Some challenges from number theory

Samuel Patterson

Universität Göttingen

Banff, 10th June, 2010

Let k be a global field containing the n^{th} roots of 1. We shall assume that $n > 1$ and even occasionally that $n > 2$.

Let S be a set of places of k containing all places v with $|n|_v \neq 1$.

Let R be the ring of S integers. It is often useful to assume that R is a principal ideal domain.

Let $k_S = \prod_{v \in S} k_v$. Let e be an additive character on k_S non–trivial but trivial on R. It is often useful to assume that

$$\{x \in k_S : e|xR = 1\} = R$$

(but this is sometimes not helpful).

Let $\mu_n(k)$ be the group of n^{th} roots of 1 in k. Let $\varepsilon : \mu_n(k) \to \mathbb{C}^\times$ be an injective character. Let $\left(\frac{\cdot}{\cdot}\right)$ be the n^{th} order Legendre-Jacobi symbol. Let $(\cdot, \cdot)$ be the Hilbert symbol on $k_S^\times \times k_S^\times$. Thus, if a and b are coprime in R we have

$$\left(\frac{a}{b}\right) = (a, b) \left(\frac{b}{a}\right)$$

Note - this is the opposite convention to others.

Let

$$g(r, \varepsilon, c) = \sum_{x \pmod c} \left(\frac{x}{c}\right) e(rx/c)$$

be the standard Gauss sum.

It is often necessary to worry about the dependence of these concepts on S - we shall supress that here.

We write

$$\psi^o(r,\epsilon,\eta,s) = \sum_{c \sim \eta} g(r,\epsilon,c) N(c)^{-s}$$

which converges in $\mathrm{Re}(s) > \frac{3}{2}$. Here we write $\sim$ to indicate that two element lie in the same coset of $k_S^\times / k_S^{\times n}$. The sum is taken modulo $R^{\times n}$.

This has an analytic continuation to the entire as a meromorphic function. There is at most one pole in $\mathrm{Re}(s) > 1$; it is at $s = 1 + \frac{1}{n}$. We denote the residue there by $\rho^o(r,\epsilon,\eta)$.

Goal: determine the $\rho^{o}(r,\epsilon,\eta)$.

Let π be a prime in S and let S' be the union of S and the valuation associated with π. Let $q = N(\pi)$. Then

$$\begin{aligned}\psi_S^o(r_o\pi^m,\epsilon,\eta,s) &= \psi_{S'}^o(r_o\pi^m,\epsilon,\eta,s)\frac{(1-q^{n-ns-1})-(1-q^{-1})q^{(n-ns)\left(\left[\frac{m}{n}\right]+1\right)}}{1-q^{n-ns}} * \\ &+\psi_{S'}^o(r_o\pi^{-m-2},\epsilon,\eta,s)g(r_o,\epsilon^{m+1},\pi)q^{(m+1)(1-s)-s}\epsilon(\eta,\pi)^{m+1}\end{aligned}$$

or, what is the same (with $(m)_n$ the least non-negative residue of $m \pmod n$)

$$\begin{aligned}\psi_S^o(r_o\pi^m,\epsilon,\eta,s) &= \tfrac{1-q^{n-ns-1}}{1-q^{n-ns}}\{\psi_{S'}^o(r_o\pi^m,\epsilon,\eta,s) \\ &-q^{(m+1)(1-s)}\{\psi_{S'}^o(r_o\pi^m,\epsilon,\eta,s)\frac{(1-q^{-1})q^{(1-s)(n-(m)_n)}}{1-q^{n-ns-1}} \\ &-\psi_{S'}^o(r_o\pi^{-m-2},\epsilon,\eta,s)\frac{g(r_o,\epsilon^{m+1},\pi)q^{-s}\epsilon(\eta,\pi)^{m+1}}{1-q^{n-ns-1}}\}\}\end{aligned}$$

From this one and the Periodicity Theorem one deduces that for $0 \leq m \leq n-2$ one has

$$\rho(r_o\pi^m, \epsilon, \eta) = N(\pi)^{-\frac{m+1}{n}} g(r_o, \epsilon^{m+1}, \pi)\epsilon((-\eta, \pi))^{m+1}\rho(r_o\pi^{n-m-2}, \epsilon, \eta\pi^{-m11})$$

and

$$\rho(r_o\pi^{n-1}, \epsilon, \eta) = 0.$$

These determine ρ essentially completely if $n = 2$ or 3. If $n > 3$ then they do not suffice. What happens is an open question.

Let Φ be an irreducible simple root system, i.e, one of $A_r, B_r, C_r, D_r, E_r, F_4, G_2$.

Let r be the rank of Φ. We assume that we have defined the positive roots and let $\alpha_1, \ldots, \alpha_r$ be the corresponding set of simple roots. Let V be the ambient vector space; let $(\cdot, \cdot)$ be the inner product. Let $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$. Let W be the Weyl group of Φ.

We shall assume that the short roots are of length 1.

Let Λ be the lattice generated by Φ. Let $\hat{\Lambda}$ be the dual lattice to Λ.

Let $\check{\alpha}_1, \ldots, \check{\alpha}_r$ be the dual basis to $\alpha_1, \ldots, \alpha_r$. Here we assume that $(\check{\alpha}_i, \alpha_j) = \delta_{ij}$.

Let $\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$.

Bump, Brubaker and Friedberg prove that if $d(\alpha) = 2\frac{(\rho,\alpha)}{(\alpha,\alpha)}$ then

$d(\alpha) = 1$ if α is simple.

Generally for $w \in W$

$$d(w\alpha_i) = 1 + \sum_{\alpha>0, w\alpha<0} 2\frac{(w\alpha, \alpha_i)}{(\alpha_i, \alpha_i)}$$

A second goal: we know that there are $g_\phi(r,\epsilon,c)$ which extend the classical Gauss sums in a natural fashion. Give a construction of them in the ring R analogous to the classical definition of the Gauss sum.

Brubaker, Bump, Chinta, Friedberg, Gunnells, Hoffstein introduce

$$g_\Phi(\mathbf{m}, \varepsilon, \mathbf{c})$$

for $\mathbf{m}, \mathbf{c} \in R^r$. This satisfies for $\mathbf{c} = (c_1, \ldots, c_r)$ and $\mathbf{c}' = (c_1', \ldots, c_r')$ with $\gcd(c_1 \ldots c_r, c_1' \ldots c_r') = (1)$

$$g_\Phi(\mathbf{m}, \varepsilon, \mathbf{cc}') = g_\Phi(\mathbf{m}, \varepsilon, \mathbf{c}) g_\Phi(\mathbf{m}, \varepsilon, \mathbf{c}') \varepsilon(\prod_i \left(\left(\frac{c_i}{c_i'}\right)\left(\frac{c_i'}{c_i}\right)\right)^{||\alpha_i||^2} \prod_{i<j} \left(\left(\frac{c_i}{c_j'}\right)\left(\frac{c_i'}{c_j}\right)\right)^{2(\alpha_i, \alpha_j)})$$

We also have if $\mathbf{m}' = (m_1', \ldots, m_r')$ and $\gcd(m_i', c_i) = (1)$ for $1 \leq i \leq r$ then

$$g_\Phi(\mathbf{mm}', \varepsilon, \mathbf{c}) = g_\Phi(\mathbf{m}, \varepsilon, \mathbf{c}) \varepsilon(\prod_i \left(\frac{m_i'}{c_i}\right)^{-||\alpha_i||^2})$$

Heuristically – but not in reality –

$$g_\Phi(\mathbf{m}, \varepsilon, \mathbf{c}) = \prod_j g(m_j \prod_{i<j} c_i^{-2(\alpha_i, \alpha_j)/||\alpha_j||^2}, \varepsilon^{||\alpha_j||^2}, c_j).$$

Let A be the ring of polynomials x^λ where λ is in Λ. Here $x^{\lambda+\lambda'} = x^\lambda x^{\lambda'}$. If we write $x_i = x^{\alpha_i}$ then we can identify A with $\mathbb{C}[x_1, \ldots, x_r, x_1^{-1}, \ldots, x_r^{-1}]$.

We let W act on A by

$$w(\sum_\lambda c(\lambda)x^\lambda) = \sum_\lambda c(w^{-1}\lambda)x^\lambda$$

We then have $w(w'\phi) = ww'(\phi)$.

It is convenient to write $\sum_\lambda c(w^{-1}\lambda)x^\lambda$ as $\sum_\lambda c(\lambda)x^{w\lambda}$.

We shall write $\tilde{A}$ for the field of fractions of A.

We now let n be as before and

$$n(\alpha) = n/\gcd(n, ||\alpha||^2)$$

and

$$n_i = n(\alpha_i).$$

Here $||\alpha||^2 = (\alpha, \alpha)$.

Let

$$M = \prod_{1 \leq i \leq r} \mu_{n_i}(k)$$

We let this group act on A by

$$(\zeta_1, \ldots, \zeta_r) : \sum_\lambda c(\lambda) x^\lambda \mapsto \sum_\lambda c(\lambda) \varepsilon(\zeta_1^{\lambda_1} \cdots \zeta_r^{\lambda_r}) x^\lambda$$

where $\lambda = \lambda_1 \alpha_1 + \cdots + \lambda_r \alpha_r$.

The dual group $\hat{M}$ to M is $\prod_{1 \leq i \leq r} \mathbb{Z}/n_i\mathbb{Z}$ and for $\ell = (\ell_1, \ldots, \ell_r)$ in $\hat{M}$ we let

$$A_\ell = \{\phi \in A : (\zeta_1, \ldots, \zeta_r)\phi = \varepsilon(\zeta_1^{\ell_1} \ldots \zeta_r^{\ell_r})\phi\}$$

Likewise we define $\tilde{A}_\ell$.

We have

$$\tilde{A} = \bigoplus \tilde{A}_\ell.$$

We allow ourselves parameters q and $(l_1, \ldots, l_r) \in \mathbb{Z}^r$. We also need a family $\gamma(1), \ldots, \gamma(n-1)$ satisfying $\gamma(k)\gamma(n-k) = q$. We set $\gamma(0) = 1$ and extend γ to $\mathbb{Z}$ by periodicity. Let $s_i \in W$ be the reflection corresponding to α_i. We let S_i map $\bigoplus \tilde{A}_\ell$ to itself by

$$\begin{aligned}(\phi_\lambda) \mapsto \quad & ((qx_i)^{l_i+1-(l_i+1)n_i}\phi_\lambda(q^{-1}(qx)^{s_i}))\tfrac{1-q^{-1}}{1-(qx_i)^{n_i}/q} \\ & -(\gamma(||\alpha_i||^2(l_i+1))(qx_i)^{l_i+1-n_i}\phi_{\lambda-(l_i+1)\check{\alpha}_i}(q^{-1}(qx)^{s_i})))\tfrac{1-(qx^{n_i}_{)}}{1-(qx_i)^{n_i}/q})\end{aligned}$$

Chinta and Gunnells prove that if we associate s_i to S_i then we obtain an action of W on $\tilde{A}$.

$$
\begin{aligned}
\psi^{o}_{S}(r_o\pi^{m},\epsilon,\eta,s) \quad = \quad & \tfrac{1-q^{n-ns-1}}{1-q^{n-ns}}\{\psi^{o}_{S'}(r_o\pi^{m},\epsilon,\eta,s) \\
& -q^{(m+1)(1-s)}\{\psi^{o}_{S'}(r_o\pi^{m},\epsilon,\eta,s)\tfrac{(1-q^{-1})q^{(1-s)(n-(m)n}}{1-q^{n-ns-1}} \\
& -\psi^{o}_{S'}(r_o\pi^{-m-2},\epsilon,\eta,s)\tfrac{g(r_o,\epsilon^{m+1},\pi)q^{-s}\epsilon(\eta,\pi)^{m+1}}{1-q^{n-ns-1}}\}\}
\end{aligned}
$$

$$
\begin{aligned}
(\phi_\lambda) \mapsto \quad & ((qx_i)^{l_i+1-(l_i+1)n_i}\phi_\lambda(q^{-1}(qx)^{s_i}))\tfrac{1-q^{-1}}{1-(qx_i)^{n_i}/q} \\
& -(\gamma(||\alpha_i||^2(l_i+1))(qx_i)^{l_i+1-n_i}\phi_{\lambda-(l_i+1)\check{\alpha}_i}(q^{-1}(qx)^{s_i})))\tfrac{1-(qx_)^{n_i}}{1-(qx_i)^{n_i}/q})
\end{aligned}
$$

Recall:

Bump, Brubaker and Friedberg prove that if $d(\alpha) = 2\frac{(\rho,\alpha)}{(\alpha,\alpha)}$ then

$d(\alpha) = 1$ if α is simple.

Generally for $w \in W$

$$d(w\alpha_i) = 1 + \sum_{\alpha>0, w\alpha<0} 2\frac{(w\alpha, \alpha_i)}{(\alpha_i, \alpha_i)}$$

Let

$$\Delta(x) = \prod_{\alpha>0}(1 - q^{n(\alpha)d(\alpha)}x^{n(\alpha)\alpha})$$

and

$$D(x) = \prod_{\alpha>0}(1 - q^{n(\alpha)d(\alpha)-1}x^{n(\alpha)\alpha}).$$

Then one has for $w \in W$

$$\Delta(x^w) = \Delta(x) \prod_{\alpha>0, w\alpha<0} -q^{-n(\alpha)d(\alpha)}x^{-n(\alpha)\alpha}.$$

Let

$$h(x; l, q) = \sum_{w\in W} \Delta(x^w)^{-1} w \circ 1(w)$$

where $\circ$ denotes the action above. Then

$$D(x)h(x; q, l)$$

lies in A. This is the kernel of the local construction of the $g_{\Phi}(\mathrm{m}, \varepsilon, \mathrm{c})$.

These formulae show how we should define the multiple Dirichlet series. In this form it is just a variant of the Chinta-Gunnells construction. What is still missing is a construction in the framework of the Dedekind ring R.

The Periodicity Theorem in one variable will extend in a straightforward way to the several variable case. This will mean that the linear relations between the residues correspond to the action of the non-identity elements of the Weyl group. Note the similarity to the theory of intertwining operators.

This means that there are cases in which one can expect a uniqueness result. Roughly – when the rank is small compared to n the residues will vanish, when large then there will be no way of deducing the values of the residues from finitely many. There will be some special values in between. One can guess what they might be but I prefer not to stick my neck out today.

Problems with general families of groups.

The general linear groups are not Chevalley groups and they are different from these in several ways. For example – the additive characters on N are in one orbit under the diagonal group.

Also – there is no Fourier synthesis in the sense of (Kirillov,) Piatetski-Shapiro and Shalika. That is – the general linear group has the very special subgroup whose bottom row is (0 0 ...0 1).

One can make various conjectures about the arithmetic nature of the residues. We shall not repeat them here – the problem of building up numerical evidence for them is extremely difficult as estimates of the scale of the computations should that they are vastly beyond what is practical at the moment.

Finally – a salutory point. If we could prove all that I have sketched could we make real progress towards the main problem, that of understanding the arithmetic nature of the residues.

Unfortunately the answer seems to be "no". To get further one will need radically new ideas.