Bijective Proofs of Schur Function and Symplectic Schur Function Identities

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The Starting Point

Given $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)\ldots$

$$\prod (x_i + y_j)$$

Two Flavours

1.
$$\prod_{i<j}(x_i + y_j)$$

2.
$$\prod_{i=1}^{n}\prod_{j=1}^{m}(x_i + y_j)$$

Background

- partitions
- tableaux
- Schur functions, s_λ
- jeu de taquin

Motivations

Motivation 1:

For all $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *and* $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ *we have*

$$\prod_{1 \le i < j \le n} (x_i + y_j) = \sum_{A \in \mathcal{A}(n)} \prod_{k=1}^{n} {x_k}^{NE_k(A)} {y_k}^{SE_k(A)} (x_k + y_k)^{NS_k(A)}.$$

where $\mathcal{A}(n)$ is the set of alternating sign matrices and NE, SE, and NS are various parameters associated to them. [Chapman 2001]

Motivation 2:

$$\prod_{i=1}^{m} \prod_{j=1}^{n} (x_i + y_j) = \sum_{\lambda \subseteq (n^m)} s_\lambda(\mathbf{x})\ s_{\lambda^\dagger}(\mathbf{y}) \qquad (1)$$

[Littlewood 1950]

Main Results

First Main Result:

$$Q_\mu(\mathbf{x}/\mathbf{y}) \quad = \quad s_\lambda(\mathbf{x}) \ \prod_{1\le i\le j\le n} (x_i + y_j).$$

Second Main Result—A new proof of:

$$\prod_{i=1}^{m}\prod_{j=1}^{n} (x_i + \overline{x}_i + y_j + \overline{y}_j) = \sum_{\lambda\subseteq(n^m)} sp_\lambda(\mathbf{x},\overline{\mathbf{x}}) \ sp_{\lambda^\dagger}(\mathbf{y},\overline{\mathbf{y}}) \qquad (2)$$

Tokuyama's Result

$$\prod_{i=1}^{n} x_i \prod_{1\le i<j\le n} (x_i + t x_j)\ s_\lambda(\mathbf{x})$$
$$= \sum_{ST\in\mathcal{ST}^\mu(n)} t^{\mathrm{hgt}(ST)}(1+t)^{\mathrm{str}(ST)-n}\ \mathbf{x}^{\mathrm{wgt}(ST)}.$$

where $\mathrm{str}(ST)$ is the total number of disjoint connected components of all the ribbon strips, $\mathrm{hgt}(ST)$ is the height of the tableau, and $\mathrm{wgt}(ST) = (w_1, w_2, \ldots, w_n)$, where w_k is the number of times k appears in ST for $k = 1, 2, \ldots, n$.

Related to . . .

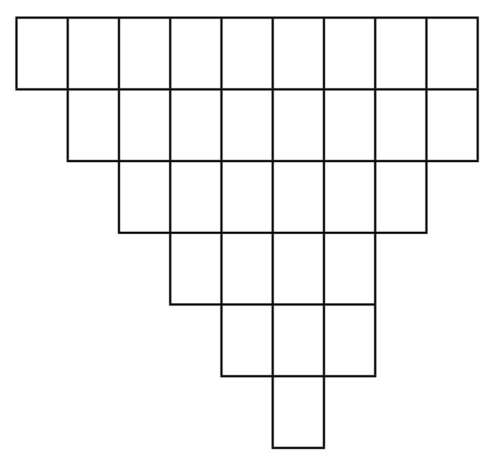
A result of Brubaker, Bump, and Friedberg (2010) showing the product of a Schur function times a deformed $x_j + x_i$ product is the partition function of a six vertex model:

$$Z(\mathfrak{S}_\lambda^\Delta) = \prod_{i<j}(t_j z_j + z_i) s_\lambda(z_1, \ldots, z_n)$$

Part One

Partitions

- A partition $\mu = (\mu_1, \mu_2, \ldots, \mu_q)$ of length $\ell(\mu) = q$ is said to be a strict partition if all the parts of μ are distinct; that is, $\mu_1 > \mu_2 > \cdots > \mu_q > 0$.
- A strict partition μ defines a shifted Young diagram SF^μ consisting of q rows of boxes of lengths $\mu_1, \mu_2, \ldots, \mu_q$ left-adjusted to a diagonal line.



(3)

Filling the tableau

Let $QST^{\mu}(n)$ be the set of all primed semistandard shifted tableaux QST obtained by numbering all the boxes of SF^{μ} with entries taken from the set $\{1', 1, 2', 2, \ldots, n', n\}$, subject to the total ordering $1' < 1 < 2' < 2 < \cdots < n' < n$. The numbering must be such that the entries are:

- QST1 weakly increasing across each row from left to right;
- QST2 weakly increasing down each column from top to bottom;
- QST3 with no two identical unprimed entries in any column;
- QST4 with no two identical primed entries in any row;

Example

$$QST = \begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 2' & 2 & 2 & 3 & 3 & 5 \\
\hline
 & 2 & 2 & 3' & 3 & 4' & 5' & 5 & 6' \\
\hline
 & & 3 & 3 & 4' & 4 & 5' & 6 & \\
\hline
 & & & 4 & 5' & 5 & 5 & & \\
\hline
 & & & & 5 & 6' & 6 & & \\
\hline
 & & & & & 6 & & & \\
\hline
\end{array} \tag{4}$$

The Symmetric Function

The weight of the tableau QST is then defined to be $\mathrm{wgt}(QST) = x_1^{u_1} x_2^{u_2} \ldots x_n^{u_n} y_1^{v_1} y_2^{v_2} \ldots y_n^{v_n}$, where u_k and v_k are the number of times k and k' appear, respectively.

$$Q_\mu(\mathbf{x}/\mathbf{y}) \quad = \quad \textstyle\sum_{QST \in \mathcal{QST}^\mu(n)} (\mathbf{x}/\mathbf{y})^{\mathrm{wgt}(QST)}.$$

- When $\mathbf{x} = \mathbf{y}$, these reduce to the Schur Q-functions.

The Breakdown

The key behind the theorem is a breakdown of QST tableaux into two pieces: one of shape $\delta = (n, n-1, \ldots, 1)$ and one of shape λ for some partition λ not necessarily of distinct parts, i.e. the usual tableau associated with Schur functions.

$QST =$

1	1	1	$2'$	2	2	3	3	5
	2	2	$3'$	3	$4'$	$5'$	5	$6'$
		3	3	$4'$	4	$5'$	6	
			4	$5'$	5	5		
				5	$6'$	6		
					6			

$\longleftrightarrow$

1	$2'$	1	$4'$	$5'$	$6'$
	2	$3'$	2	$5'$	2
		3	$4'$	3	3
			4	$5'$	$6'$
				5	5
					6

,

1	2	3
3	5	5
4	6	
5		
6		

The δ shaped tableaux

Let $\delta = (n, n-1, \ldots, 1)$ and let $\mathcal{QD}^{\delta}(n)$ be the set of all primed shifted tableaux, QD, of shape δ, obtained by numbering the boxes of SF^{δ} with entries taken from the set $\{1', 1, 2', 2, \ldots, n', n\}$ in such a way that

QD1 each unprimed entry k appears only in the kth row;

QD2 each primed entry k' appears only in the kth column;

QD

The weight of the tableau QD is defined by
$\mathrm{wgt}(QD) = x_1^{u_1} x_2^{u_2} \ldots x_n^{u_n} y_1^{v_1} y_2^{v_2} \ldots y_n^{v_n}$, where u_k and v_k are the numbers of times k and k', respectively, appear in QD for $k = 1, 2, \ldots, n$. For example, for $n = 6$ we have

$$QD = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2' & 1 & 4' & 5' & 6' \\ \hline & 2 & 3' & 2 & 5' & 2 \\ \cline{2-6} & & 3 & 4' & 3 & 3 \\ \cline{3-6} & & & 4 & 5' & 6' \\ \cline{4-6} & & & & 5 & 5 \\ \cline{5-6} & & & & & 6 \\ \cline{6-6} \end{array}$$

The QD Symmetric Function

$$Q_\mu(\mathbf{x}/\mathbf{y}) = \sum_{QD \in \mathcal{QD}^\delta(n)} (\mathbf{x}/\mathbf{y})^{\mathrm{wgt}(QD)} = \prod_{1 \le i \le j \le n} (x_i + y_j).$$

First Main Result

Proposition 1 *Let* $\mu = \lambda + \delta$ *be a strict partition of length* $\ell(\mu) = n$*, with* λ *a partition of length* $\ell(\lambda) \leq n$ *and* $\delta = (n, n-1, \ldots, 1)$*. In addition, let* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *and* $\mathbf{y} = (y_1, y_2, \ldots, y_n)$*. Then*

$$Q_\mu(\mathbf{x}/\mathbf{y}) \quad = \quad s_\lambda(\mathbf{x}) \ \prod_{1 \leq i \leq j \leq n} (x_i + y_j).$$

Corollaries

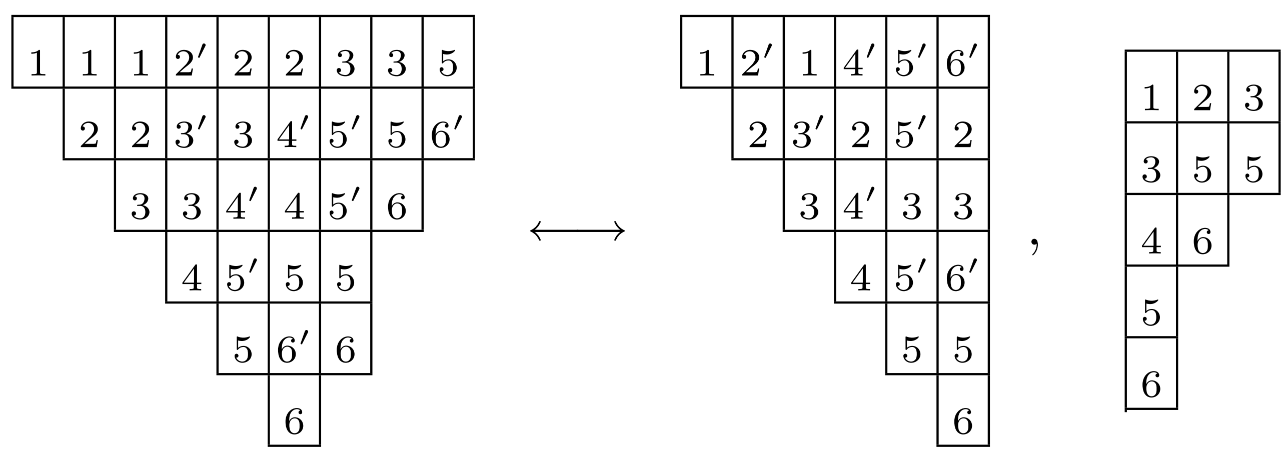
Corollary 2 (Tokuyama 1988)

$$\prod_{i=1}^{n} x_i \prod_{1\le i<j\le n} (x_i + t x_j)\ s_\lambda(\mathbf{x}) = \sum_{ST \in \mathcal{ST}^\mu(n)} t^{\mathrm{hgt}(ST)} (1+t)^{\mathrm{str}(ST)-n}\ \mathbf{x}^{\mathrm{wgt}(ST)}.$$

Proof: Set $\mathbf{y} = t\mathbf{x}$.

Main Idea of the Proof of Main Result

There exists a weight preserving, bijective map Θ from $\mathcal{Q}ST^{\mu}(n)$ to $\big(\mathcal{Q}D^{\delta}(n), \mathcal{T}^{\lambda}(n)\big)$.



Example of Proof

- The primed entries in the tableau migrate left and up to reside completely in the QD portion of the tableau.
- The key move involves sliding each k' in the north-west direction by a sequence of interchanges with either its *unprimed* northern or western neighbour until it reaches a position in the kth column either in the topmost row, or immediately below another k', or immediately below some unprimed entry i in the ith row. This amounts to playing jeu de taquin, treating k' to be strictly less than all the unprimed entries.
- The paths traced out by the primed entries k' of QST as they move northwest as far as but no further than the kth column are illustrated by means of boldface entries in the tableaux shown below.

First moving the single $2'$ gives:

1	**1**	**1**	**2′**	2	2	3	3	5
	2	2	3′	3	4′	5′	5	6′
		3	3	4′	4	5′	6	
			4	5′	5	5		
				5	6′	6		
					6			

$\longrightarrow$

1	**2′**	**1**	**1**	2	2	3	3	5
	2	2	3′	3	4′	5′	5	6′
		3	3	4′	4	5′	6	
			4	5′	5	5		
				5	6′	6		
					6			

Next the only $3'$ moves just one step west where it has, as required, reached the 3rd column. It does not move north because the entry 1 immediately above already lies in its own row:

1	$2'$	1	1	2	2	3	3	5
	2	**2**	$\mathbf{3}'$	3	$4'$	$5'$	5	$6'$
		3	3	$4'$	4	$5'$	6	
			4	$5'$	5	5		
				5	$6'$	6		
					6			

$\longrightarrow$

1	$2'$	1	1	2	2	3	3	5
	2	$\mathbf{3}'$	**2**	3	$4'$	$5'$	5	$6'$
		3	3	$4'$	4	$5'$	6	
			4	$5'$	5	5		
				5	$6'$	6		
					6			

There are two $4'$s. Under the definition of the action, the upper one must be moved first and then the lower one:

1	$2'$	1	$\mathbf{1}$	$\mathbf{2}$	2	3	3	5
	2	$3'$	2	$\mathbf{3}$	$\mathbf{4'}$	$5'$	5	$6'$
		3	3	$4'$	4	$5'$	6	
			4	$5'$	5	5		
				5	$6'$	6		
					6			

$\longrightarrow$

1	$2'$	1	$\mathbf{4'}$	$\mathbf{1}$	2	3	3	5
	2	$3'$	2	$\mathbf{2}$	$\mathbf{3}$	$5'$	5	$6'$
		3	3	$4'$	4	$5'$	6	
			4	$5'$	5	5		
				5	$6'$	6		
					6			

$=$

1	$2'$	1	$4'$	1	2	3	3	5
	2	$3'$	2	2	3	$5'$	5	$6'$
		3	$\mathbf{3}$	$\mathbf{4'}$	4	$5'$	6	
			4	$5'$	5	5		
				5	$6'$	6		
					6			

$\longrightarrow$

1	$2'$	1	$4'$	1	2	3	3	5
	2	$3'$	2	2	3	$5'$	5	$6'$
		3	$\mathbf{4'}$	$\mathbf{3}$	4	$5'$	6	
			4	$5'$	5	5		
				5	$6'$	6		
					6			

There are three $5'$s to deal with in turn from top to bottom, but the last of these is already in the 3rd column and directly below a 3 in the 3rd row, and so does not move:

1	$2'$	1	$4'$	**1**	**2**	**3**	3	5
	2	$3'$	2	2	3	$\mathbf{5'}$	5	$6'$
		3	$4'$	3	4	$5'$	6	
			4	$5'$	5	5		
				5	$6'$	6		
					6			

$\longrightarrow$

1	$2'$	1	$4'$	$\mathbf{5'}$	**1**	**2**	3	5
	2	$3'$	2	2	3	**3**	5	$6'$
		3	$4'$	3	4	$5'$	6	
			4	$5'$	5	5		
				5	$6'$	6		
					6			

=

1	$2'$	1	$4'$	$5'$	1	2	3	5
	2	$3'$	2	**2**	**3**	3	5	$6'$
		3	$4'$	3	**4**	$\mathbf{5'}$	6	
			4	$5'$	5	5		
				5	$6'$	6		
					6			

$\longrightarrow$

1	$2'$	1	$4'$	$5'$	1	2	3	5
	2	$3'$	2	$\mathbf{5'}$	**2**	3	5	$6'$
		3	$4'$	3	**3**	**4**	6	
			4	$5'$	5	5		
				5	$6'$	6		
					6			

Then we deal with the two $6'$s to give

1	2′	1	4′	5′	**1**	**2**	**3**	**5**
	2	3′	2	5′	2	3	5	**6′**
		3	4′	3	3	4	6	
			4	5′	5	5		
				5	6′	6		
					6			

$\longrightarrow$

1	2′	1	4′	5′	**6′**	**1**	**2**	**3**
	2	3′	2	5′	2	3	5	**5**
		3	4′	3	3	4	6	
			4	5′	5	5		
				5	6′	6		
					6			

=

1	2′	1	4′	5′	6′	1	2	3
	2	3′	2	5′	2	3	5	5
		3	4′	3	3	4	6	
			4	5′	**5**	5		
				5	**6′**	6		
					6			

$\longrightarrow$

1	2′	1	4′	5′	6′	1	2	3
	2	3′	2	5′	2	3	5	5
		3	4′	3	3	4	6	
			4	5′	**6′**	5		
				5	**5**	6		
					6			

This results in the juxtaposition of QD and T as claimed:

1	1	1	$2'$	2	2	3	3	5
	2	2	$3'$	3	$4'$	$5'$	5	$6'$
		3	3	$4'$	4	$5'$	6	
			4	$5'$	5	5		
				5	$6'$	6		
					6			

$\longleftrightarrow$

1	$2'$	1	$4'$	$5'$	$6'$
	2	$3'$	2	$5'$	2
		3	$4'$	3	3
			4	$5'$	$6'$
				5	5
					6

,

1	2	3
3	5	5
4	6	
5		
6		

Part Two

Symplectic tableaux

- Let $\mathcal{S}pT^{\lambda}(n)$ be the set of $sp(2n)$-tableaux T obtained by filling the boxes of F^{λ} with entries from $\{\overline{1} < 1 < \overline{2} < 2 < \cdots < \overline{n} < n\}$ such that they

  S1 weakly increase across each row from left to right;

  S2 strictly increase down each column from top to bottom;

  S3 k and $\overline{k}$ appear no lower than the kth row.

- Ex: For $n = 4$, $\lambda = (3, 3, 2, 1)$

$$T = \begin{array}{|c|c|c|} \hline \overline{1} & \overline{2} & \overline{3} \\ \hline 2 & 3 & 3 \\ \hline \overline{3} & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \quad \in \mathcal{S}pT^{3321}(4)$$

Symplectic characters and tableaux

- Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ with $\overline{\mathbf{x}}_k = x_k^{-1}$ for $k = 1, 2, \ldots, n$
- Then

$$\mathrm{ch}\, V^{\lambda}_{Sp(2n)} = sp_{\lambda}(\mathbf{x}, \overline{\mathbf{x}}) = \sum_{T \in \mathcal{S}p T^{\lambda}(n)} \mathbf{x}^{\mathrm{wgt}(T)}$$

where $\mathrm{wgt}(T)_k \;=\; \#k \in T - \#\overline{k} \in T \quad$ for $k = 1, 2, \ldots, n$

- Ex: For $n = 4$, $\lambda = (3, 3, 2, 1)$

$$T \;=\; \begin{array}{|c|c|c|} \hline \overline{1} & \overline{2} & \overline{3} \\ \hline 2 & 3 & 3 \\ \hline \overline{3} & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad \mathrm{wgt}(T) = x_1^{0-1}\, x_2^{1-1}\, x_3^{2-2}\, x_4^{2-0} = x_1^{-1}\, x_4^{2}$$

Second Main Result

Classical expression (Littlewood):

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i+y_j)=\sum_{\lambda\subseteq(n^m)} s_\lambda(\mathbf{x})\ s_{\lambda^\dagger}(\mathbf{y}) \tag{5}$$

Symplectic expression:

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i+\overline{x}_i+y_j+\overline{y}_j)=\sum_{\lambda\subseteq(n^m)} sp_\lambda(\mathbf{x},\overline{\mathbf{x}})\ sp_{\lambda^\dagger}(\mathbf{y},\overline{\mathbf{y}}) \tag{6}$$

Various proofs by . . . King; Hasegawa; Jimbo and Miwa; Terada; Bump and Gamburd

Pairs of symplectic tableaux

Let $\mathcal{R}(n,p)$ be the set of tableaux $R=(TS^\dagger)$ composed, for some $\lambda\subseteq(p^n)$, of $T\in\mathcal{S}pT^{\lambda}(n)$ and $S\in\mathcal{S}pT^{\lambda^\dagger}(p)$ reoriented so as to constitute a rectangular tableaux of shape $F^{(p^n)}$

Ex: $n=4$, $p=5$, $\lambda=(3,3,2,1)$, $\lambda^\dagger=(4,4,2,1,0)$

$T=$

$\overline{1}$	$\overline{2}$	$\overline{3}$
2	3	3
$\overline{3}$	4	
4		

$S=$

$\overline{1}'$	$1'$	$1'$	$2'$
$\overline{2}'$	$\overline{4}'$	$\overline{4}'$	$4'$
$\overline{4}'$	$4'$		
$5'$			

$R=$

$\overline{1}$	$\overline{2}$	$\overline{3}$	$4'$	$2'$
2	3	3	$\overline{4}'$	$1'$
$\overline{3}$	4	$4'$	$\overline{4}'$	$1'$
4	$5'$	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

Observation

$$\sum_{\lambda \subseteq p^n} sp_\lambda(\mathbf{x}, \overline{\mathbf{x}})\ sp_{\lambda^\dagger}(\mathbf{y}, \overline{\mathbf{y}})$$

$$= \sum_{\lambda \subseteq p^n} \sum_{T \in \mathcal{S}pT^\lambda(n)} \mathbf{x}^{\mathrm{wgt}(T)} \sum_{S \in \mathcal{S}pT^{\lambda^\dagger}(n)} \mathbf{y}^{\mathrm{wgt}(S)}$$

$$= \sum_{R \in \mathcal{R}(n,p)} (\mathbf{x}\,\mathbf{y})^{\mathrm{wgt}(R)}$$

Ex: $n = 4,\ \ p = 5,\ \ \lambda = (3,3,2,1),\ \ \lambda^\dagger = (4,4,2,1,0)$

$$R \;=\; \begin{array}{|c|c|c|c|c|}\hline \overline{1} & \overline{2} & \overline{3} & 4' & 2' \\ \hline 2 & 3 & 3 & \overline{4}' & 1' \\ \hline \overline{3} & 4 & 4' & \overline{4}' & 1' \\ \hline 4 & 5' & \overline{4}' & \overline{2}' & \overline{1}' \\ \hline \end{array} \qquad (\mathbf{x}\,\mathbf{y})^{\mathrm{wgt}(R)} = x_1^{-1}\, x_4^2\; y_1\, y_4^{-1}\, y_5$$

New rectangular tableaux

Let $\mathcal{D}(n,p)$ be the set of tableaux D obtained by filling the boxes of $F^{(p^n)}$ with entries from
$\{\overline{1} < 1 < \overline{2} < \cdots < \overline{n} < n < \overline{1}' < 1' < \overline{2}' < \cdots < \overline{p}' < p'\}$
in such a way that:

D1 each unprimed entry k or $\overline{k}$ lies in the kth row counted from top to bottom;

D2 each primed entry k' or $\overline{k}'$ lies in the kth column counted from right to left.

Typically

$$D \;=\; \begin{array}{|c|c|c|c|c|}\hline \overline{1} & 1 & \overline{1} & 2' & 1' \\ \hline 5' & 4' & 2 & \overline{2}' & \overline{2} \\ \hline \overline{3} & \overline{4}' & 3 & 2' & 1' \\ \hline 4 & \overline{4}' & 4 & \overline{2}' & \overline{1}' \\ \hline \end{array} \;\in\; \mathcal{D}(4,5)\,.$$

$$\prod_{i=1}^{n}\prod_{j=1}^{p}(x_i+\overline{x}_i+y_j+\overline{y}_j) = \sum_{D\in\mathcal{D}(n,p)} (\mathbf{x}\,\mathbf{y})^{\mathrm{wgt}(D)}$$

- $(\mathbf{x},\mathbf{y}) = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_p)$
- $\mathrm{wgt}(D)_i = \#k - \#\overline{k}$ for $i = k$ with $k = 1, 2, \ldots, n$
- $\mathrm{wgt}(D)_i = \#k' - \#\overline{k}'$ for $i = n + k$ with $k = 1, 2, \ldots, p$

Ex:

$$D \;=\; \begin{array}{|c|c|c|c|c|c} \hline \overline{1} & 1 & \overline{1} & 2' & 1' & \multicolumn{1}{c}{-1} \\ \hline 5' & 4' & 2 & \overline{2}' & \overline{2} & \multicolumn{1}{c}{0} \\ \hline \overline{3} & \overline{4}' & 3 & 2' & 1' & \multicolumn{1}{c}{0} \\ \hline 4 & \overline{4}' & 4 & \overline{2}' & \overline{1}' & \multicolumn{1}{c}{2} \\ \hline \multicolumn{1}{c}{1} & \multicolumn{1}{c}{-1} & \multicolumn{1}{c}{0} & \multicolumn{1}{c}{0} & \multicolumn{1}{c}{1} & \end{array} \;\Rightarrow\; \begin{array}{l} (\mathbf{x},\mathbf{y})^{\mathrm{wgt}(D)} \\ = x_1^{-1}\,x_4^2\;y_1\,y_4^{-1}\,y_5 \end{array}$$

Note: Entry in the (i,j)th box is any one of $\{i, \overline{i}', j', \overline{j}'\}$

Tableau Rules

$\mathcal{S}pT^{\lambda}(n)$ tableaux:

S1 weakly increase across each row from left to right;

S2 strictly increase down each column from top to bottom;

S3 k and $\overline{k}$ appear no lower than the kth row.

$\mathcal{S}pT^{\lambda^\dagger}(n)$ tableaux:

$S1^\dagger$ weakly increase up each column from bottom to top;

$S2^\dagger$ strictly increase across each row from right to left;

$S3^\dagger$ k and $\overline{k}$ appear no further left than the kth column.

Lemma

For all $n, p \in \mathbb{N}$

$$\sum_{R \in \mathcal{R}(n,p)} (\mathbf{x}\,\mathbf{y})^{\mathrm{wgt}(R)} = \sum_{D \in \mathcal{D}(n,p)} (\mathbf{x}\,\mathbf{y})^{\mathrm{wgt}(D)}$$

Proof

- Construct a weight preserving bijection between $\mathcal{R}(n,p)$ and $\mathcal{D}(n,p)$
- Use jeu de taquin to map each $R \in \mathcal{R}(n,p)$ to corresponding $D \in \mathcal{D}(n,p)$
- Move each primed entry k' or $\overline{k'}$ north-west to its own column, the kth, and then north while moving each unprimed entry i or $\overline{i}$ to its own row, the ith.
- To right of kth column maintain S1-S3 and S1†-S3†

Legitimate moves for k'

For k' in position (i, j) with $i > 1$ and $j < k$

$$\begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{b} \\ \hline \multicolumn{1}{|c|}{a} & \multicolumn{1}{c|}{k'} \\ \hline \end{array} \quad \Leftrightarrow \quad \begin{cases} \begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{k'} \\ \hline \multicolumn{1}{|c|}{a} & \multicolumn{1}{c|}{b} \\ \hline \end{array} & \text{if } a \leq b \\ \\ \begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{b} \\ \hline \multicolumn{1}{|c|}{k'} & \multicolumn{1}{c|}{a} \\ \hline \end{array} & \text{if } a > b \end{cases}$$

For k' in position $(1, j)$ with $j < k$:

$$\begin{array}{|c|c|} \hline a & k' \\ \hline \end{array} \quad \Leftrightarrow \quad \begin{array}{|c|c|} \hline k' & a \\ \hline \end{array}$$

Legitimate moves for k'

For k' in position (i, k) with $i > 1$:

$$\begin{array}{|c|}\hline b \\ \hline k' \\ \hline\end{array} \quad \Leftrightarrow \quad \begin{array}{|c|}\hline k' \\ \hline b \\ \hline\end{array} \text{if } b \leq i$$

Legitimate moves for $\overline{k}'$

For $\overline{k}'$ in position (i, j) with $i > 1$ and $j < k$:

$$\begin{array}{|c|c|} \hline \multicolumn{1}{c|}{} & b \\ \hline a & \overline{k}' \\ \hline \end{array} \quad \Leftrightarrow \quad \begin{cases} \begin{array}{|c|c|} \hline \multicolumn{1}{c|}{} & \overline{k}' \\ \hline a & b \\ \hline \end{array} & \text{if } a \leq b \\[2ex] \begin{array}{|c|c|} \hline \multicolumn{1}{c|}{} & b \\ \hline \overline{k}' & a \\ \hline \end{array} & \text{if } a > b \end{cases}$$

For $\overline{k}'$ in position $(1, j)$ with $j < k$:

$$\begin{array}{|c|c|} \hline a & \overline{k}' \\ \hline \end{array} \quad \Leftrightarrow \quad \begin{array}{|c|c|} \hline \overline{k}' & a \\ \hline \end{array}$$

Legitimate moves for $\overline{k}'$

For $\overline{k}'$ in position (i, k) with $i > 1$:

$$\begin{array}{|c|}\hline b \\ \hline \overline{k}' \\ \hline \end{array} \quad \Leftrightarrow \quad \begin{array}{|c|}\hline \overline{k}' \\ \hline b \\ \hline \end{array} \text{if } b \leq i$$

Weight preserving transformations

For k' in position (i,k) so that k' is in kth column, but blocks $\overline{k}'$ from moving to kth column:

$$\begin{array}{|c|c|}\hline k' & \overline{k}' \\ \hline \end{array} \quad \Leftrightarrow \quad \begin{array}{|c|c|}\hline i & \overline{i} \\ \hline \end{array}$$

For i in position (i,k) so that i is in ith row, but blocks $\overline{i}$ from moving to ith row:

$$\begin{array}{|c|}\hline \overline{i} \\ \hline i \\ \hline \end{array} \quad \Leftrightarrow \quad \begin{array}{|c|}\hline \overline{k}' \\ \hline k' \\ \hline \end{array}$$

Map from $R \in \mathcal{R}(n,p)$ to $D \in \mathcal{D}(n,p)$

Procedure

Identify largest primed entries. Move topmost such entry, k' or $\overline{k}'$, North-West by a sequence of interchanges with nearest neighbours until it reaches kth column and then North as far as possible in this column, while moving unprimed entries, i or $\bar{i}$, South to the ith row and changing any vertical pair $\bar{i}\,i$ to $\overline{k}'\,k'$.

$\overline{1}$	$\overline{2}$	$\overline{3}$	$4'$	$2'$
2	3	3	$\overline{4}'$	$1'$
$\overline{3}$	4	$4'$	$\overline{4}'$	$1'$
4	$5'$	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	$\overline{2}$	$\overline{3}$	$4'$	$2'$
2	3	3	$\overline{4}'$	$1'$
$\overline{3}$	4	$4'$	$\overline{4}'$	$1'$
4	$5'$	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	$\overline{2}$	$\overline{3}$	$4'$	$2'$
2	3	3	$\overline{4}'$	$1'$
$\overline{3}$	$5'$	$4'$	$\overline{4}'$	$1'$
4	4	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	$\overline{2}$	$\overline{3}$	$4'$	$2'$
2	$5'$	3	$\overline{4}'$	$1'$
$\overline{3}$	3	$4'$	$\overline{4}'$	$1'$
4	4	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	$\overline{2}$	$\overline{3}$	$4'$	$2'$
$5'$	2	3	$\overline{4}'$	$1'$
$\overline{3}$	3	$4'$	$\overline{4}'$	$1'$
4	4	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	$\overline{2}$	$\overline{3}$	$4'$	$2'$
$5'$	2	3	$\overline{4}'$	$1'$
$\overline{3}$	3	$4'$	$\overline{4}'$	$1'$
4	4	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	$\overline{2}$	$4'$	$\overline{3}$	$2'$
$5'$	2	3	$\overline{4}'$	$1'$
$\overline{3}$	3	$4'$	$\overline{4}'$	$1'$
4	4	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	$4'$	$\overline{2}$	$\overline{3}$	$2'$
$5'$	2	3	$\overline{4}'$	$1'$
$\overline{3}$	3	$4'$	$\overline{4}'$	$1'$
4	4	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	$4'$	$\overline{2}$	$\overline{3}$	$2'$
$5'$	2	3	$\overline{4}'$	$1'$
$\overline{3}$	3	$4'$	$\overline{4}'$	$1'$
4	4	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	$4'$	$\overline{2}$	$\overline{3}$	$2'$
$5'$	2	$4'$	$\overline{4}'$	$1'$
$\overline{3}$	3	3	$\overline{4}'$	$1'$
4	4	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	$4'$	$\overline{2}$	$\overline{3}$	$2'$
$5'$	$4'$	2	$\overline{4}'$	$1'$
$\overline{3}$	3	3	$\overline{4}'$	$1'$
4	4	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	$4'$	$\overline{2}$	$\overline{3}$	$2'$
$5'$	$4'$	2	$\overline{4}'$	$1'$
$\overline{3}$	3	3	$\overline{4}'$	$1'$
4	4	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	$4'$	$\overline{2}$	$\overline{4}'$	$2'$
$5'$	$4'$	2	$\overline{3}$	$1'$
$\overline{3}$	3	3	$\overline{4}'$	$1'$
4	4	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	$4'$	$\overline{4}'$	$\overline{2}$	$2'$
$5'$	$4'$	2	$\overline{3}$	$1'$
$\overline{3}$	3	3	$\overline{4}'$	$1'$
4	4	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	$4'$	$\overline{4}'$	$\overline{2}$	$2'$
$5'$	$4'$	2	$\overline{3}$	$1'$
$\overline{3}$	3	3	$\overline{4}'$	$1'$
4	4	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	1	$\overline{1}$	$\overline{2}$	$2'$
$5'$	$4'$	2	$\overline{3}$	$1'$
$\overline{3}$	3	3	$\overline{4}'$	$1'$
4	4	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	1	$\overline{1}$	$\overline{2}$	$2'$
$5'$	$4'$	2	$\overline{3}$	$1'$
$\overline{3}$	3	3	$\overline{4}'$	$1'$
4	4	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	1	$\overline{1}$	$\overline{2}$	$2'$
$5'$	$4'$	2	$\overline{3}$	$1'$
$\overline{3}$	3	$\overline{4}'$	3	$1'$
4	4	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	1	$\overline{1}$	$\overline{2}$	$2'$
$5'$	$4'$	2	$\overline{3}$	$1'$
$\overline{3}$	$\overline{4}'$	3	3	$1'$
4	4	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	1	$\overline{1}$	$\overline{2}$	$2'$
$5'$	$4'$	2	$\overline{3}$	$1'$
$\overline{3}$	$\overline{4}'$	3	3	$1'$
4	4	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	1	$\overline{1}$	$\overline{2}$	$2'$
$5'$	$4'$	2	$\overline{3}$	$1'$
$\overline{3}$	$\overline{4}'$	3	3	$1'$
4	$\overline{4}'$	4	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	1	$\overline{1}$	$\overline{2}$	$2'$
$5'$	$4'$	2	$\overline{3}$	$1'$
$\overline{3}$	$\overline{4}'$	3	3	$1'$
4	$\overline{4}'$	4	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	1	$\overline{1}$	$2'$	$\overline{2}$
$5'$	$4'$	2	$\overline{3}$	$1'$
$\overline{3}$	$\overline{4}'$	3	3	$1'$
4	$\overline{4}'$	4	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	1	$\overline{1}$	$2'$	$\overline{2}$
$5'$	$4'$	2	$\overline{3}$	$1'$
$\overline{3}$	$\overline{4}'$	3	3	$1'$
4	$\overline{4}'$	4	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	1	$\overline{1}$	$2'$	$\overline{2}$
$5'$	$4'$	2	$\overline{3}$	$1'$
$\overline{3}$	$\overline{4}'$	3	3	$1'$
4	$\overline{4}'$	4	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	1	$\overline{1}$	$2'$	$\overline{2}$
$5'$	$4'$	2	$\overline{2}'$	$1'$
$\overline{3}$	$\overline{4}'$	3	2	$1'$
4	$\overline{4}'$	4	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	1	$\overline{1}$	$2'$	$\overline{2}$
$5'$	$4'$	2	$\overline{2}'$	$1'$
$\overline{3}$	$\overline{4}'$	3	$2'$	$1'$
4	$\overline{4}'$	4	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	1	$\overline{1}$	$2'$	$1'$
$5'$	$4'$	2	$\overline{2}'$	$\overline{2}$
$\overline{3}$	$\overline{4}'$	3	$2'$	$1'$
4	$\overline{4}'$	4	$\overline{2}'$	$\overline{1}'$

$\overline{1}$	1	$\overline{1}$	$2'$	$1'$
$5'$	$4'$	2	$\overline{2}'$	$\overline{2}$
$\overline{3}$	$\overline{4}'$	3	$2'$	$1'$
4	$\overline{4}'$	4	$\overline{2}'$	$\overline{1}'$

Bijection

Thus we have a map from $R \in \mathcal{R}(n,p)$ to $D \in \mathcal{D}(n,p)$ illustrated by:

$$R \; = \; \begin{array}{|c|c|c|c|c|} \hline \overline{1} & \overline{2} & \overline{3} & 4' & 2' \\ \hline 2 & 3 & 3 & \overline{4}' & 1' \\ \hline \overline{3} & 4 & 4' & \overline{4}' & 1' \\ \hline 4 & 5' & \overline{4}' & \overline{2}' & \overline{1}' \\ \hline \end{array} \quad \Leftrightarrow \quad \begin{array}{|c|c|c|c|c|} \hline \overline{1} & 1 & \overline{1} & 2' & 1' \\ \hline 5' & 4' & 2 & \overline{2}' & \overline{2} \\ \hline \overline{3} & \overline{4}' & 3 & 2' & 1' \\ \hline 4 & \overline{4}' & 4 & \overline{2}' & \overline{1}' \\ \hline \end{array} \; = \; D$$

- Every step is reversible - the map is bijective
- The map is weight preserving
- Hence our identity is proved

Based on Two Papers ...

- A.M.Hamel and R.C.King, Bijective proofs of shifted tableau and alternating sign matrix identities, *J. Algebraic Combinatorics*, 25 (2007), 417–458.
- A.M. Hamel and R.C. King, Bijective proof of a symplectic dual pair identity, 2010, preprint.