Department of Mathematics, Peking University

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Normalized Ricci flow on asymptotically hyperbolic manifolds

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1. Some Backgrounds of Problem

A complete and noncompact manifold (M^n, g) **is called** $C^{k,\alpha}$ **conformally compact if**

- $M \cong Int(\bar{M})$, $\partial \bar{M} \neq \phi$
- τ**:** $M \mapsto R^+$, **s.t.** $\tau|_{\partial \bar{M}} = 0$, $d\tau|_{\partial \bar{M}} \neq 0$, $\bar{g} \triangleq \tau^2 g \in C^{k,\alpha}(\bar{M})$

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$\Rightarrow$ ***conformal class of g on $\partial\bar{M}$ make sense,***

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Example 1.2 : $\mathbb{H}^3 = \left(\mathbb{B}^3, \frac{4dS_0^2}{(1-|x|^2)^2}\right)$.

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Remark 1.4 : ***All*** C^2***- conformally compact Einstein manifolds are AH manifolds***

Problem 1.5 : ***Given $\bar{M}^n$ and conformal structure $[\psi]$ on $\partial\bar{M}$, find an AH metric g on M with***

- $Ric(g) = -(n-1)g$;
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- **J.Lee (2006): Find AHE metric which is $C^{2,\alpha}$ perturbation of non-degenerate AHE at infinity boundary.**

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- $h(g_1 + \xi) = 0 \Leftrightarrow \Delta_L \xi + 2(n-1)\xi =$**higher order terms of** τ
- **non-degenerate condition:** $\Delta_L \xi + 2(n-1)\xi = 0$**,** $tr_{g_0}\xi = 0$**,** $\xi \in L^2(M, g_0) \Rightarrow \xi = 0$**.**

$$
\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2h_{ij} \triangleq -2(R_{ij} + (n-1)g_{ij}) \\ g(x,0) = g_0(x) \end{cases}
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- **R.Ye (1993): compact case;**
- **H.Li, H.Yin (2010);**
- **O.C.Schnürer, F. Schulze, M.Simon (2010);**
- **E. Bahuaud;**
- **……**

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Problem 1.7 ***For general AHE metric*** g***, "non-degenerate" ⇒ "stability under NRF"?***

Problem 1.8 ***Can we reprove Graham, J.Lee's or J.Lee's results by NRF?***

2. The main results

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Definition 2.1 $(\mathcal{M}^n,\ g)$ ***is said to satisfy condition*** $B(k_0, \delta_0, \lambda)$ ***, if:***

- ***bounded curvature condition***

$$\|Rm(g)\|_g \leq k_0,$$

- ***the injectivity radius condition***

$$inj(\mathcal{M}) \geq \delta_0,$$

  where $inj(\mathcal{M})$ ***is the injective radius on*** $\mathcal{M}$ ***with respect to*** g***.***

- ***non-degenerate condition***

$$\int_{\mathcal{M}} \langle \big(\Delta_L + 2(n-1)\big) u_{ij}, u_{ij} \rangle \geq \lambda \int_{\mathcal{M}} \|u\|^2$$

  holds for any symmetric 2***-tensor*** u ***such that*** $\int_{\mathcal{M}} (|\nabla u|^2 + |u|^2) dv < \infty$***.***

$(\mathcal{M}^n, g)$ is said to be of $B(k_0, k_1, \delta_0, \lambda)$, if it further satisfies

$$\sup_{\mathcal{M}} \|\nabla Rm(g)\| \leq k_1$$

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g ***is said to be*** ε***-Einstein of order*** δ ***if,***

$$\|h(g) = Ric(g) + (n-1)g\|_g(x) \leq \varepsilon e^{-\delta d(x, x_0)}.$$

Theorem 2.3 : ***Suppose***

- g ***is AH and*** $B(k_0, k_1, \delta_0, \lambda)$
- $\|h(g)\| \leq \varepsilon r^\delta$
- $\|\nabla h\| \leq C r^\delta$, $\delta \in (\frac{n-1}{2}, , \ \frac{n-1}{2} + \sqrt{\frac{(n-1)^2}{4}} - 2)$.

Then

$$\begin{cases} \frac{\partial g_{ij}}{\partial t} = -2h_{ij} \triangleq -2(R_{ij} + (n-1)g_{ij}) \\ g(x,0) = g(x) \end{cases}$$

exists on $[0, +\infty)$, ***and*** $g(t) \to g_\infty$ ***with***

- g_∞ ***is AHE;***
- g_∞ ***is*** C^2***-conformally compact***
- $[g_\infty] = [g]$

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$$g = r^{-2}(dr^2 + g_r),$$

$$\Rightarrow g_r = \hat{g} + g^{(2)}r^2 + \cdots + g^{(n-3)}r^{n-3} + hr^{n-1}\log r + g^{(n-1)}r^{n-1} + \cdots = \hat{g} + g^{(2)}r^2 + \cdots + g^{(k)}r^k + t^{(k)}[g]$$

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$$\Rightarrow g_{k,\nu}^{\phi} = \begin{cases} g_r^{k,\nu}, & \text{near the boundary;} \\ g, & \text{in a compact set.} \end{cases}$$

$g_{k,\nu}^{\phi}$ **s.t. all assumptions of Theorem.**

Theorem 2.4 : *Let* $(\mathcal{M}^n,\ g)$, $n \geq 5$, *be a smooth conformally compact Einstein manifold and* $(\partial\mathcal{M},\ [g])$ *be the conforal infinity. Assume that* g *is of the non-degeneracy* λ_0 *as defined in Definition 2.1 Suppose that*

$$\max\{2, \frac{n-1}{2} - \sqrt{\lambda_0}\} < k + 2.$$

Then, if a smooth metric $[\hat{g}_\nu]$ *is a sufficiently small* C^{k+3}*-perturbation of* $[g]$, *then there is a smooth conformally compact Einstein metric on* $\mathcal{M}$ *whose conformal infinity is* $[\hat{g}_\nu]$.

Theorem 2.5 : ***Let $\mathbf{B}^n$ be the ball in $\mathbb{R}^n$ and $\hat{g}$ the standard metric on the $\mathbf{S}^{n-1}$. For any smooth Riemannian metric $\hat{g}_\nu$ on $\mathbf{S}^{n-1}$ which is sufficiently close to $\hat{g}$ in $C^{2,\alpha}$, for some $\alpha \in (0,1)$, then there is a smooth metric g in $\mathbf{B}^n$ satisfying***

(1). $Ric(g) = -(n-1)g$

(2). g ***can be C^1 conformally compactfied with the conformal infinity $[\hat{g}_\nu]$***

If in addition, $\hat{g}_\nu$ is $C^{3,\alpha}$ close to $\hat{g}$ then g can be C^2 conformally compactfied.

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- **Let** $[0, T]$ **be the maximal interval s.t.** $\forall t \in [0, T]$ **is** $B(2k_0, \frac{\delta_0}{2}, \frac{\lambda}{2})$**,** 2ϵ**-Einstein with order** δ**;**

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- **Show that** $g(T)$ **is** $B(\frac{3}{2}k_0, \frac{2\delta_0}{3}, \frac{2\lambda}{3})$, $\frac{3}{2}\epsilon$**-Einstein with order** δ**;**

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- $\lambda > 0 \Rightarrow \|h\| \leq Ce^{-\frac{1}{2}\lambda t}$

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$$\frac{\partial}{\partial t} R^l_{ijk} = -g^{lp}\{\nabla_i \nabla_j h_{kp} + \nabla_i \nabla_k h_{jp} - \nabla_i \nabla_p h_{jk} - \nabla_j \nabla_i h_{kp} - \nabla_j \nabla_k h_{ip} + \nabla_j \nabla_p h_{ik}\}$$

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$$\|Rm(g(\tau)) - Rm(g(0))\| \leq C\sqrt{\tau}$$
$$\|\Gamma^i_{jk}(g(\tau)) - \Gamma^i_{jk}(g(0))\| \leq C\tau$$

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$\Rightarrow g(\tau)$ **is** $B(\frac{4}{3}k_0, \frac{3\delta_0}{5}, \frac{3\lambda}{5})$**, and is** $\frac{4}{3}\varepsilon$**-Einstein of order** δ

$$\frac{\partial}{\partial t} h_{ij} = -(\Delta_L + 2(n-1))h_{ij}$$

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$$\Rightarrow$$

$$\frac{d}{dt} \int_{\mathcal{M}} \|h\|^2 = \int_{\mathcal{M}} \langle -(\Delta_L + 2(n-1)) h, h \rangle + C(\varepsilon) \int_{\mathcal{M}} \|h\|^2$$
$$\leq -(2\lambda - C(\varepsilon)) \int_{\mathcal{M}} \|h\|^2$$

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$\Rightarrow \forall t \in [\frac{\tau}{2}, T]$

$$\sup_{B_0(x,\sqrt{\frac{\tau}{2}})\times[t-\frac{\tau}{2},t]} \|h\|^2 \leq Ce^{-(2\lambda - C(\varepsilon))t}\int_{\mathcal{M}} \|h\|^2 \mathrm{dv}_{\mathrm{g}(0)}$$
$$\leq Ce^{-(2\lambda - C(\varepsilon))t}\varepsilon^2$$

$$\Rightarrow \forall t \in [\tfrac{\tau}{2}, T]$$

$$\sup_{B_0(x,\sqrt{\frac{\tau}{2}})\times[t-\frac{\tau}{2},t]} \|h\|_{C^{2,\alpha}} \leq Ce^{-(\lambda - C(\varepsilon))t}\Big(\int_{\mathcal{M}} \|h\|^2 \mathrm{dv}_{\mathrm{g}(0)}\Big)^{\frac{1}{2}}$$

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$$\frac{\partial}{\partial t} R^l_{ijk} = -g^{lp}\{\nabla_i\nabla_j h_{kp} + \nabla_i\nabla_k h_{jp} - \nabla_i\nabla_p h_{jk} - \nabla_j\nabla_i h_{kp} - \nabla_j\nabla_k h_{ip} + \nabla_j\nabla_p h_{ik}\}$$

$$\Rightarrow \forall t \in [\tfrac{\tau}{2}, T]$$

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$$\|Rm(g(T)) - Rm(g(\tau))\| \le C\varepsilon$$

$$\|\Gamma^i_{jk}(g(T)) - \Gamma^i_{jk}(g(\tau))\| \le C\varepsilon$$

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$\Rightarrow g_\infty$ **can be** C^2 **conformally compactified if** $\delta > 2$**.**