{-1}\boldsymbol{z}^\alpha)$. More precisely, we may define $W_{\boldsymbol{z}}^{\circ}$ as an integral, thus:

$$W_{\boldsymbol{z}}^{\circ}(g) = \int_{U_F} f^\circ(w_0 u g)\psi(u)^{-1} du, \qquad w_0 = \text{long } W \text{ element},$$

where $f^\circ(bk) = \delta^{1/2}\chi(b)$, $b \in B_F$, $k \in K$. Then $\text{const} = \prod_{\alpha \in \Phi^+} (1 - q^{-1}\boldsymbol{z}^\alpha)$.

Why Seek Other Models?

The Casselman-Shalika formula is the complete story for the spherical Whittaker function. **Why look any further?**

- The constant $\prod_{\alpha \in \Phi^+} (1 - q^{-1}\boldsymbol{z}^{\alpha})$ is a deformation of Weyl's denominator. So we seek a deformation of the Weyl character formula.
- The study of such deformations leads us to crystal bases and statistical (ice-type) models.
- Furthermore such models work for **metaplectic Whittaker functions** where the Casselman-Shalika formula does not apply.

Suppose that $F \supset \mu_n$ (the n-th roots of unity). Weil, Kubota and Matsumoto defined a **metaplectic cover** which is a central extension

$$1 \longrightarrow \mu_n \longrightarrow \tilde{G}(F) \longrightarrow G(F) \longrightarrow 1.$$

The cover splits over $U(F)$ so one may still consider Whittaker models.

- Uniqueness of Whittaker models fails. **Still spherical Whittaker functions have expressions in terms of crystal or ice models.**

Deformations of the Weyl Character formula

A deformations of the Weyl character formula was found by **Tokuyama** (1988). Others considered deformations of the Weyl denominator.

- Kuperberg, Okada, Simpson, Hamel and King.
- Beineke, Brubaker, Bump, Chinta, Friedberg, Frechette, Gunnells, Ivanov, Tabony.

There are different ways of writing Tokuyama's formula.

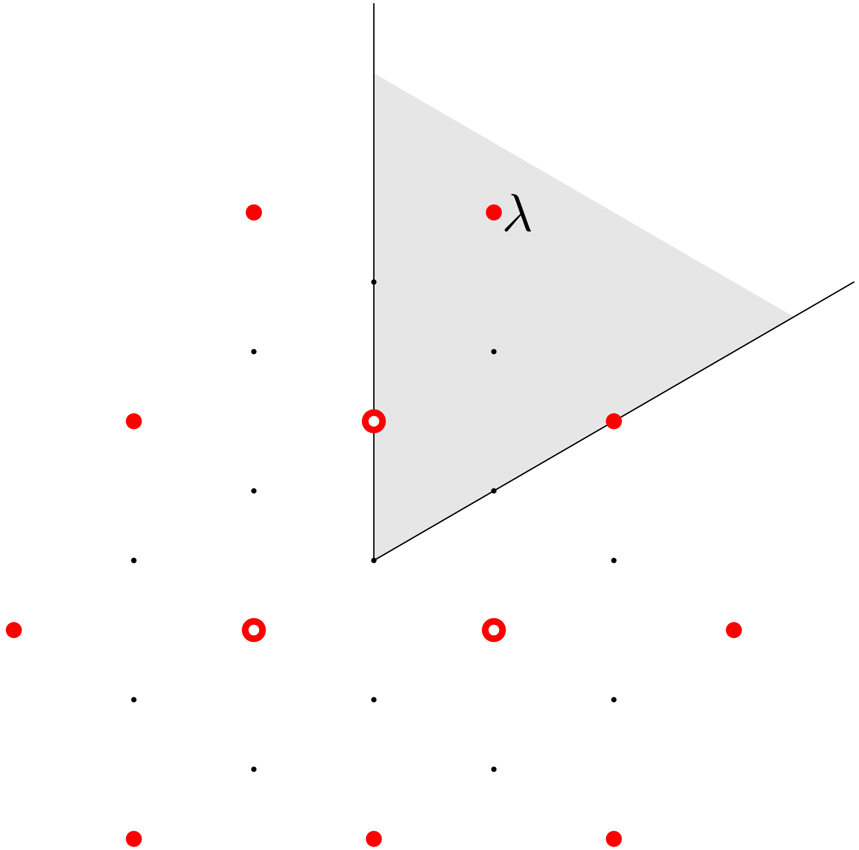
- Sum over strict Gelfand-Tsetlin patterns (original paper).
- Sum over crystal $\mathcal{B}_{\lambda+\rho}$. $\left(\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha\right)$
- Six-vertex model.

The last two approaches are subtly different suggesting different tools.

Weyl Characters

Let $\mathcal{G}$ be a complex Lie group. **Note:** **Eventually $\mathcal{G}$ will be $\hat{G}(\mathbb{C})$ so Φ will become $\Phi^\vee$ (coroots) and P will become $P^\vee$ coweights.**

- Let $\lambda \in P$ be dominant. Let ξ_λ be the irr character of highest weight λ.
- Decompose ξ_λ into a sum of weights with multiplicities.



Example: $G = \mathrm{GL}_3(\mathbb{C})$, $P = \mathbb{Z}^3$.
$\lambda = (3, 1, 0)$

.	Elements of P
shaded area	Positive Weyl Chamber (dominant weights)
●	Weights with multiplicity 1
○	Weights with multiplicity 2

Observe that the "weight diagram" is invariant under W (which is the group generated by the reflections in the two hyperplanes bounding the positive Weyl chamber).

Root operators

Let $\mathcal{G}$ be a complex Lie group. Let P be the weights (char's of max'l torus $\mathcal{T}$).
Note: Eventually $\mathcal{G}$ will be $\hat{G}(\mathbb{C})$ so Φ will become $\Phi^\vee$ (coroots) and P will become $P^\vee$ coweights.

- Φ – The root system
- Φ^+ – The positive roots
- $\Sigma = \{\alpha_1, \cdots, \alpha_r\}$ – The simple roots
- V – A $\mathcal{G}$-module
- $\mu \in P$ – a weight of $\mathcal{G}$.
- $V(\mu)$ – The weight space

A positive root is called **simple** if it cannot be decomposed as a sum of other positive roots.

We have $V = \bigoplus_{\mu \in P} V(\mu)$

If $X \in \mathrm{Lie}(\mathcal{G})$ then X acts on V. Let $\alpha \in \Phi$ and $X_\alpha \in \mathrm{Lie}(\mathcal{G})$ be in the one-dimensional root eigenspace. Then

$$X_\alpha : V(\mu) \longrightarrow V(\mu + \alpha).$$

We choose $E_i = X_{\alpha_i}$ and $F_i = X_{-\alpha_i}$ to be the Chevalley generators. Then

$$E_i : V(\mu) \longrightarrow V(\mu + \alpha_i), \qquad F_i : V(\mu) \longrightarrow V(\mu - \alpha_i).$$

Crystals

A **(Kashiwara) crystal** is a combinatorial substitute for $V(\mu)$. The crystal $\mathcal{B}_\lambda$ of highest weight λ is a set with cardinality $\dim(V(\mu))$.

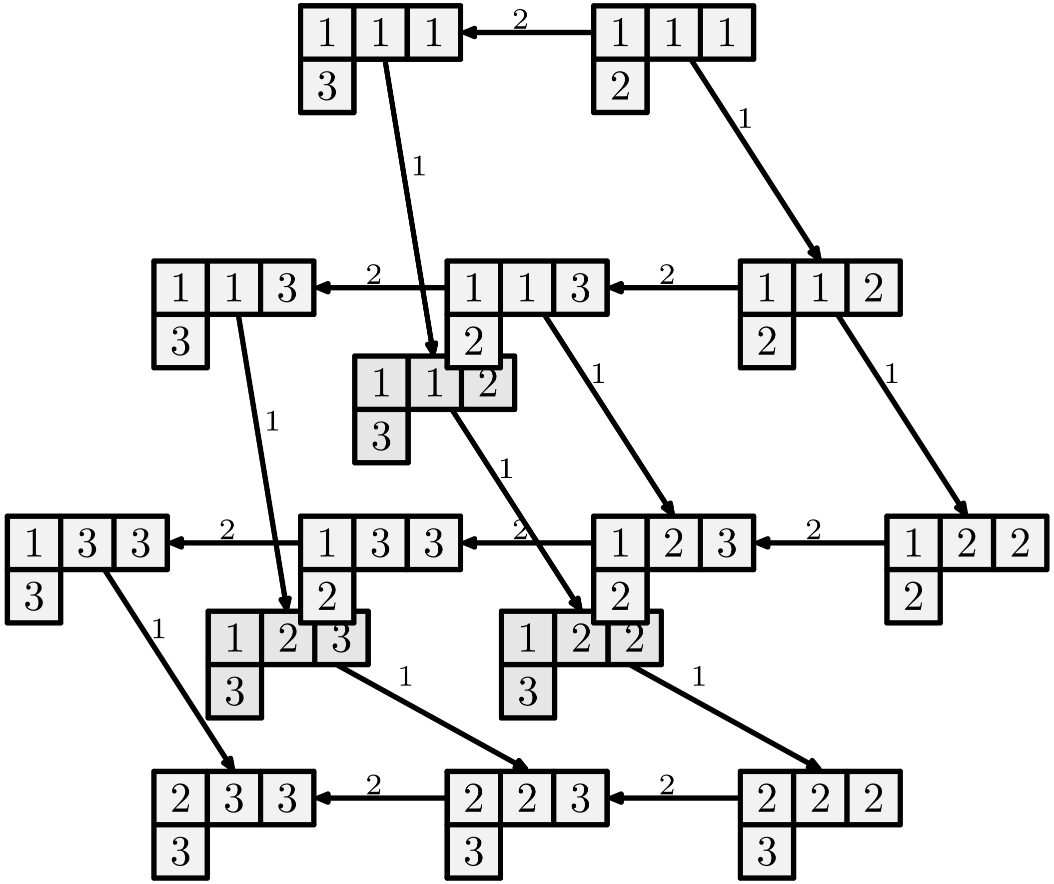
- It is equipped with a **weight map** $\mathrm{wt}\colon \mathcal{B}_\lambda \longrightarrow P$.
- The number of $\mathcal{B}_\lambda$ with weight μ is $m(\mu) = \dim V(\mu)$
- **Root operators** $E_i, F_i\colon \mathcal{B}_\lambda \longrightarrow \mathcal{B}_\lambda \cup \{0\}$ are defined.
- If $E_i(v) = w \neq 0$ then $F_i(w) = v$ and $\mathrm{wt}(v) = \mathrm{wt}(w) + \alpha_i$.

Following Kashiwara and Nakashima, if $\Phi = A_r$ the elements of $\mathcal{B}_\lambda$ are **semistandard Young tableaux** of shape λ in the alphabet $\{1, 2, 3, \cdots, r\}$. These are fillings of the Young diagram with shape λ by elements of the alphabet with weakly increasing rows and strictly increasing columns, like this:

1	1	2	3	4
2	3			
3				

Example: GL_3

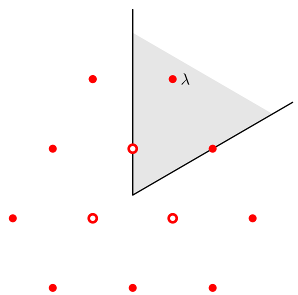
Here is the crystal with highest weight $\lambda = (3, 1, 0)$. Compare it with the weight diagram (above) for $V(\lambda)$.



Kashiwara: elements of $\mathcal{B}_\lambda$ are labeled by tableaux of shape λ in $\{1, 2, 3\}$.

If $F_i(v) = w$ and $E_i(w) = v$ we draw an arrow $v \xrightarrow{i} w$.

We have drawn the crystal so that the elements of equal weight overlap.



The crystal is mapped to the weight diagram (left) by $\mathrm{wt}\colon \mathcal{B}_\lambda \longrightarrow P$.

Tokuyama functions

By a **Tokuyama function** on the crystal $\mathcal{B}_{\lambda+\rho}$ we mean a function

$$G\colon \mathcal{B}_{\lambda+\rho} \times \mathbb{C} \longrightarrow \mathbb{C}$$

such that $\displaystyle\sum_{v \in \mathcal{B}_{\lambda+\rho}} G(v,t)\boldsymbol{z}^{\mathrm{wt}(v)} = \left[\prod_{\alpha \in \Phi^+} (1 + t\boldsymbol{z}^{\alpha})\right] \xi_\lambda(\boldsymbol{z})$.

- If $t = -1$ the formula should reduce to the Weyl character formula.
- If $t = 0$, then $G(v, t)$ should vanish unless v is in the image of a map $\mathcal{B}_\lambda \longrightarrow B_{\lambda+\rho}$, and the formula should reduce to $\xi_\lambda(\boldsymbol{z}) = \displaystyle\sum_{v \in \mathcal{B}_\lambda} \boldsymbol{z}^{\mathrm{wt}(v)}$.
- If $t = -q^{-1}$ the formula should give the Casselman-Shalika formula (with deformed Weyl denominator.)
- There are also **metaplectic Tokuyama functions**. These produce not characters but **metaplectic Whittaker functions**.
- "Natural" Tokuyama functions can be given in many cases, beginning with Tokuyama (1988). The Tokuyama function is not unique. Using results of McNamara one gets **one Tokuyama function for each reduced word** decomposing the long Weyl group element into simple reflections.

Statistical Models

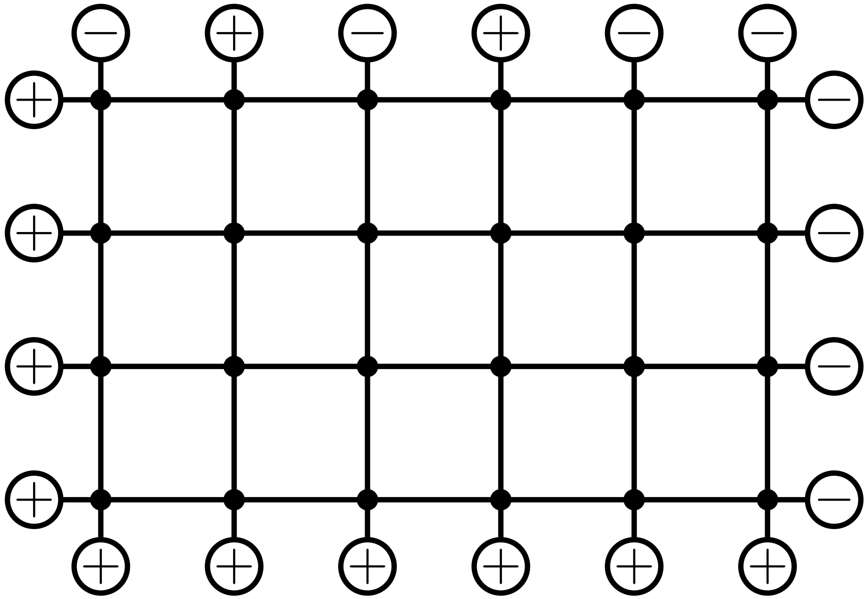
Solvable lattice models in statistical mechanics are 2-dimensional systems in which the partition function can be evaluated explicitly. The first example was the Ising model, solved by Onsager (1944). The six-vertex model is an important example.

- Solved by Lieb and Sutherland in the 1960's.
- Baxter developed the **star-triangle relation** or **Yang-Baxter equation** as a powerful tool.
- Hamel and King showed how characters (together with deformed Weyl denominators) are **partition functions** of systems of this type.
- Brubaker, Bump and Friedberg showed how to use the Yang-Baxter equation to investigate these models.
- Metaplectic Whittaker functions can also be represented as such partition functions.

Six-Vertex Model

We describe a statistico-physical system $\mathfrak{S}$. Take a square lattice of finite size.

For Example :

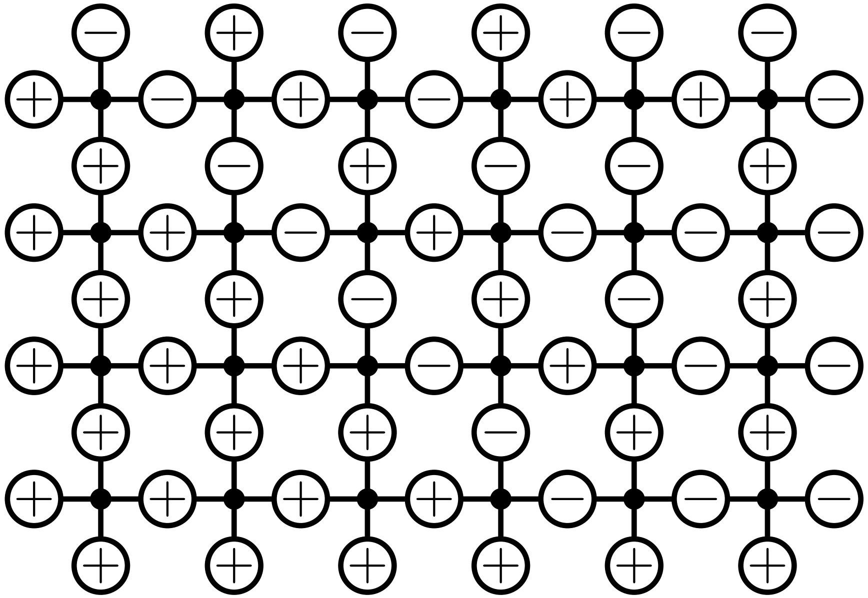


To specify the system, we require some further data.

- Signs or spins $\pm$ on the boundary edges are fixed.
- At each vertex v there are assigned six values $a_1(v)$, $a_2(v)$, $b_1(v)$, $b_2(v)$, $c_1(v), c_2(v)$ which are also part of the data defining the system.

States

- A **state** $\mathfrak{s}$ of the system $\mathfrak{S}$ consists of an assignment of signs $\pm$ to the **interior edges**.
- Recall that the signs of the boundary edges are fixed.

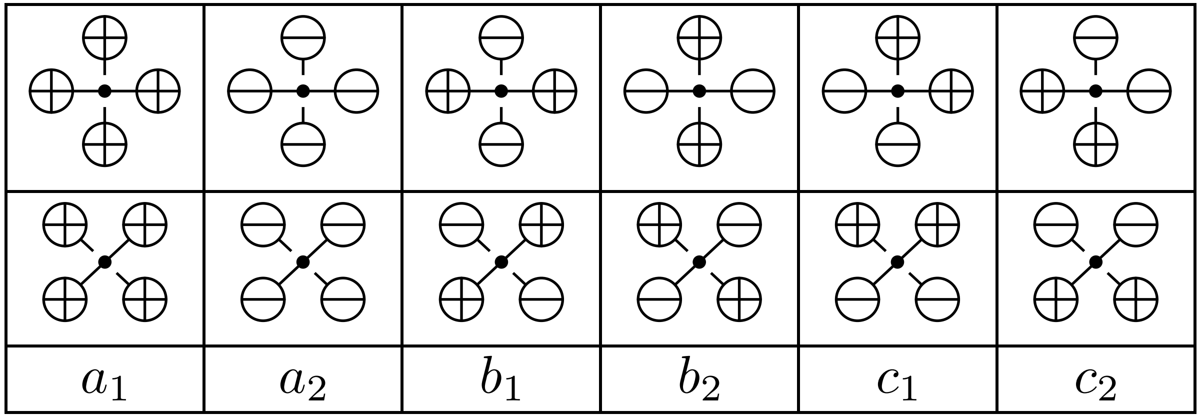


For example, here is a state of the system shown earlier.

We will also consider more general planar graphs in which some of the edges are rotated.

The Partition Function

Given a state of the system, every vertex v is assigned a value $\beta_{\mathfrak{s}}(v)$, its **Boltzmann weight**. This is either zero or one of the six values $a_1(v), a_2(v), b_1(v), b_2(v), c_1(v), c_2(v)$.



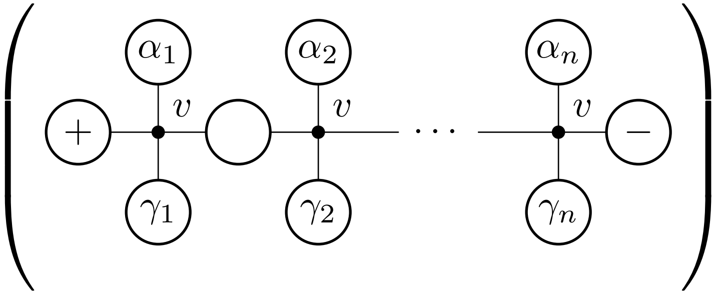
(We are also showing the weights in a rotated configuration.)

- If the weight does not appear in the table it is **zero**.
- Given the state $\mathfrak{s}$, the **Boltzmann weight** $\beta(\mathfrak{s}) = \prod_v \beta_{\mathfrak{s}}(v)$.
- The **Partition function** $Z(\mathfrak{S}) = \sum_{\text{states } \mathfrak{s}} \beta(\mathfrak{s})$.

Transfer Matrices

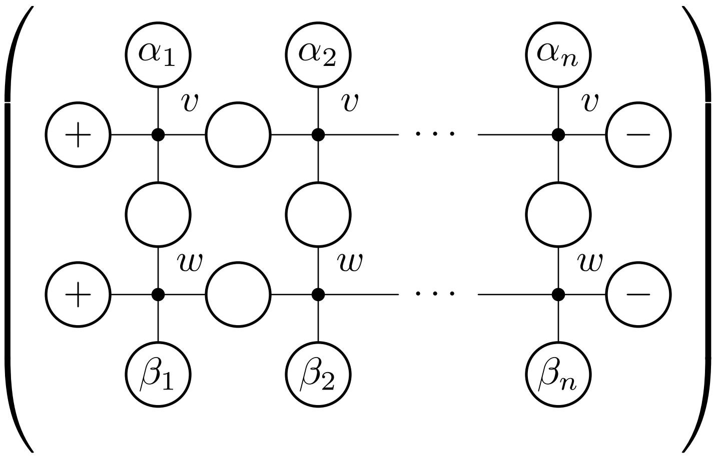
Let v be a vertex type with Boltzmann weights $a_i(v), b_i(v), c_i(v)$. Define

$$V_v(\alpha, \gamma) = Z$$



where $\alpha_i, \gamma_i \in \{\pm\}$. There are 2^n possibilities for $\alpha = (\alpha_1, \cdots, \alpha_n)$, so we think of V_v as being a $2^n \times 2^n$ matrix, the **row transfer matrix** for v. Clearly

$$Z$$



$$= \sum_{\gamma} V_v(\alpha, \gamma) V_w(\gamma, \beta).$$

We may compute the partition function by multiplying transfer matrices!

Baxter

- Baxter: organize transfer matrices into commuting families.
- A maximal commuting family of operators is like a maximal torus.
- This leads to evaluation of the partition function.

Example: the Field-Free Case

Suppose $a_1(v) = a_2(v) = a(v)$, $b_1(v) = b_2(v) = b(v)$, $c_1(v) = c_2(v) = c(v)$. Let

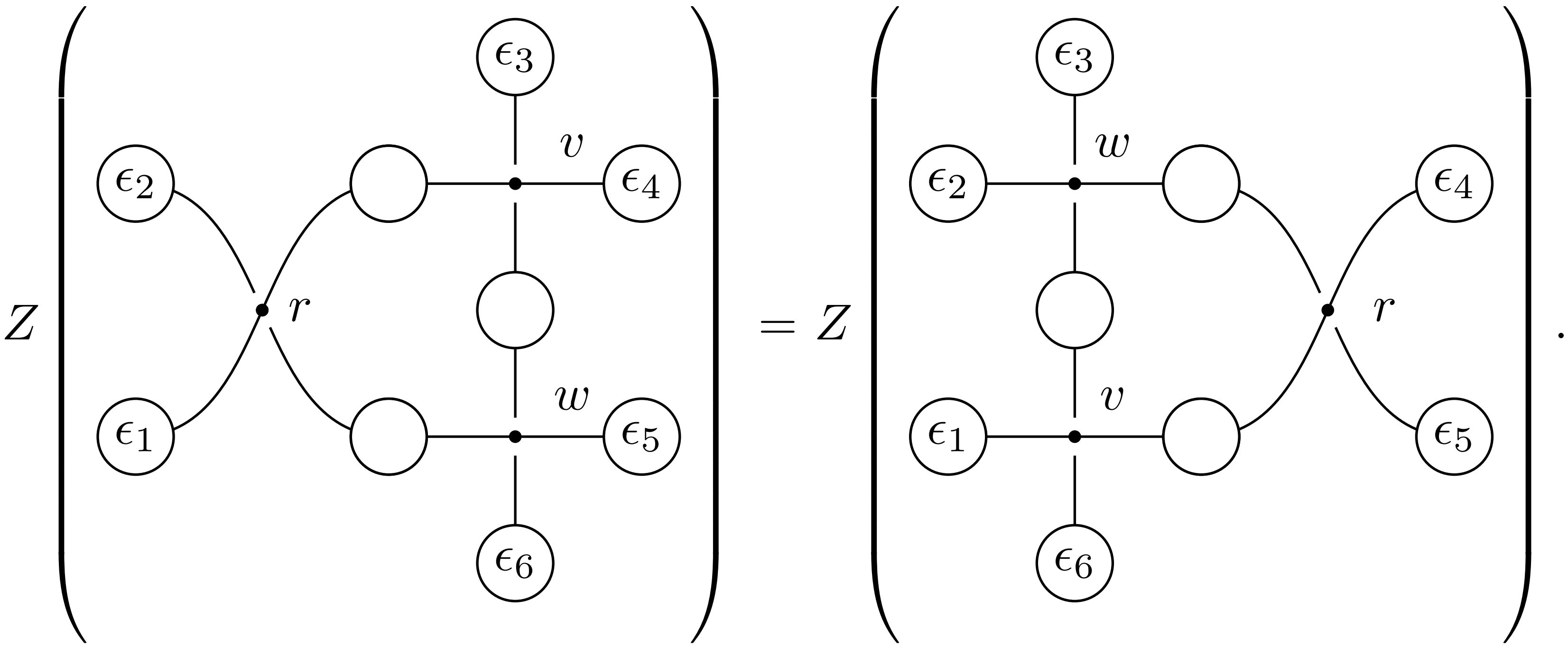
$$\Delta(v) = \frac{a(v)^2 + b(v)^2 - c(v)^2}{2\,a(v)\,b(v)}.$$

Theorem. (Baxter) *If* $\Delta(v) = \Delta(w)$ *then* V_v *and* V_w *commute.*

Proof. Use the **Yang-Baxter equation.** □

The Yang-Baxter Equation

Let v, w, r be three types of vertices, with Boltzmann weights $a_i(x)$, $b_i(x)$, $c_i(x)$ for $x \in \{v, w, r\}$. Then we write $[\![\boldsymbol{r}, v, w]\!] = 0$ if for all $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6 \in \{\pm\}$:

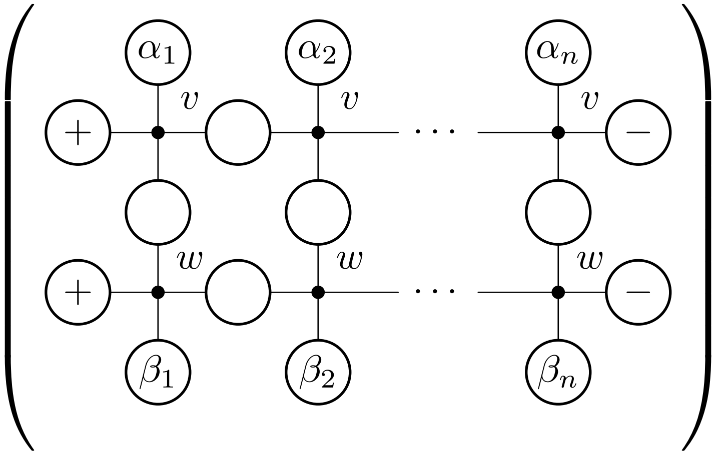


This means that summing over the three unlabeled edges gives the same result on both sides.

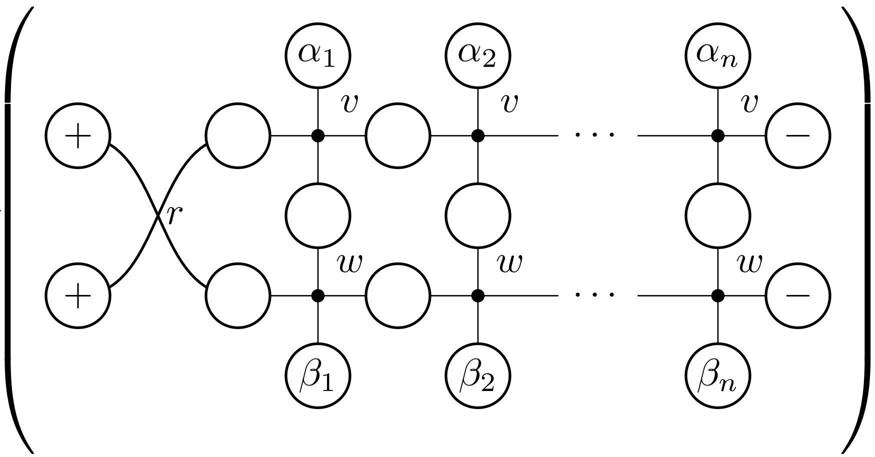
Lemma. (Baxter) *If* $\Delta(v) = \Delta(w) = \Delta$ *there exists a third field-free vertex* r *with* $\Delta(r) = \Delta$ *such that* $[\![r, v, w]\!] = 0$.

The R-matrix in action ...

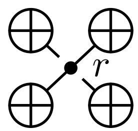
To prove Baxter's commutativity in the field-free case, that $Z(\mathfrak{S})$,

$\mathfrak{S} =$ 

is unchanged if v and w are interchanged, attach the **R-matrix** vertex r:

Z 

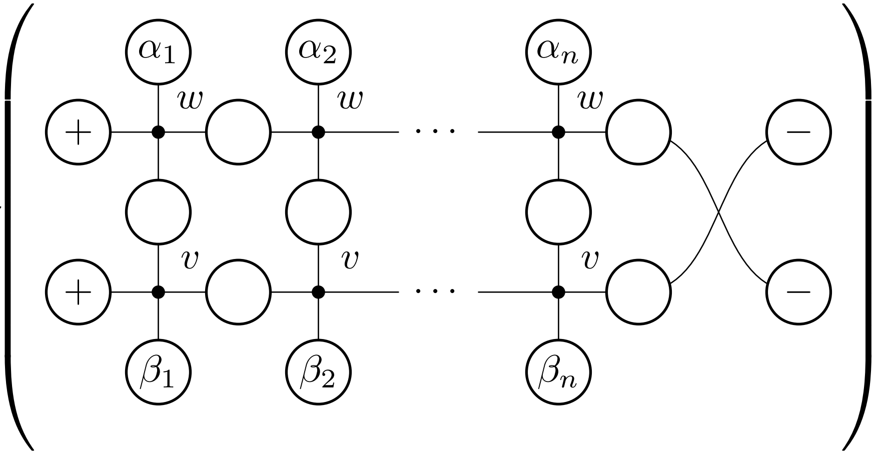
$= a_1(r) Z(\mathfrak{S}),$

because $a_1(r)$ is the value of 

, the only legal configuration at r.

... Yang-Baxter equation n times ...

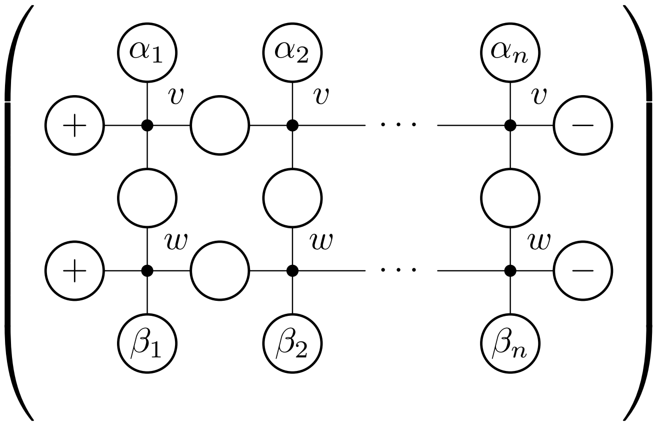
$= Z$



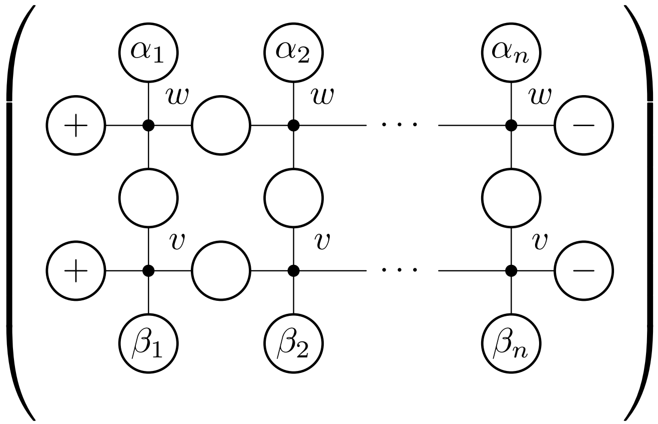
$= a_2(r) Z(\mathfrak{S}'),$

where $\mathfrak{S}'$ is the system $\mathfrak{S}$ with **v and w interchanged**. Since $a_1(r) = a_2(r)$ we may cancel them and get $Z(\mathfrak{S}) = Z(\mathfrak{S}')$.

Z



$= Z$



That is, **the transfer matrices V_v and V_w commute**, **as promised**.

Parametrized Yang-Baxter Equation

Let $\Delta \in \mathbb{C}$ be fixed and let $\boldsymbol{R}_\Delta$ be the set of field-free Boltzmann weights

$$a_1 = a_2 = a, \qquad b_1 = b_2 = b, \qquad c_1 = c_2 = c, \qquad \frac{a^2 + b^2 - c^2}{2ab} = \Delta. \qquad \text{Recall:}$$

Lemma. (Baxter) *If $\Delta(v) = \Delta(w) = \Delta$ there exists a third field-free vertex r with $\Delta(r) = \Delta$ such that $[\![r, v, w]\!] = 0$.*

We have actually a **parametrized Yang-Baxter equation.**

Theorem. (Baxter) *There is a map $R: \mathbb{C}^\times \longrightarrow R_\Delta$ such that*

$$[\![R(t), R(tu), R(u)]\!] = 0. \qquad \text{So } r = R(t), v = R(tu), w = R(u).$$

Discard the field free assumption and impose **free Fermionic condition**. Let:

$$\boldsymbol{R}_{\text{ff}} = \{v | a_1(v) a_2(v) + b_1(v) b_2(v) - c_1(v) c_2(v) = 0\}.$$

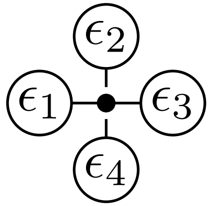
Lemma. (BBF) *There is a map $R: \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \longrightarrow \boldsymbol{R}_{\text{ff}}$ such that*

$$[\![R(t), R(tu), R(u)]\!] = 0.$$

The Yang-Baxter Commutator

Let $V=\mathbb{C}^2$ with basis $+$ and $-$. Given $T\in\mathrm{End}(V\otimes V)$ let

$$T(\varepsilon_i\otimes\varepsilon_j)=\sum_{k,l}T_{ij}^{kl}\cdot\varepsilon_k\otimes\varepsilon_l \qquad (\varepsilon_i\in\{\pm\}).$$

We interpret the coefficients T_{ij}^{kl} as a Boltzmann weight of 

. With respect to basis $+\otimes+$, $+\otimes-$, $-\otimes+$, $-\otimes-$ of $V\otimes V$ the vertex v is

$$\text{the linear transformation with matrix} \begin{pmatrix} a_1(v) & & & \\ & b_1(v) & c_1(v) & \\ & c_2(v) & b_2(v) & \\ & & & a_2(v) \end{pmatrix}.$$

If $T\in\mathrm{End}(V\otimes V)$ let $T_{ij}\in\mathrm{End}(V\otimes V\otimes V)$ be T acting on the i-th and j-th components and I_V acts on the k-th component $(k\neq i,\ j)$. The **Yang-Baxter commutator** is

$$[\![A,B,C]\!]=A_{12}\,B_{13}C_{23}-C_{23}B_{13}A_{12}.$$

Quantum Groups

With this framework many (Faddeev, Kulish, Sklyanin, Kirillov, Reshetikhin, Takhtadjan, **Jimbo**, Miwa, **Drinfeld**, ...) sought an explanation for the **Yang-Baxter equation**. This led to the invention of **Quantum groups**.

The explanation for Baxter's parametrized YBE

$$[\![R(t), R(tu), R(u)]\!] = 0$$

is that $V(t)$ is a module for the Hopf algebra $H = U_q(\hat{\mathfrak{sl}}_2)$ (completed) and there is an element $R \in H \otimes H$ that induces an endomorphism $V(t) \otimes V(u)$ for every pair of modules. The quantum group H is a **quasitriangular Hopf algebra** which means that R satisfies conditions implying the Yang-Baxter equation.

Question 1: Give a similar treatment of the Free fermionic case.

Question 2: Extend the free Fermionic story to the eight vertex model.

Schur Polynomials

- Hamel and King extended Tokuyama's deformation for Cartan Type A_r by giving a generalized deformation. They also treated Cartan Type C_r.
- Brubaker, Bump and Friedberg found two families of deformations, one of which is Hamel and King's. These are called **Gamma ice** and **Delta ice.**
- They gave proofs based on the Yang-Baxter equation.
- Fix a partition $\lambda = (\lambda_1, \cdots, \lambda_n)$.
- Let $z_1, \cdots, z_n \in \mathbb{C}^\times$ be **spectral parameters**.
- Let $t_1, \cdots, t_n \in \mathbb{C}$ be **deformation parameters.**

The character $\xi_\lambda(\boldsymbol{z})$ is the **Schur polynomial** $s_\lambda(\boldsymbol{z})$.

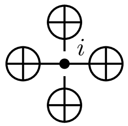
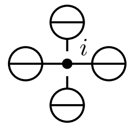
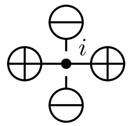
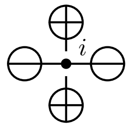
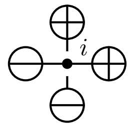
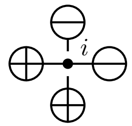
There are two statistical systems $\mathfrak{S}_\lambda^\Gamma$ and $\mathfrak{S}_\lambda^\Delta$ with

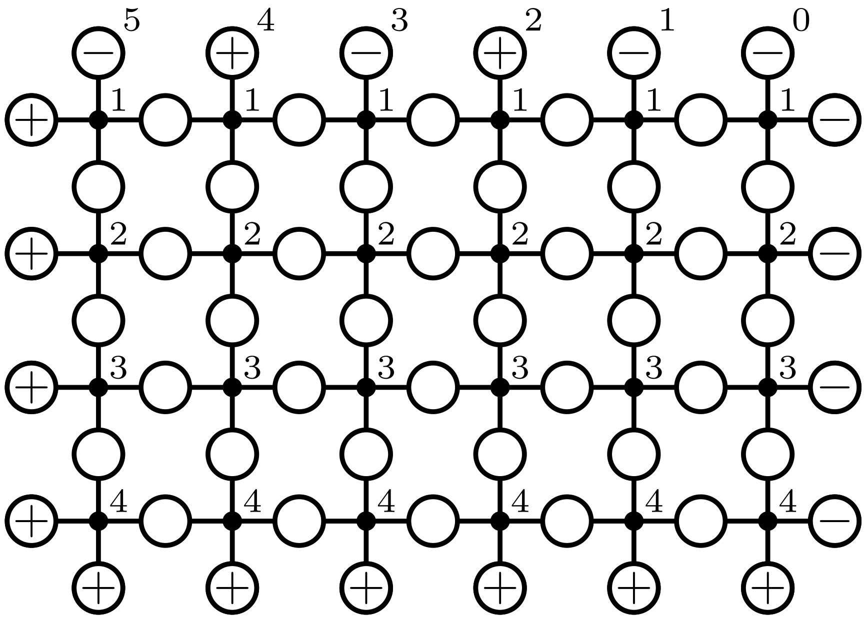
$$Z(\mathfrak{S}_\lambda^\Gamma) = \prod_{i<j} (t_i z_j + z_i) s_\lambda(z_1, \cdots, z_n), \qquad Z(\mathfrak{S}_\lambda^\Delta) = \prod_{i<j} (t_j z_j + z_i) s_\lambda(z_1, \cdots, z_n).$$

Gamma Ice

Label columns $0, 1, 2, \cdots$ **from right to left**, rows $1, 2, 3, \cdots, n$ top to bottom.

Use these weights in the i-th row:

					
$a_1(i)$	$a_2(i)$	$b_1(i)$	$b_2(i)$	$c_1(i)$	$c_2(i)$
1	z_i	t_i	z_i	$z_i(t_i+1)$	1



$+$ on left and bottom boundary edges,
$-$ on right boundary edges.

On top edges in $\rho+\lambda$ put $-$,
On remaining top edges put $+$.

Theorem: $Z(\mathfrak{S}^{\Gamma}_{\lambda}) = \prod_{i<j} (t_i z_j + z_i) s_\lambda(z_1, \cdots, z_n)$.

Example: $\boldsymbol{\lambda = (2,1,0,0)}$
$\boldsymbol{\lambda + \rho = (5,3,1,0)}$

Tokuyama Function

Recall that a **Tokuyama function** is a map $G\colon \mathcal{B}_{\lambda+\rho} \times \mathbb{C} \longrightarrow \mathbb{C}$ such that

$$\left[\prod_{\alpha\in\Phi^+} (1+t\boldsymbol{z}^{\alpha})\right]\xi_\lambda(\boldsymbol{z}) = \sum_{v\in\mathcal{B}_{\lambda+\rho}} G(v,t)\boldsymbol{z}^{\mathrm{wt}(v)}.$$

- There are different Tokuyama functions (one for every reduced decomposition of the long Weyl group element) but we discuss a particular one.
- This Tokuyama function has another description using an embedding of Berenstein, Zelevinsky, Lusztig, Littelmann (**BZL**) of $\mathcal{B}_{\lambda+\rho}$ into cones.

We will describe an injection

$$c\colon \{\text{states of } \mathfrak{S}_\lambda^\Gamma\} \longrightarrow \mathcal{B}_{\lambda+\rho}.$$

When all $t_i = t$

$$\beta(\mathfrak{s}) = G(v,t)\boldsymbol{z}^{\mathrm{wt}(v)}, \qquad v = c(\mathfrak{s}),$$

and $G(v,\, t) = 0$ if v is not in the image of c. Thus the **nonzero** terms in the crystal description coincide with the states of the ice.

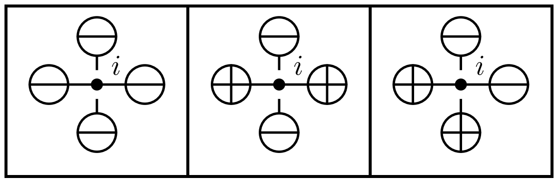
A subtle shift of viewpoint

The **nonzero** terms in the crystal description coincide with the states of the statistical model. So we might think that the two descriptions are identical. However there is a **subtle shift of viewpoint** between the two pictures.

- The image of c is (in some sense) **most** but **not all** of $\mathcal{B}_{\lambda+\rho}$.
- The tool sets are **different** in the two pictures **since c is not bijective**.
- The image of c is not stable under the Schützenberger involution of $\mathcal{B}_{\lambda+\rho}$, so that involution has no significance in the statistical picture.
- But the Yang-Baxter equation is not available in the crystal picture.
- An aggravating fact about the crystal picture is that the terms in the sum are **usually invariant** under the Schützenberger involution, yet there are some **on the boundary of the BZL polytopes** that are **not invariant** under the involution. Using the involution to understand the sums leads one to **group these exceptional terms together in packets** resulting in COMBINATORIAL DIFFICULTIES.
- **These are surmountable** but the Yang-Baxter equation is a **welcome alternative**.

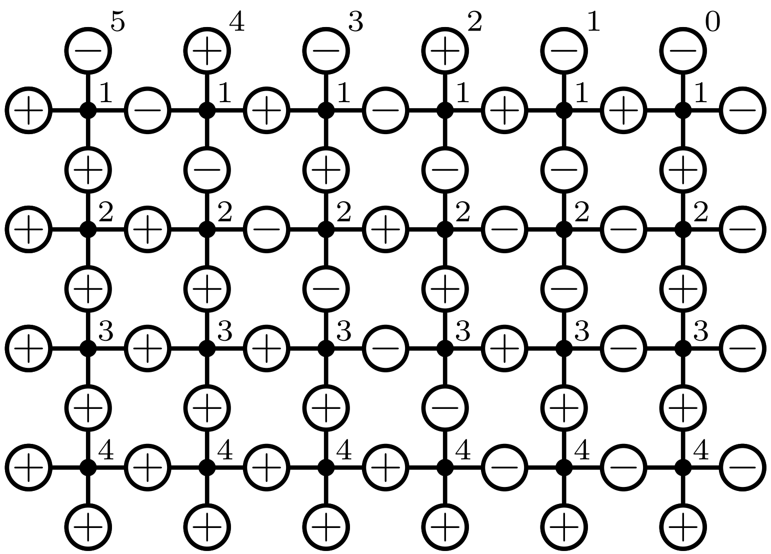
Associate a Gelfand-Tsetlin pattern with a State

- Identify states with **strict Gelfand-Tsetlin Patterns**.



(A **Gelfand-Tsetlin Pattern** is a triangular array of partitions of descending length whose rows interleave.)

For each row, write down the column numbers of vertices in the above 3 configurations (having a $-$ above the vertex). **Example:**



$\Longrightarrow$

$$\left\{ \begin{array}{ccccccc} 5 & & 2 & & 1 & & 0 \\ & 4 & & 2 & & 1 & \\ & & 3 & & 1 & & \\ & & & 2 & & & \end{array} \right\}$$

The pattern is **strict** meaning each row is strictly decreasing.

Associate a tableaux with that Gelfand-Tsetlin P.

Striking all n's, then all $n-1$'s, etc. from a tableau gives a sequence of shapes.

$$\begin{array}{|c|c|c|c|c|}\hline 1&1&2&3&4\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array} \Rightarrow \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array} \Rightarrow \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2\\\cline{1-1}\end{array} \Rightarrow \begin{array}{|c|c|}\hline 1&1\\\hline\end{array}$$

$$\{5,2,1,0\} \qquad \{4,2,1\} \qquad \{3,2\} \qquad \{2\}$$

Taking those shapes and arranging them gives a Gelfand-Tsetlin pattern:

$$\left\{\begin{array}{ccccccc} 5 & & 2 & & 1 & & 0 \\ & 4 & & 2 & & 1 & \\ & & 3 & & 1 & & \\ & & & 2 & & & \end{array}\right\}$$

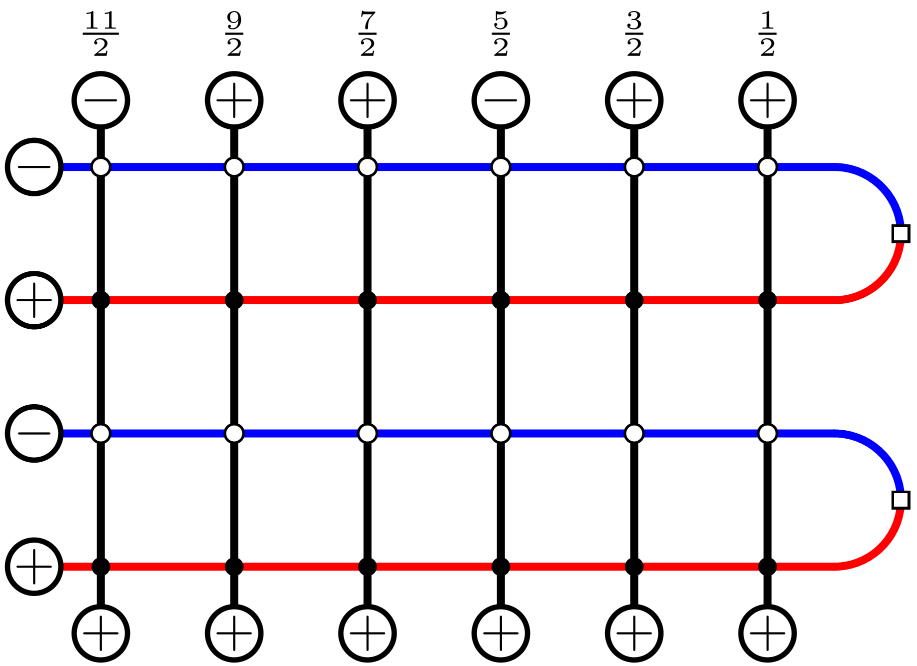
Thus

- States correspond to **strict** Gelfand-Tstelin patterns with top row $\lambda + \rho$.
- Gelfand-Tsetlin patterns biject with tableaux with shape $\lambda + \rho$.
- Not all patterns are strict so the map c is an injection but not a bijection.

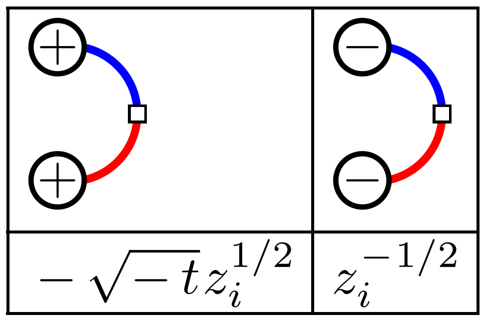
Metaplectic Ice

- For Type A and **arbitrary** metaplectic covers, there are ice models.
- Key facts amount to **commutativity of transfer matrices.**
- Still the **Yang-Baxter equation remains elusive for Type A.**

But there is a model for the Whittaker function **on the metaplectic double cover of $\mathbf{Sp_4(F)}$** where the **Yang-Baxter equation** plays a **significant role.**



Use Delta ice on **Blue rows**
Use Gamma ice on **Red rows**
For the “cap vertices” use the **metaplectic weights:**



(for the i-th pair of rows)

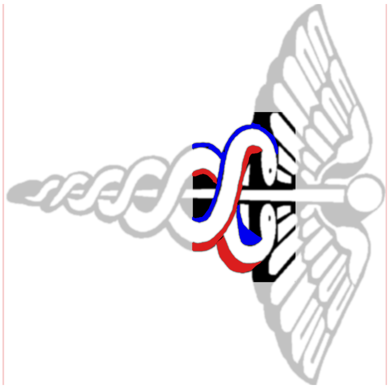
Related Nonmetaplectic Work

- Related to **U-Turn Ice** and **Alternating sign matrices** of Kuperberg, Okada and Hamel and King.
- Those models are related to work of Beineke, Brubaker and Frechette on crystal models for Type C (nonmetaplectic).
- Thesis of Dmitriy Ivanov introduces **Yang-Baxter equation** in such models introducing a novel **caduceus relation** which we will discuss.

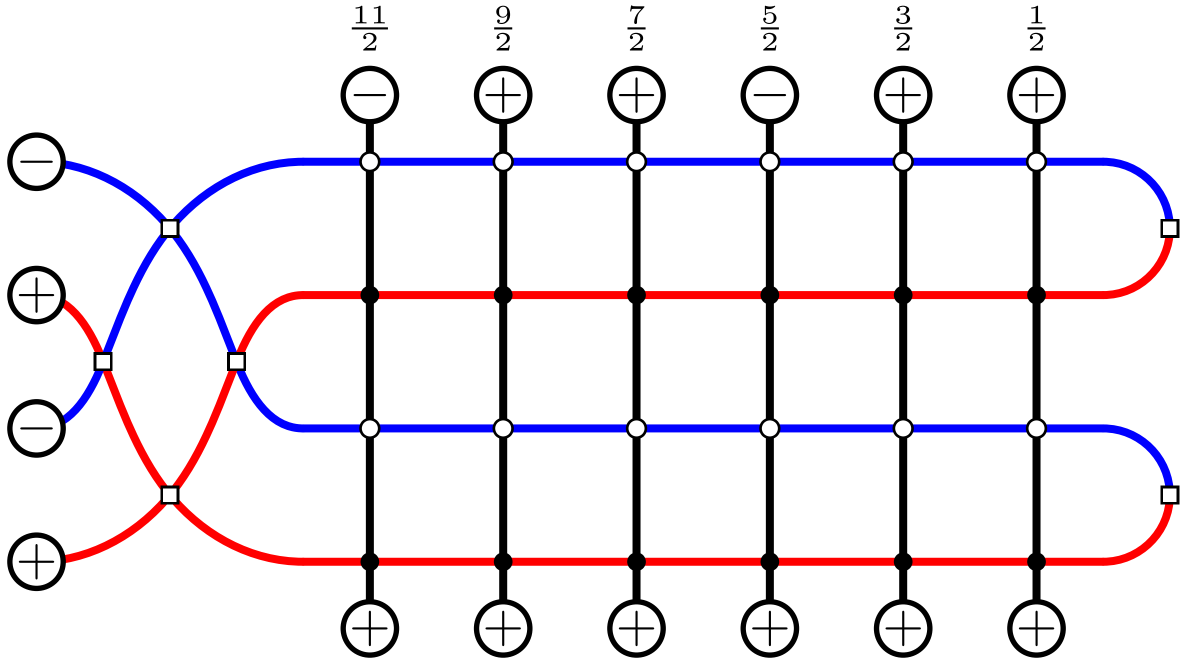
The Caduceus

The so-called

Caduceus braid



bears a noted resemblance
to the fabled “staff of Hermes”

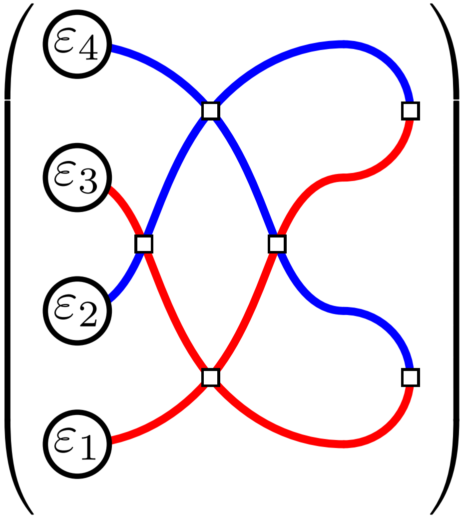


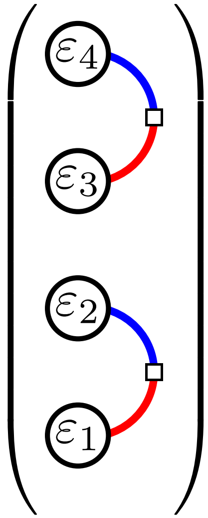
We have attached a **caduceus braid** preparing to prove a functional equation with respect to the **first simple reflection in the Weyl group.**

- The caduceus braid first appeared in the thesis of D. Ivanov.
- This multiplies $Z(\mathfrak{S})$ by $(tz_j + z_i^{-1})(z_i + tz_j)(tz_i^{-1} + z_j^{-1})(tz_i + z_j^{-1})$.
- Using the Yang-Baxter equation, the caduceus moves to the right.

The Caduceus Identity

Lemma. *For any* $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ *we have*

Z 

$= \text{const} \times Z$ 

where the constant is $(t z_i + z_j^{-1})(t z_i + z_j)(t z_i^{-1} + z_j^{-1})(t z_j + z_i^{-1})$, *independent of the* ε_i.

- Discarding the caduceus this way shows how the partition function changes under the interchange of spectral parameters.
- We are aware of caduceus identities for three different sets of cap weights. (The archetype is in Ivanov's thesis.)