Descartes’ rule of signs.

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1 Introduction.

2 Descartes’ rule of signs is exact!

3 Some questions.
Descartes’ rule of signs is easy.

Let \( f = \sum_{i=0}^{d} a_i x^i \in \mathbb{R}[x] \) be a non-zero polynomial of degree \( d \).

- \( R(f) \) is the number of positive roots of \( f \) counted with multiplicities.
- \( S(f) \) is the number of changes of signs in the sequence of coefficients of \( f \), ignoring the zeros.

**Theorem (Descartes (1637) - Gauss (1828))**

\( R(f) \leq S(f) \) and \( S(f) - R(f) \) is even.
Descartes’ rule of signs is correct.

Proven by Gauss (1828), Albert (1943), Wang (2004), ...

The proofs are based on:

**Lemma**

\[ S((x - 1)f(x)) \geq S(f) + 1. \]

**Lemma**

- \( a_0a_d > 0 \implies \) \( S(f) \) and \( R(f) \) are both even.
- \( a_0a_d < 0 \implies \) \( S(f) \) and \( R(f) \) are both odd.
Descartes’ rule of signs is sharp.

- If \( f = (x - r_1) \cdots (x - r_n) \in \mathbb{R}[x] \) where \( r_i > 0 \ \forall \ i \), then 
  \[ S(f) = R(f) = n. \]

- [Grabiner (1999)] For any sequence of signs (no zeros), there exists a non-zero \( f \in \mathbb{R}[x] \) with coefficients of the given signs and \( S(f) = R(f) \).
Descartes’ rule of signs is inexact.

- If \( f = x^2 + bx + c \in \mathbb{R}[x] \) where \( b < 0 \) and \( c > b^2/4 \), then \( S(f) = 2 \) and \( R(f) = 0 \).
- \([\text{Anderson, Jackson, Sitharam (1998)}]\) For any sequence of signs or zeros with \( n \) changes of signs and an even integer \( k \) such that \( 0 \leq k \leq n \), there exists a non-zero \( f \in \mathbb{R}[x] \) with coefficients of the given signs and \( R(f) = n - k \).
Descartes’ rule of signs is almost exact.

- [Poincare (1888)] There exists $g \in \mathbb{R}[x]$, that depends on $f$, such that $R(f) = S(fg)$.

- [Polya (1928)] If $f$ has no positive roots, then there exists $n \in \mathbb{N}_0$ such that $S((x + 1)^nf(x)) = 0$.

- [Powers, Reznick (2007)] If $f$ has no positive roots and

$$n > \left( \binom{d}{2} \right) \frac{\max_{0 \leq i \leq d} \left\{ \frac{a_i}{\binom{d}{i}} \right\}}{\min_{\lambda \in [0,1]} \{(1 - \lambda)^d f \left( \frac{\lambda}{1-\lambda} \right) \}} - d$$

then $S((x + 1)^nf(x)) = 0$. 
Descartes’ rule of signs is exact!

**Theorem (Avendano (2009))**

*For any non-zero \( f \in \mathbb{R}[x] \), the sequence \( S((x + 1)^n f(x)) \) is monotone decreasing and it stabilizes at \( R(f) \).*
Recall that \( f = a_dx^d + \cdots + a_1x + a_0 \).

Then \( (x + 1)^nf(x) = c_n^{n+d}x^{n+d} + \cdots + c_n^1x + c_n^0 \) where

\[
    c_n^k = \sum_{i=0}^d a_i \binom{n}{k-i}.
\]

Encode the (signs of the) coefficients \( c_n^k \) in the piecewise constant functions \( g_n : [0, 1) \rightarrow \mathbb{R} \) given by

\[
    g_n(\lambda) = \left( \frac{n+d}{[\lambda(n+d+1)]} \right)^{-1} c_n^{[\lambda(n+d+1)]}.
\]

\[
    \text{sgn}(c_n^k) = \text{sgn}(g_n(k/(n+d+1))).
\]
Example 1

Consider the polynomial

\[ f = (x - 2)(x - 7)(9x^6 - x^5 + 2x^4 - 4x^3 + 2x^2 + 4x + 1) \]

\[ = 9x^8 - 82x^7 + 137x^6 - 36x^5 + 66x^4 - 70x^3 - 7x^2 + 47x + 14. \]

**Figure:** Functions \( g_0(\lambda), g_1(\lambda) \) and \( g_5(\lambda) \) compared with \( g(\lambda) \).
Figure: Functions $g_{10}(\lambda)$, $g_{25}(\lambda)$ and $g_{100}(\lambda)$ compared with $g(\lambda)$. 
Show that the sequence of functions \( \{g_n\}_{n \geq 0} \) converge uniformly to

\[
g(\lambda) = (1 - \lambda)^d f \left( \frac{\lambda}{1 - \lambda} \right)
\]

in the interval \([0, 1)\).

Note that the homography \( \lambda \mapsto \frac{\lambda}{1 - \lambda} \) is a bijection from \([0, 1)\) to \([0, \infty)\). Its inverse is given by \( x \mapsto \frac{x}{x + 1} \).

For large enough \( n \), the number of sign alternations in \( c_n^k \) is equal to the number of changes of signs of \( g(\lambda) \), i.e. the number of positive roots of \( f \).
What else?

- The $n$ required to get $S((x + 1)^nf) = R(f)$ is usually (very) large. An analysis of the optimal $n$ is in progress.
- Can we change $x + 1$ by some other polynomial?
- For large enough $n$, the coefficients of $(x + 1)^nf(x)$ and the values of $f$, after some normalization, almost coincide. Can we use this for finding roots?
- The proof uses that a Binomial probability distribution can be approximated well by a Poisson distribution. Also, we are multiplying by powers of $(x + 1)$. Is this technique related with random walks?
What is a Descartes’ rule of signs?

Let $\mathcal{M}$ be the set of sequences of real numbers indexed by the non-negative integers, with finite support. We use this sequences to encode the coefficients of polynomials in $\mathbb{R}[x]$.

Consider a function $\hat{S} : \mathcal{M} \rightarrow \mathbb{N}_0$ such that:

1. $\hat{S}(\square \ast a) \leq \hat{S}(a)$
2. $\hat{S}(a) \geq \text{"positive regions in } a\text{"} + \text{"negative regions in } a\text{"} - 1$

for all $a \in \mathcal{M}$. Then $\hat{S}$ is a DRS, i.e. $R(f) \leq \hat{S}(f)$ for all $f \in \mathbb{R}[x]$.

Here $\ast$ denotes convolution of sequences (or multiplication of polynomials) and $\square$ corresponds to the binomial $1 + x$. 

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Descartes’ rule of signs.
Is there any other Descartes’ rule of signs?

Yes, sure!

Define \( \hat{S}(a) \) as the number of times the sequence changes from \(+\) to \(−\) plus twice the number of changes from \(−\) to \(+\). This gives a DRS.

Want more?

For any sequence \( a \in M \) define \( \hat{a} \in M \) by

\[
\hat{a}_n = \sum_{i=n}^{\infty} a_i \binom{i}{n} (-1)^{i-n}.
\]

Then the function \( \hat{S} : M \rightarrow \mathbb{N}_0 \) given by \( \hat{S}(a) = S(\hat{a}) \) is a DRS.
Let $\mathcal{M}_2$ denote the set of two-dimensional sequences (indexed by $\mathbb{N}_0 \times \mathbb{N}_0$) of real numbers with finite support. Consider a function $\hat{S} : \mathcal{M}_2 \to \mathbb{N}_0$ such that

1. $\hat{S}(□□ * a) \leq \hat{S}(a)$
2. $\hat{S}(□ □ * a) \leq \hat{S}(a)$
3. $\hat{S}(a) \geq \text{“positive regions in } a\text{”} + \text{“negative regions in } a\text{”}$

for all $a \in \mathcal{M}_2$. Then $\hat{S}$ gives a DRS in two variables, i.e. for any non-zero $f \in \mathbb{R}[x, y]$, it gives an upper bound for the number of connected components of the complement of the zero set of $f$. 

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Descartes’ rule of signs.
Yes, sure!

For any \( a \in \mathcal{M}_2 \) define \( Q(a) = \text{“positive regions in } a \text{”} + \text{“negative regions in } a \text{”} \) and

\[
\hat{S}(a) = \max_{n, m \geq 0} Q(\square^n \ast \square^m \ast a).
\]

The function \( \hat{S} \) is a DRS in two variables.
Is there any DRS in two variables with a simple formula?