Spherical Whittaker functions on Metaplectic $GL(r)$

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joint with Gautam Chinta
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For $v \in V^K$ and $\ell \in (V^*)^{U,\psi}$

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Earlier, Shintani (1976) obtained the formula for $\frac{W(\varpi^\lambda : y)}{W(e:y)}$.

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Yumiko Hironaka applied the Casselman-Shalika method to compute spherical functions in a case where multiplicity one fails. (on a p-adic space of Hermitian matrices).

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The groups U, K split in $\tilde{G}$ and we may consider them as subgroups of $\tilde{G}$. (we may choose a splitting of K that agrees on $U \cap K$ with the canonical splitting of U.)

A function f on $\tilde{G}$ with values in a complex vector space is called genuine if

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Let $B_* = A_*U$. For $y \in (\mathbb{C}^*)^r$ we can associate a genuine spherical character χ_y of $\tilde{B}_*$ and define the genuine spherical principal series representation $I(y) = \mathrm{Ind}_{\tilde{B}_*}^{\tilde{G}}(\chi_y)$.

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For y in general position $\{\Omega_a(y) : a \in \tilde{A}/\tilde{A}_*\}$ is a basis of the space $I(y)^{U,\psi}$.

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The coefficients were computed explicitly by K-P.

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They are also equivariant in a and b,

$$\tau_{a_* a, b_* b}(w, y) = \delta_{\tilde{B}}^{1/2}(a_* b_*^{-1})\chi_{wy}(a_*)\chi_y(b_*)^{-1}\tau_{a,b}(w, y)$$

for $a_*,\ b_* \in \tilde{A}_*$.

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$$\mathfrak{g}(m) = \sum_{u \in \mathcal{O}_F^\times / 1+\mathfrak{p}_F} (u, \varpi^m)_n \psi(\varpi^{-1} u).$$

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Theorem (Chinta-O)

For $a \in \tilde{A}$, $b \in \widetilde{A^-}$ let $b^\sharp = w_0 b^{-1} w_0^{-1}$. We have

$$W_a(b : y) = \delta_{\tilde{B}}(b) \sum_{w \in S_r} \frac{c_{w_0}(w^{-1}y)}{c_w(w^{-1}y)} \tau_{a,b^\sharp}(w, w^{-1}y).$$

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$$N(y,\lambda) = y^{-\lambda} c_{w_0}(y) \sum_{w \in S_r} j(w,y)(m_\lambda|_{\mathrm{CG}}\ w)(y)$$

where $\lambda = (0, l_2, l_2 + l_3, \ldots, l_2 + \cdots + l_r)$, $j(w,y) = \frac{e(y)}{e(wy)}$ and $e(y) = \prod_{i<j}(1 - (y_i/y_j)^n)$.

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Theorem (Chinta-O)

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The two independent computations of the spherical whittaker functions serve as a bridge between the constructions of BBF and of CG.

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Step 3: Apply the KP functional equations to obtain the other terms.

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Let $\varphi_y \in I(y)^{\mathcal{I}}$ be the unique element with support $\tilde{B}_* w_0 \mathcal{I}$ normalized by $\varphi_y(w_0) = 1$.

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Corollary

$$W_a(b : y) = \sum_{w \in S_r} \frac{c_{w_0}(w^{-1}y)}{c_w(w^{-1}y)} \Omega_a(I(b, y) T_w \varphi_{w^{-1}y}).$$

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$$\Omega_a(I(b,y)\varphi_y) = \begin{cases} \delta_{\tilde{B}}(b)\ (\delta_{\tilde{B}}^{1/2}\chi_y)(a(b^\sharp)^{-1}) & b \in \widetilde{A}^- \text{ and } \tilde{A}_* a = \tilde{A}_* b^\sharp \\ 0 & \text{otherwise} \end{cases}$$

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Now each summand over c is of the form computed in Step 2. By Step 2, only $c \in b^\sharp \tilde{A}_*$ contributes and finally we get our Theorem

$$W_a(b:y) = \sum_{w \in S_r} \frac{c_{w_0}(w^{-1}y)}{c_w(w^{-1}y)} \tau_{a,b^\sharp}(w, w^{-1}y).$$