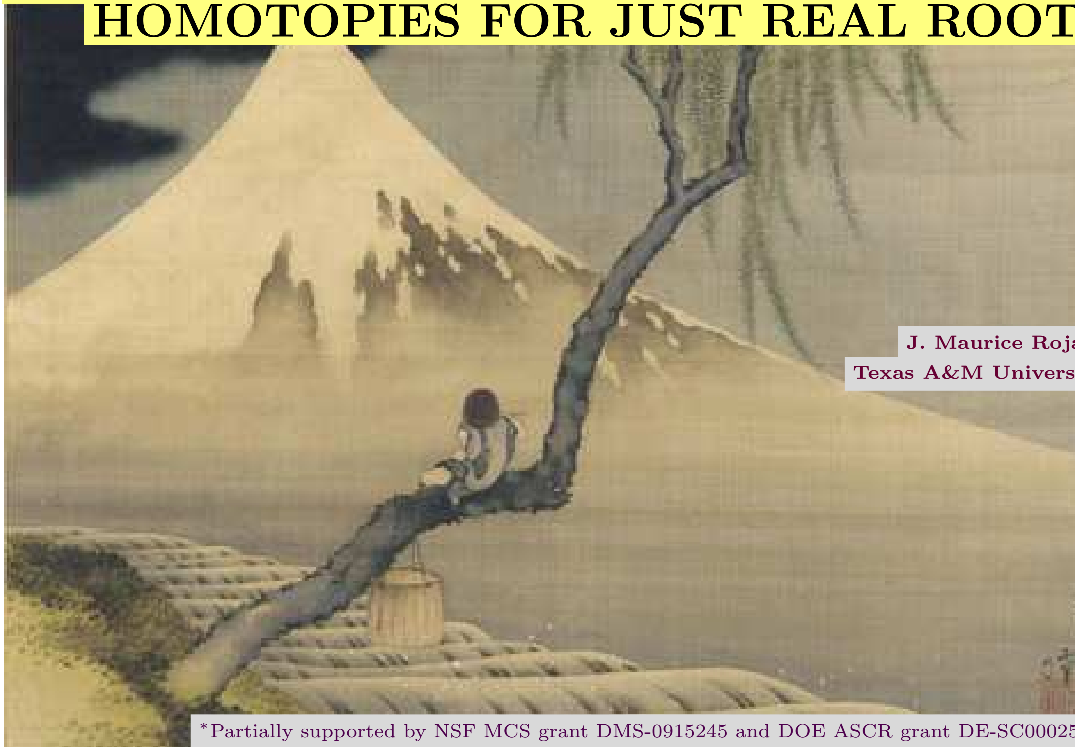
CHAMBER CONES AND SIMPLE HOMOTOPIES FOR JUST REAL ROOT



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OUTLINE

Let's see how **chamber cones** can be used to deal with real solutions of polynomial equations. Specifically...

- Estimating their number...
- Deciding their existence...
- Approximating their coordinates...

We begin by discussing approximation first...

SMALE'S $17^{\underline{\text{th}}}$ PROBLEM (2000)

"Can a solution of n complex polynomial equations in n unknowns be found approximately, on the average, in polynomial time with a uniform algorithm?"

HEDGING YOUR BETS...

"Can **a** *solution of n complex polynomial equations in n unknowns be found* **approximately***,* **on the average***, in polynomial time with a uniform algorithm?"*

SMALE'S $17^{\underline{\text{th}}}$ PROBLEM (2000)

"Can a solution of n complex polynomial equations in n unknowns be found approximately, on the average, in polynomial time with a uniform algorithm?"

...major recent progress Beltran and Pardo [FoCM 2007, JAMS 2008] and Bürgisser and Cucker [STOC 2010].

EXAMPLE

[Beltran, Pardo, 2008] $\Longrightarrow$ given a random* $n \times n$ system of degree d polynomials, on average, you can find a "good" start point for Newton's method, with probability 99.9%, using just

$$O\left(d^3 n^7 \left(\frac{d+n}{e}\right)^{3\min\{d,n\}} \log^2 d\right)$$

arithmetic operations.

...in spite of d^n complex solutions with probability 1, and the existence of systems with arbitrarily bad numerical conditioning...

* ...via the usual $U(n+1)$-invariant measure...

"SPARSADELIC" SMALE'S $17^{\underline{\text{th}}}$

Sparsity and **real** (and ***p*-adic**) solutions have been ignored so far in Smale's 17th Problem, so let us consider the following new complement...

MAIN CONJECTURE

1. A **real** *solution of a random* **(feasible) real** $n \times n$ *polynomial system can be found approximately, on the average, in polynomial time* **in the sparse encoding**, *with a uniform algorithm.*

2. In particular, one can count **exactly** *the number of positive roots, with high probability, in polynomial time.*

UNIVARIATE BINOMIALS

If you let c_1, c_2 be i.i.d. real Gaussians, then for $c_1 + c_2 x_1^d$...

- There are ≤ 2 isolated real roots...
- You can count exactly the number positive roots in **constant** time...
- You can find an approximate **real** root within $O(\log d)$ arithmetic operations on average. (See, e.g., [Ye, '94] and then estimate some integrals...)

UNIVARIATE TRINOMIALS

If you let c_1, c_2, c_3 be independent real Gaussians* then for $c_1 + c_2 x_1^d + c_3 x_1^D$...

- There are ≤ 4 isolated real roots...
- You can count exactly the number positive roots within $(\log(c_1) + \log(c_2) + \log(c_3) + \log(D))^{O(1)}$ bit operations [Bihan, Rojas, Stella, 2010].
- You can find an approximate **real** root within $O(\log D)$ arithmetic operations on average [Faria, Popov, Rojas, 2010].

HERE'S HOW...

You can decide whether $1 - cx^{196418} + x^{317811}$ has 0, 1, or 2 positive roots, just by checking whether $196418^{196418} 121393^{121393} c^{317811} - 317811^{317811}$ is <0, $=0$, or >0.

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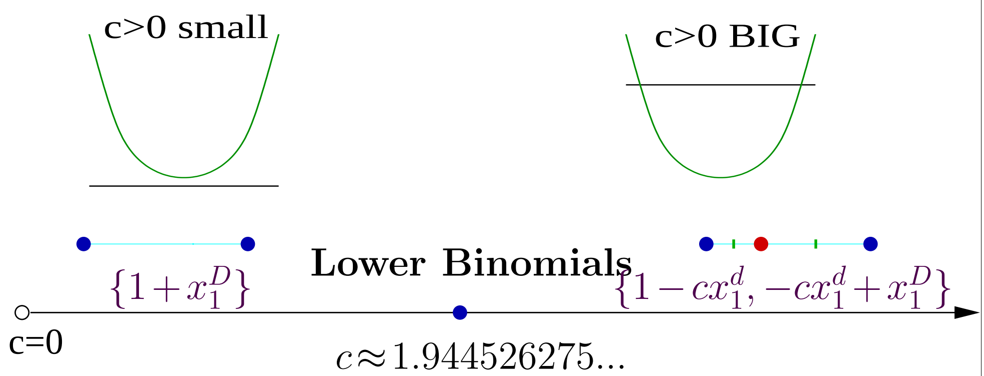


THEOREM 1

[Bihan-Rojas-Stella] *Fix n. Then for any "honest" n-variate $(n+2)$-nomial f, one can decide $Z_+(f) \stackrel{?}{=} \emptyset$ in* $\mathbf{P}$.

Note: All earlier algorithms (even much more general results of Basu, Gabrielov, and Zell) yield singly exponential time at best. Our use of Diophantine Approximation appears to be unavoidable and leads to interesting connections to the *abc*-Conjecture.

n-VARIATE $(n+k)$-NOMIALS?

Obstruction:

THEOREM 2 [Bihan-Rojas-Stella] *Fix any ε. Then deciding $Z_+(f) \stackrel{?}{=} \emptyset$ for* **general** *n-variate $(n+n^\varepsilon)$-nomials f (with $n \in \mathbb{N}$ part of the input) is* **NP***-hard.*

...but there is a way out!

Chamber Cones and Randomization...

2×2 TRINOMIAL SYSTEMS

Consider
$$\begin{matrix} x_1^{82} + \frac{31}{50}x_2^{41} - x_2 \\ x_2^{82} + 55x_1^{41} - x_1 \end{matrix}$$

This system has exactly $82^2 - 1 = 6723$ roots in $\mathbb{C}^2$; and exactly 1 (resp. 2, 2, 0) roots in $\mathbb{R}^2_+$ (resp. $\mathbb{R}_- \times \mathbb{R}_+$, $\mathbb{R}^2_-$, $\mathbb{R}_+ \times \mathbb{R}_-$)...

`realroot` applied to the x-eliminant on `Maple 13` dies, so how do we find **certifiable** information about the real roots quickly?

$\mathcal{A}$-DISCRIMINANTS?

Consider
$$\begin{matrix} x_1^{82} + ax_2^{41} - x_2 \\ x_2^{82} + bx_1^{41} - x_1 \end{matrix}$$

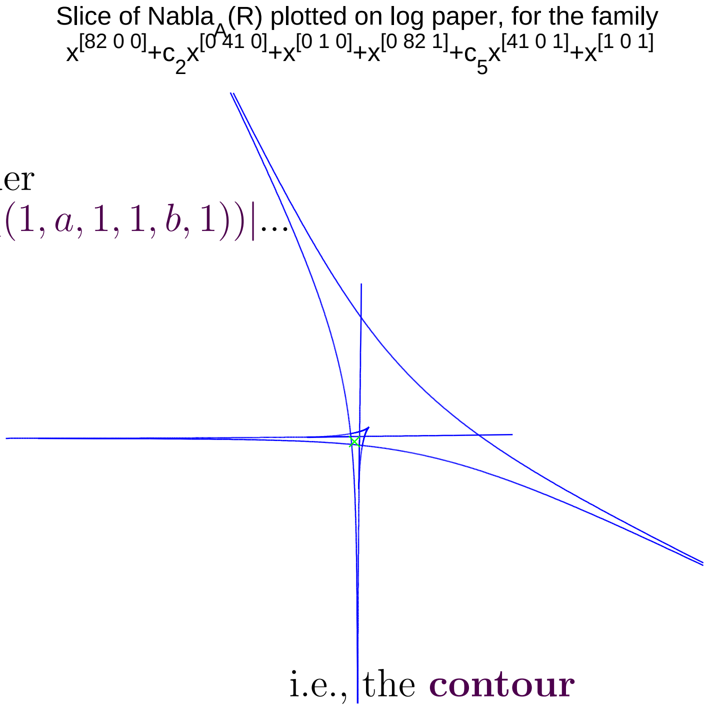
The underlying **discriminant variety** could give valuable information, but defining polynomial has coefficients of over 6000 digits (and likely thousands of such coefficients).

Nevertheless, the **Horn-Kapranov Uniformization** gives us a **one-line parametrization**!:

$$\varphi(\lambda, t) := [\lambda_1, \lambda_2]\begin{bmatrix} -40 & 6723 & -6683 & -3280 & 0 & 3280 \\ -40 & 163 & -123 & -80 & 80 & 0 \end{bmatrix} \odot \left(1, \frac{t_2^{41}}{t_1^{82}}, \frac{t_2}{t_1^{82}}, \frac{t_2^{82}t_3}{t_1^{82}}, \frac{t_3}{t_1^{41}}, \frac{t_3}{t_1^{81}}\right)$$

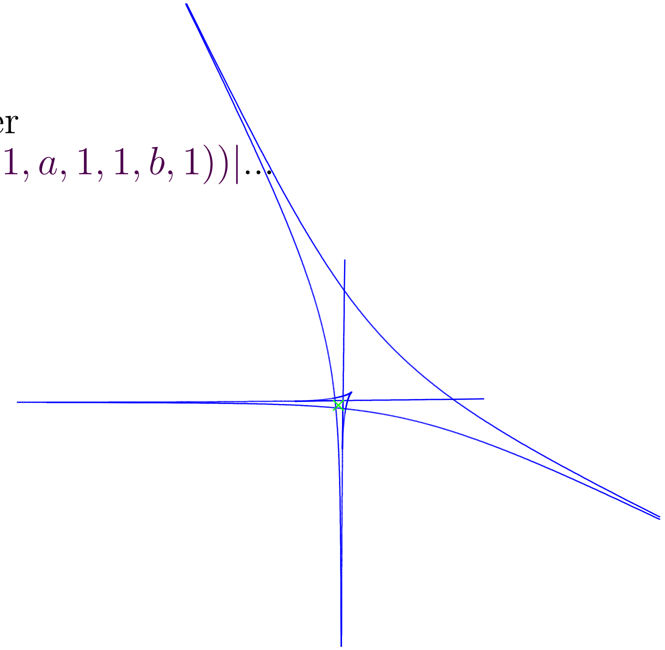
INNER/OUTER CHAMBERS

Now consider
$\mathrm{Log}|Z_{\mathbb{R}}(\Delta_{\mathcal{A}}(1,a,1,1,b,1))|\ldots$

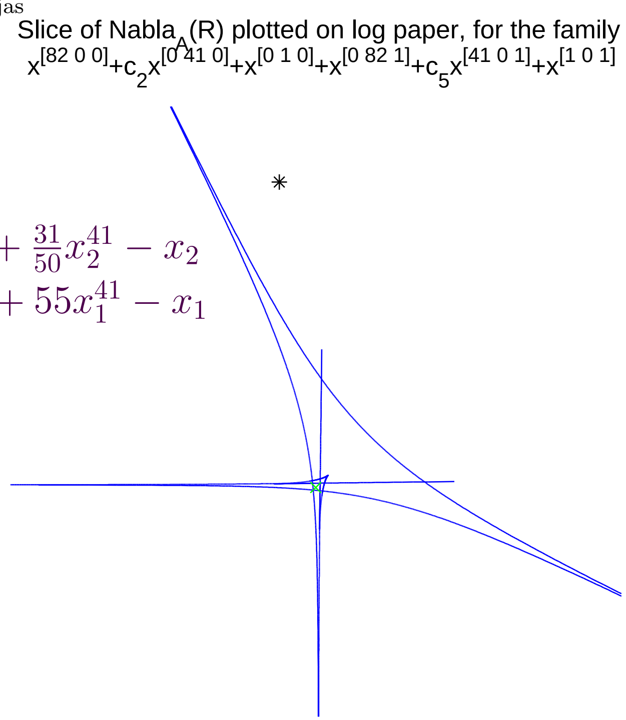


i.e., the **contour**
of $\mathrm{Amoeba}(\Delta_{\mathcal{A}}(1,a,1,1,b,1))$

Now consider
$\mathrm{Log}|Z_{\mathbb{R}}(\Delta_{\mathcal{A}}(1,a,1,1,b,1))|\ldots$



Consider $x_1^{82}+\frac{31}{50}x_2^{41}-x_2$
$x_2^{82}+55x_1^{41}-x_1$



Consider $x_1^{82}+\frac{31}{50}x_2^{41}-x_2$
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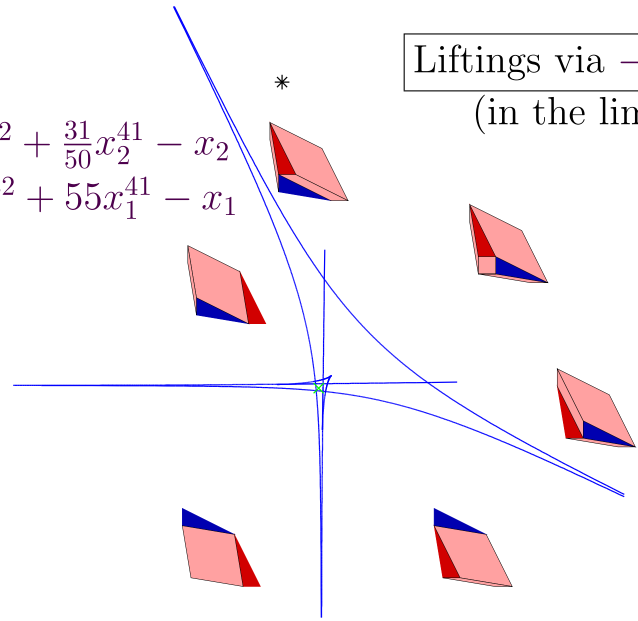
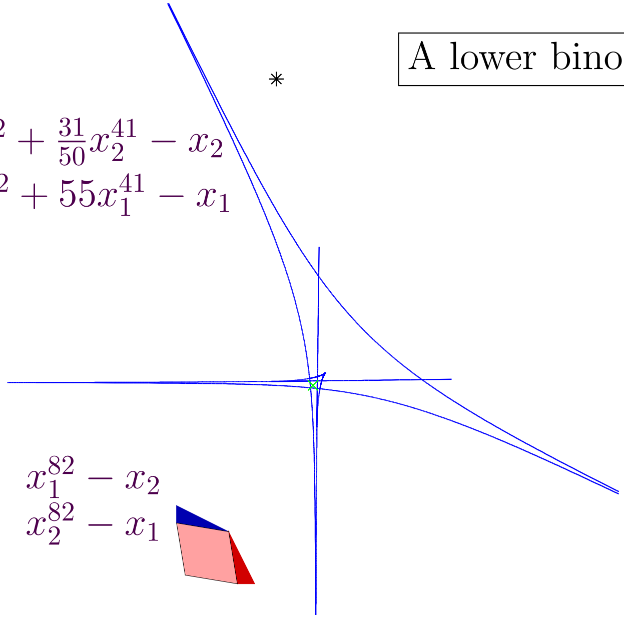
$\boxed{\text{Liftings via } -\log|\text{coeff}|}$
(in the limit)



A lower binomial system...

Consider $x_1^{82} + \frac{31}{50}x_2^{41} - x_2$

$x_2^{82} + 55x_1^{41} - x_1$



$x_1^{82} - x_2$

$x_2^{82} - x_1$

Lower binomial systems...

Consider $x_1^{82} + \frac{31}{50}x_2^{41} - x_2$

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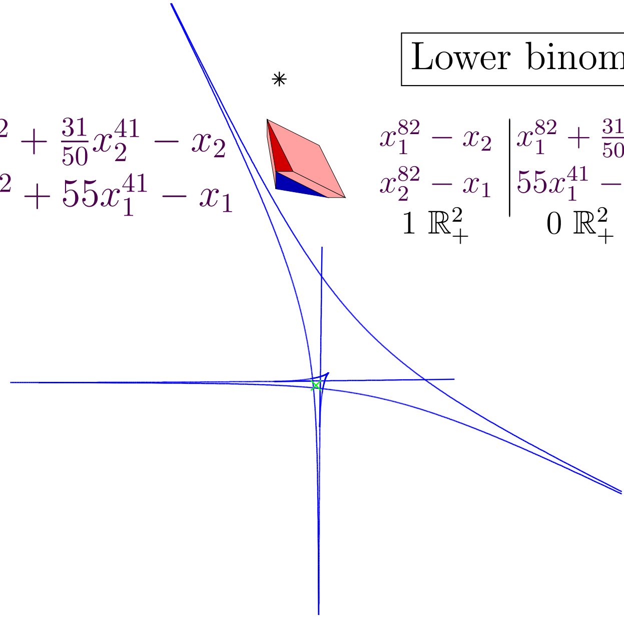


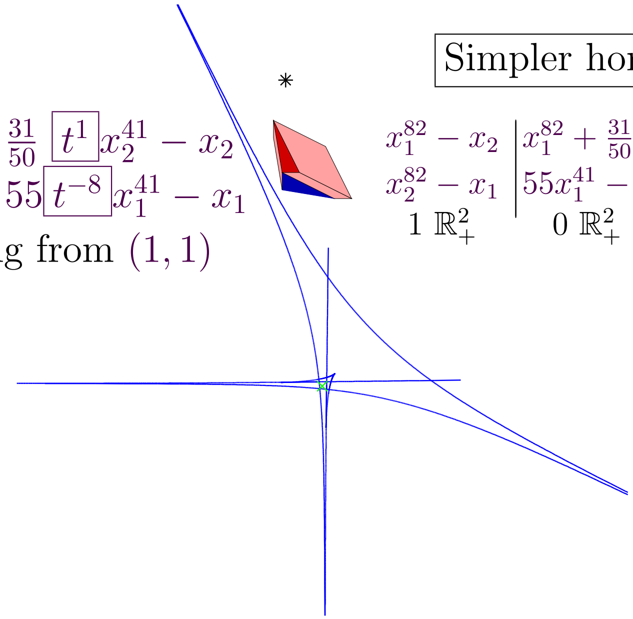
$x_1^{82} - x_2$	$x_1^{82} + \frac{31}{50}x_2^{41}$	$x_1^{82} + \frac{31}{50}x_2^{41}$
$x_2^{82} - x_1$	$55x_1^{41} - x_1$	$x_2^{82} + 55x_1^{41}$
$1\ \mathbb{R}_+^2$	$0\ \mathbb{R}_+^2$	$0\ \mathbb{R}_+^2$

Simpler homotopy!

Use $x_1^{82} + \frac{31}{50}\boxed{t^1}x_2^{41} - x_2$

$x_2^{82} + 55\boxed{t^{-8}}x_1^{41} - x_1$

...starting from $(1,1)$



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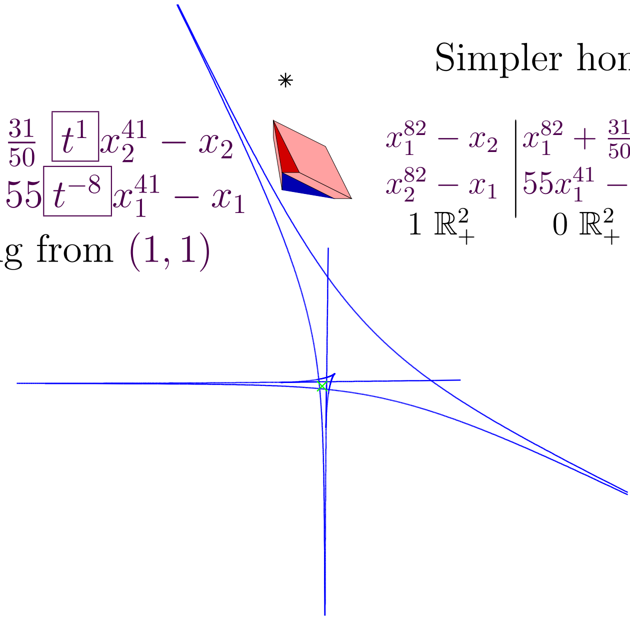

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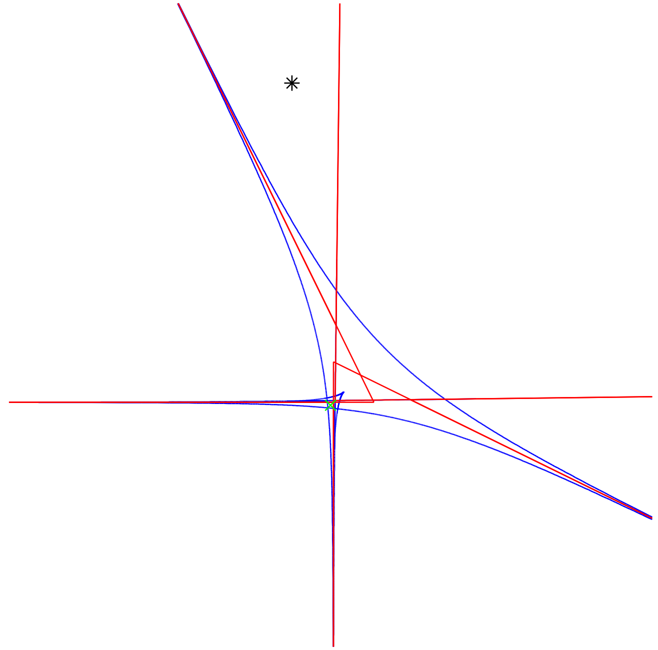
...but how do you know where you are?!

CHAMBER CONES



LARGER EXAMPLE

Consider $\begin{array}{l} x^6 \ + \alpha y^3 \quad + 1 \\ y^{14} + \beta x^3 y^8 + xy^8 + \gamma x^{133} \ldots \end{array}$

...what would the chambers and cones look like?

`./movie2`

THEOREM 3

[Pébay, Rojas, Rusek, Thompson, 2010] *Fix n and let $\mathcal{A} = \{a_i\} \subset \mathbb{Z}^n$ have cardinality m. Then, in time polynomial in the sparse encoding, we can determine the unique chamber cone containing $f(x) = \sum_{i=1}^m c_i x^{a_i}$, or obtain a true declaration that f lies in ≥ 2 chamber cones.*

Geometrically, chamber cone membership is like LP redundancy, but applied to an **oriented** hyperplane arrangement. One then proceeds via a careful application of an interior-point of [Vavasis & Ye, 1996] and Baker's Theorem...

THEOREM 3

[Pébay, Rojas, Rusek, Thompson, 2010] *Fix n... Then, in time polynomial in the sparse encoding, we can determine the unique chamber cone...*

Corollary. *For fixed n, real feasibility for "most" n-variate $(n+k)$-nomials lies in* **NP***!*

...p-adic analogue now in progress [Avendaño, Ibrahim, Rojas, Rusek, 2010].

♡ Thank you for listening!

Please see...

`www.math.tamu.edu/~rojas`

for on-line papers and further information.