Feedback Laws: Results and Methods, with Applications to Quantum Control Systems

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Outline

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1. Some abstract results in finite dimension

2. Control Lyapunov function

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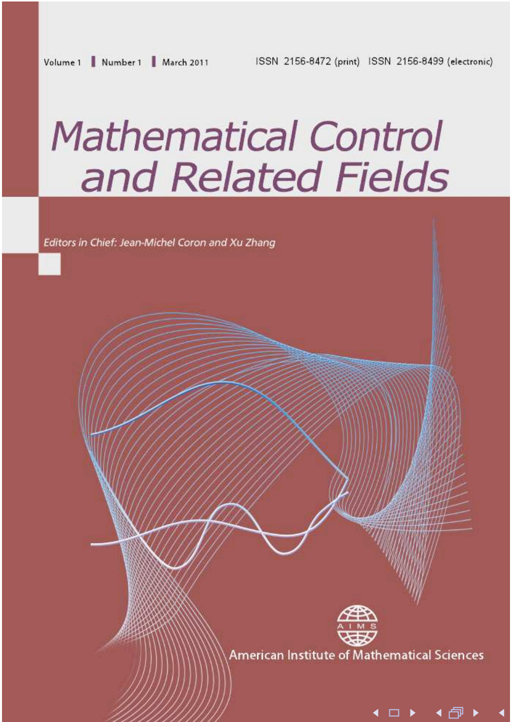
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The stabilizability problem

We consider the control system $\dot{x} = f(x,u)$ where the state is x in $\mathbb{R}^n$ and the control is u in $\mathbb{R}^m$. We assume that $f(0,0) = 0$.

Problem

Does there exists $u : \mathbb{R}^n \to \mathbb{R}^m$ vanishing at 0 such that $0 \in \mathbb{R}^n$ is (locally) asymptotically stable for $\dot{x} = f(x,u(x))$? (If the answer is yes, one says that the control system is locally asymptotically stabilizable.)

Remark

The map $u : x \in \mathbb{R}^n \mapsto \mathbb{R}^m$ is called a feedback (or feedback law). The dynamical system $\dot{x} = f(x,u(x))$ is called the closed loop system.

Regularity of feedback laws

The regularity of $x \mapsto u(x)$ is an important point. With u continuous, asymptotic stability implies the existence of a smooth strict Lyapunov function and one has robustness with respect to small actuator errors as well as small measurement errors.
If u is discontinuous, one needs to define the notion of solution of the closed loop system $\dot{x} = f(x, u(x))$ and study carefully the robustness of the closed loop system.

Discontinuous feedback laws

- Filippov solution: H. Hermes (1967); JMC and Lionel Rosier (1994), E. Ryan (1994).
- Euler solutions: F. Clarke, Y. Ledyaev, E. Sontag and A. Subbotin (1997); E. Sontag (1999).
- Carathéodory solutions: F. Ancona and A. Bressan (1999, 2004).
- Add extra variables in order to have more robustness (hybrid feedback laws): Y. Ledyaev and E. Sontag (1997); C. Prieur (2005); R. Goebel, C. Prieur and A. Teel (2007, 2009).

Unless otherwise specified, from now on we assume that the feedback laws are continuous.

Stabilizability of linear controllable systems

Notations. For a matrix $M \in \mathbb{R}^{n\times n}$, P_M denotes the characteristic polynomial of M: $P_M(z) := \det\,(zI - M)$.
Let us denote by $\mathcal{P}_n$ the set of polynomials of degree n in z such that the coefficients are all real numbers and such that the coefficient of z^n is 1. One has the following theorem

Theorem (Pole shifting theorem)

Let us assume that the linear control system $\dot{x} = Ax + Bu$ is controllable. Then

$$\left\{P_{A+BK};\, K \in \mathbb{R}^{m\times n}\right\} = \mathcal{P}_n.$$

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Corollary

If the linear control system $\dot{x} = Ax + Bu$ is controllable, there exists a linear feedback $x \mapsto u(x) = Kx$ such that $0 \in \mathbb{R}^n$ is (globally) asymptotically stable for the closed loop system $\dot{x} = Ax + Bu(x)$.

Application to nonlinear controllable systems

Let us consider the linearized control system $\dot{x} = Ax + Bu$ of $\dot{x} = f(x, u)$ at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$:

$$A := \frac{\partial f}{\partial x}(0, 0),\ B := \frac{\partial f}{\partial u}(0, 0).$$

Let us assume that the linearized control system $\dot{x} = Ax + Bu$ is controllable. Then, by the pole-shifting theorem, there exists $K \in \mathbb{R}^{m \times n}$ such that $\sigma(A + BK) = \{-1\}$. Let us consider the feedback $u(x) = Kx$. Then, if $X(x) := f(x, u(x))$, $X'(0) = A + BK$. Hence $0 \in \mathbb{R}^n$ is locally asymptotically stable for the closed loop system $\dot{x} = f(x, u(x))$.
In conclusion, if the the linearized control system is controllable, then

- The control system $\dot{x} = f(x, u)$ is small-time locally controllable at $(0, 0)$.
- The control system $\dot{x} = f(x, u)$ is locally asymptotically stabilizable (at the equilibrium $(0, 0)$).

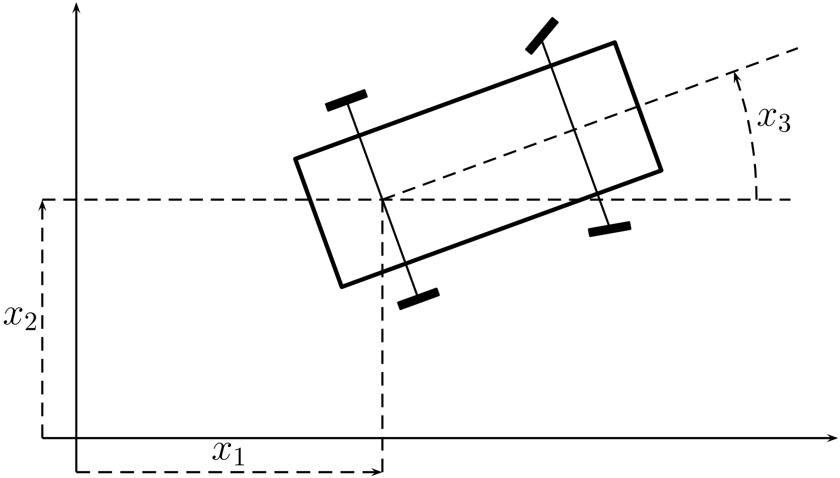
Obstruction to the stabilizability

Theorem (R. Brockett (1983))

If the control system $\dot{x} = f(x, u)$ can be locally asymptotically stabilized then

(N) *the image by f of every neighborhood of $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$ is a neighborhood of $0 \in \mathbb{R}^n$.*

The baby stroller: The model



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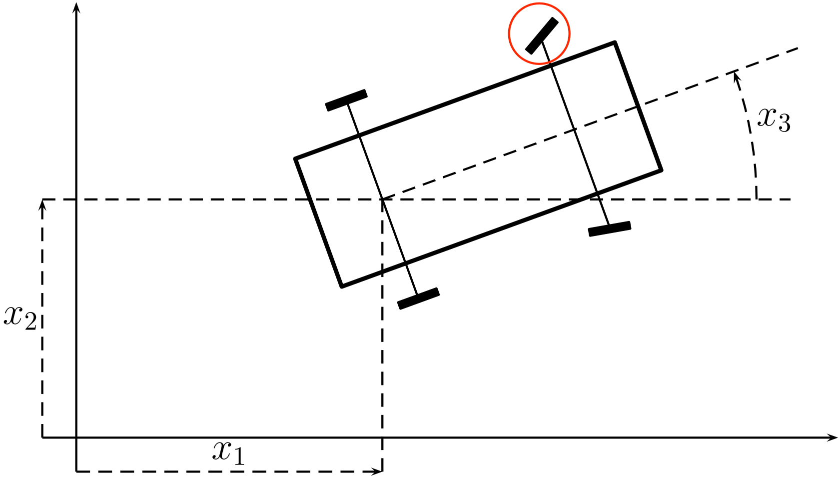

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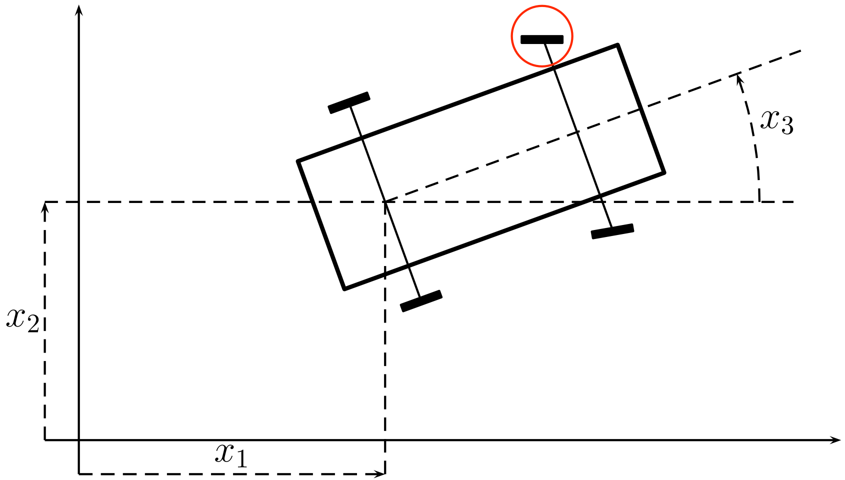

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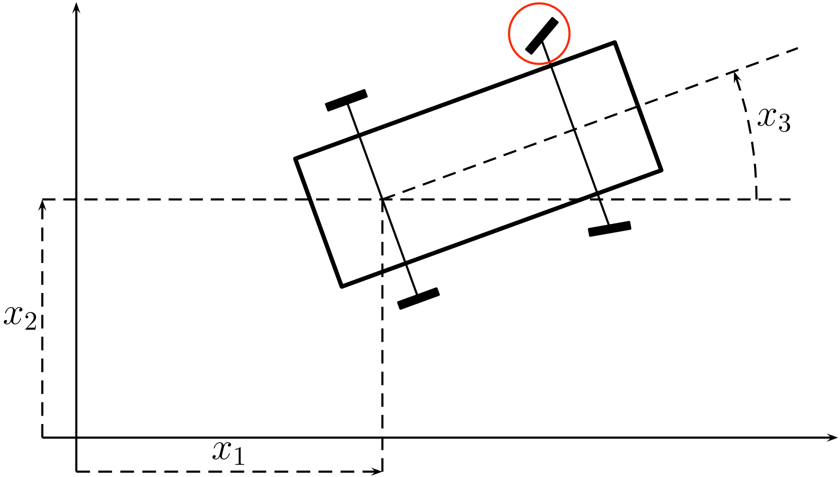

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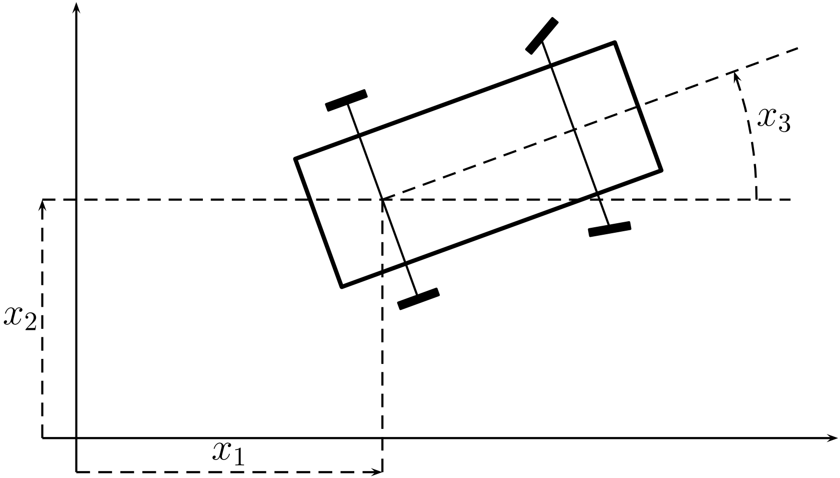


$$\dot{x}_1 = u_1 \cos x_3,\ \dot{x}_2 = u_1 \sin x_3, \dot{x}_3 = u_2,\ n = 3,\ m = 2.$$

The baby stroller and the Brockett condition

The baby stroller control system

$$\dot{x}_1 = u_1 \cos x_3,\ \dot{x}_2 = u_1 \sin x_3, \dot{x}_3 = u_2$$

is small-time locally controllable at $(0, 0)$. However (N) does not hold for the baby stroller control system. Hence the baby stroller control system cannot be locally asymptotically stabilized.

Example: The under-actuated satellite

$$\dot{\omega} = J^{-1}S(\omega)J\omega + \sum_{i=1}^{m} u_i b_i,\ \dot{\eta} = A(\eta)\omega, \tag{1}$$

We consider again the case where $m = 2$. Using the fact that $A(0) = \text{Id}$, one easily sees that (N) never holds. However

$$\text{Span } \{b_1, b_2, S(\omega)J^{-1}\omega;\ \omega \in \text{ Span } \{b_1, b_2\}\} = \mathbb{R}^3.$$

then the control system (1) is small-time locally controllable at $(0,0) \in \mathbb{R}^6 \times \mathbb{R}^2$.

A solution: Time-varying feedback laws

Instead of $u(x)$, use $u(t, x)$. Note that asymptotic stability for time-varying feedback laws is also robust (there exists again a strict Lyapunov function).

Time-varying feedback laws for driftless control systems

Theorem (JMC (1992))

Assume that

$$\{g(x);\ g \in \mathit{Lie}\{f_1, \dots, f_m\}\} = \mathbb{R}^n,\ \forall x \in \mathbb{R}^n \setminus \{0\}. \qquad (2)$$

Then, for every $T > 0$, there exists u in $C^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m)$ such that

$$u(t, 0) = 0,\ \forall t \in \mathbb{R},$$
$$u(t + T, x) = u(t, x),\ \forall x \in \mathbb{R}^n,\ \forall t \in \mathbb{R},$$
$$0 \text{ is globally asymptotically stable for } \dot{x} = \sum_{i=1}^{m} u_i(t, x) f_i(x).$$

Sketch of proof

Sketch of the proof of the theorem. Let $T > 0$. Assume that there exists $\bar{u}$ in $C^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m)$ T-periodic with time, vanishing for $x = 0$, and such that, if $\dot{x} = f(x, \bar{u}(t,x))$, then

(i) $x(T) = x(0)$,

(ii) If $x(0) \neq 0$, then the linearized control system around the trajectory $t \in [0,T] \mapsto (x(t), \bar{u}(t, x(t)))$ is controllable on $[0,T]$.

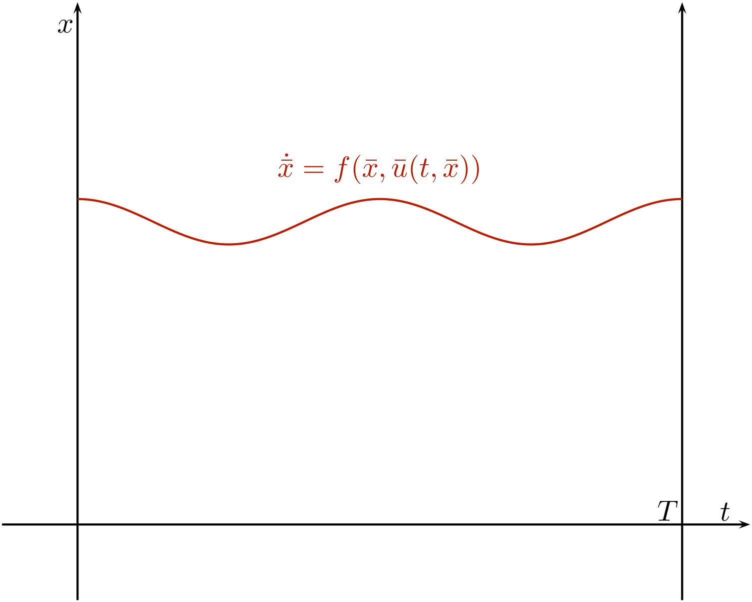
Using (i) and (ii) one easily sees that one can construct a "small" feedback v in $C^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m)$ T-periodic with time and vanishing for $x = 0$ such that, if $\dot{x} = f(x, (\bar{u} + v)(t,x))$ and $x(0) \neq 0$, then $|x(T)| < |x(0)|$, which implies that 0 is globally asymptotically stable for $\dot{x} = f(x, (\bar{u} + v)(t,x))$.

Construction of $\bar{u}$

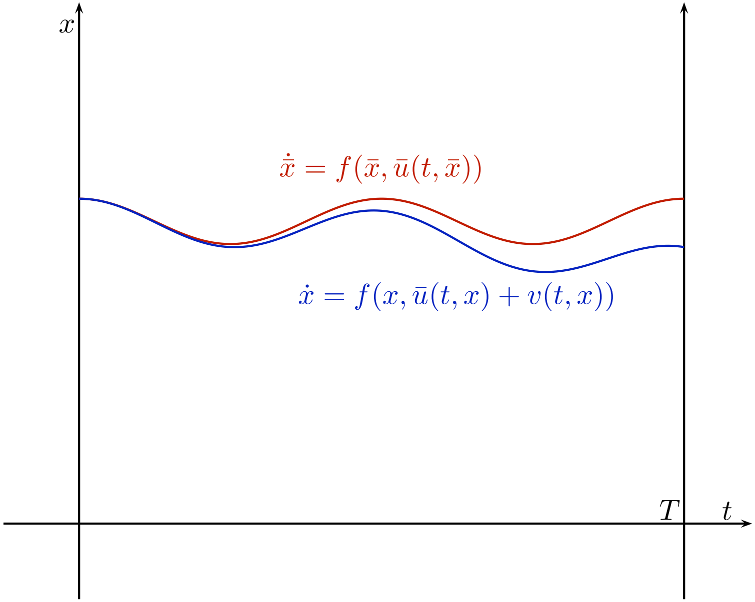
In order to get (i), one just imposes on $\bar{u}$ the condition

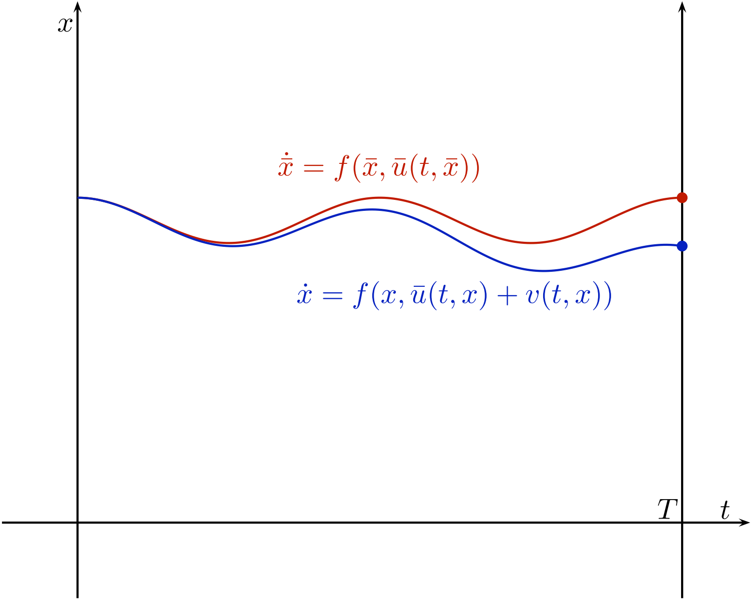
$$\bar{u}(t,x) = -\bar{u}(T-t,x),\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n,$$

which implies that $x(t) = x(T-t)$, $\forall t \in [0,T]$, for every solution of $\dot{x} = f(x, u(t,x))$, and therefore gives $x(0) = x(T)$. Finally, one proves that (ii) holds for "many" $\bar{u}$'s (this is the difficult part of the proof).



x
$\dot{\bar{x}} = f(\bar{x}, \bar{u}(t, \bar{x}))$
T
t


x
$\dot{\bar{x}} = f(\bar{x}, \bar{u}(t, \bar{x}))$
$\dot{x} = f(x, \bar{u}(t, x) + v(t, x))$
T
t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T
t

General control systems

Definition

The origin (of $\mathbb{R}^n$) is *locally continuously reachable in small time* for the control system $\dot{x} = f(x, u)$ if, for every positive real number T, there exist a positive real number ε and an element u in $C^0\left(\bar{B}_\varepsilon; L^1\left((0,T);\mathbb{R}^m\right)\right)$ such that

$$\mathrm{Sup}\{|u(a)(t)|; t \in (0,T)\} \to 0 \text{ as } a \to 0,$$
$$((\dot{x} = f(x, u(a)(t)),\ x(0) = a) \Rightarrow (x(T) = 0)), \forall a \in \bar{B}_\varepsilon.$$

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Open problem

Assume that f is analytic and that $\dot{x} = f(x,u)$ is small-time locally controllable at $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$. Is the origin (of $\mathbb{R}^n$) locally continuously reachable in small time for the control system $\dot{x} = f(x,u)$?

Theorem

Let us assume that 0 *is locally continuously reachable in small-time for the control system* $\dot{x} = f(x, u)$*. Then the control system* $\dot{x} = f(x, u)$ *is small-time locally controllable at* $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$*.*

Theorem

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Theorem (JMC (1995))

Assume that $0 \in \mathbb{R}^n$ *is locally continuously reachable in small time for the control system* $\dot{x} = f(x, u)$*, that* f *is analytic and that* $n \notin \{2, 3\}$*. Then, for every positive real number* T*, there exist* ε *in* $(0, +\infty)$ *and* u *in* $C^0(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m)$*, of class* C^∞ *on* $\mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$*,* T*-periodic with respect to time, vanishing on* $\mathbb{R} \times \{0\}$ *and such that, for every* $s \in \mathbb{R}$*,*

$$((\dot{x} = f(x, u(t, x)) \text{ and } x(s) = 0) \Rightarrow (x(\tau) = 0, \forall \tau \geqslant s)),$$
$$(\dot{x} = f(x, u(t, x)) \text{ and } |x(s)| \leqslant \varepsilon) \Rightarrow (x(\tau) = 0, \forall \tau \geqslant s + T)).$$

Stabilization of the under-actuated satellite

$$\dot{\omega} = J^{-1}S(\omega)J\omega + \sum_{i=1}^{m} u_i b_i,\ \dot{\eta} = A(\eta)\omega, \tag{3}$$

We consider again the case where $m = 2$ and assume that

$$\text{Span } \{b_1, b_2, S(\omega)J^{-1}\omega;\ \omega \in \text{ Span } \{b_1, b_2\}\} = \mathbb{R}^3.$$

Then $0 \in \mathbb{R}^6$ is locally continuously reachable in small-time for the control system the control system (3) and therefore can be locally asymptotically stabilized by means of periodic time-varying feedback laws.

Stabilization of the under-actuated satellite

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Then $0 \in \mathbb{R}^6$ is locally continuously reachable in small-time for the control system the control system (3) and therefore can be locally asymptotically stabilized by means of periodic time-varying feedback laws. Construction of explicit time-varying stabilizing feedback laws:

- Special cases: G. Walsh, R. Montgomery and S. Sastry (1994); P. Morin, C. Samson, J.-B. Pomet and Z.-P. Jiang (1995).
- General case: JMC and E.-Y. Keraï (1996); P. Morin and C. Samson (1997).

Output feedback laws

In most practical situations, only part of of the state (called the observation $y = h(x) \in \mathbb{R}^p$) is measured. Hence, one cannot use $u(x)$ or $u(t, x)$. At a first glance, one would like to use $u(h(x))$ or $u(t, h(x))$.

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Question (separation principle): Do controllability and a good observability condition imply stabilizability by means of output feedback laws?

Observability

As least two possible definitions are possible (even for small time)
(i) For every $T > 0$ and for every $u : [0, T] \to \mathbb{R}^m$,
$$(\dot{x}_1 = f(x, u(t)),\ \dot{x}_2 = f(x, u(t)),\ h(x_1) = h(x_2) \text{ on } [0, T]) \Rightarrow (x_1(0) = x_2(0)).$$
(ii) For every $T > 0$, for every $(a_1, a_2) \in \mathbb{R}^n \times \mathbb{R}^n$, there exists $u : [0, T] \to \mathbb{R}^m$ such that
$$(\dot{x}_1 = f(x, u(t)),\ \dot{x}_2 = f(x, u(t)),\ x_1(0) = a_1,\ x_2(0) = a_2) \Rightarrow (\exists \tau \in [0, T] \text{ such that } h(x_1(\tau)) \neq h(x_1(\tau))).$$
Of course (ii) is weaker than (i). We shall use (ii). Moreover, since we want to use a control which vanishes if the output is 0, it is natural to require that, for every $T > 0$
(iii) $(\dot{x} = f(x, 0),\ |x(0)|$ small and $h(x(t)) = 0,\ \forall t \in [0, T]) \Rightarrow (x(0) = 0).$
From now on, by “observable” we mean (ii) and (iii).

$u(h(x))$ and $u(t, h(x))$ give poor results

Example (A 1-D car)

Let us consider the control system

$$\dot{x}_1 = x_2,\ \dot{x}_2 = u,\ y = x_1. \tag{C}$$

Note that (C) is controllable and observable. However there is no $y \mapsto u(y)$ or more generally no $(t, y) \mapsto u(t, y)$ such that $(0, 0)$ is asymptotically stable for the closed loop system $\dot{x}_1 = x_2,\ \dot{x}_2 = u(t, x_1)$. (Proof: Take the divergence of $(x_2, u(t, x_1))^{\text{tr}}$...)

Dynamic output feedback laws

Definition

The control system

$$\dot{x} = f(x,u),\ y = h(x),\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ y \in \mathbb{R}^p$$

is locally stabilizable by means of dynamic output feedback laws if there exists $k \in \mathbb{N}$ such that the control system

$$\dot{x} = f(x,u),\ \dot{z} = v,\ \tilde{y} = (h(x); z), x \in \mathbb{R}^n,\ z \in \mathbb{R}^k,$$

where the state is $(x; z) \in \mathbb{R}^{n+k}$, the control is $(u; v) \in \mathbb{R}^{n+m}$ and the observation is $\tilde{y} \in \mathbb{R}^{p+k}$.

Dynamic output feedback laws and the 1-D car

The control system is

$$\dot{x}_1 = x_2,\ \dot{x}_2 = u,\ y = x_1, \tag{C}$$

where the state is $(x_1, x_2)^2 \in \mathbb{R}^2$, the control is $u \in \mathbb{R}$ and the observation is $y \in \mathbb{R}$. Let us take $k = 1$. Let us choose the following general dynamic linear output feedback law

$$\dot{x} = x_2,\ \dot{x}_2 = \alpha x_1 + \beta z,\ \dot{z} = \gamma x_1 + \delta z. \tag{*}$$

The point $0 \in \mathbb{R}^3$ is asymptotically stable if and only if the zeroes of the polynomial

$$P(\lambda) := \lambda^3 - \delta\lambda^2 - \alpha\lambda + \alpha\delta - \gamma\beta.$$

One can choose $(\alpha, \beta, \gamma, \delta)^{\text{tr}} \in \mathbb{R}^4$ such that $P(\lambda) = (\lambda + 1)^3$.

Interest of time-varying output feedback laws

We go back to the control system

$$\dot{x} = u,\ y = x^2. \tag{*}$$

This control system is controllable and observable. However one has the following proposition.

Proposition

Let $k \in \mathbb{N}$. There are no $u \in C^0(\mathbb{R}^{1+k}; \mathbb{R})$, $(y; z) \mapsto u(y; z)$, and $v \in C^0(\mathbb{R}^{1+k}; \mathbb{R}^k)$, $(y; z) \mapsto v(y; z)$, such that $(0; 0) \in \mathbb{R}^{1+k}$ is locally asymptotically stable for the closed loop system

$$\dot{x} = u(x^2; z),\ \dot{z} = v(x^2; z),\ x \in \mathbb{R},\ z \in \mathbb{R}^k.$$

(One uses the convention that, if $k = 0$, the closed system is just $\dot{x} = u(x^2)$ and $(0; 0) \in \mathbb{R}^{1+k}$ is just $0 \in \mathbb{R}$.)

Sketch of the proof of the proposition

Let $X \in C^0(\mathbb{R}^{1+k};\mathbb{R}^{1+k})$ be defined by

$$X(x;z) := (u(x^2;z);v(x^2;z)),\ x \in \mathbb{R},\ z \in \mathbb{R}^k. \tag{4}$$

By a theorem due to Krasnosel′skiĭ, the fact that 0 is locally asymptotically stable for $\dot{x} = X(x)$ implies de existence of $\varepsilon > 0$ such that, with $B_\varepsilon := \left\{(x;z) \in \mathbb{R}^{1+k};\ x^2 + |z|^2 < \varepsilon^2\right\}$,

$$X(x;z) \neq 0,\ \forall (x,z) \in \mathbb{R} \times \mathbb{R}^k \text{ such that } x^2 + |z|^2 = \varepsilon^2, \tag{5}$$

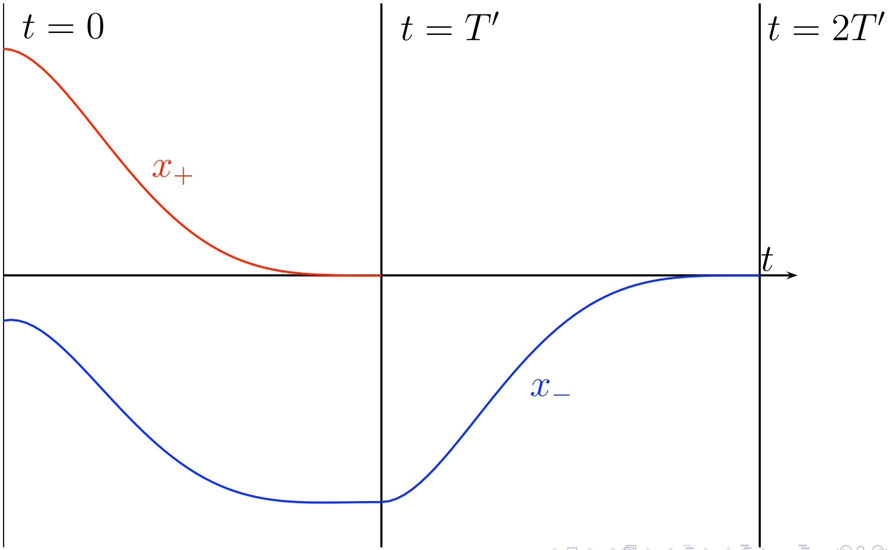
$$\operatorname{degree}(X, B_\varepsilon, 0) = (-1)^{k+1}. \tag{6}$$

Note that, by (4),

$$X(x;z) = X(-x;z),\ \forall (x;z) \in B_\varepsilon, \tag{7}$$

from which we get that $\operatorname{degree}(X, B_\varepsilon, 0) = 0$, a contradiction with (6).

Output stabilization of $\dot{x} = u$, $y = x^2$ by means of time-varying feedback laws



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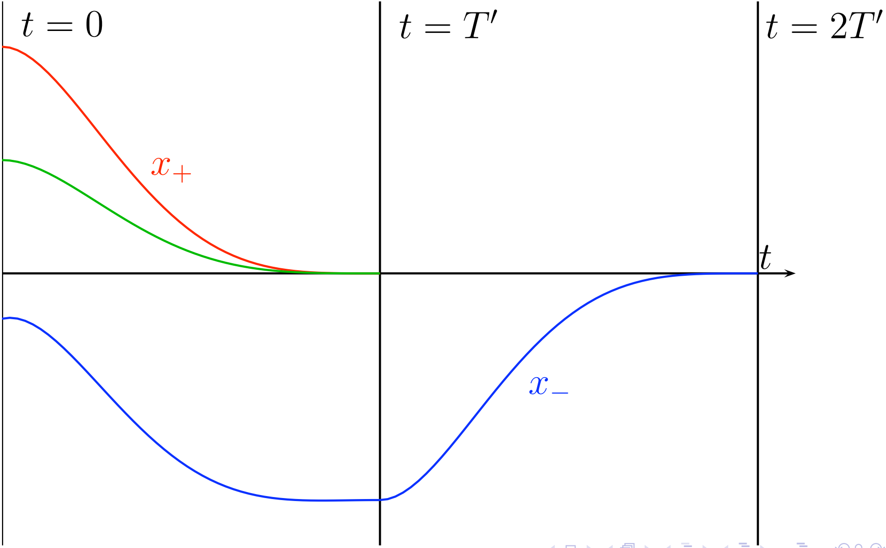

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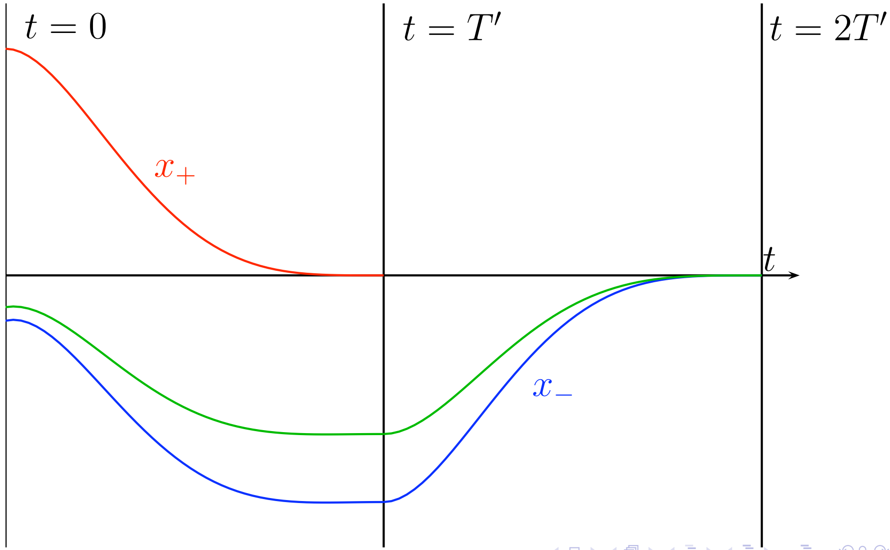

Separation principle and time-varying feedback laws

Theorem (JMC (1994))

Assume that f and h are analytic. Assume that $0 \in \mathbb{R}^n$ is locally continuously reachable in small time for $\dot{x} = f(x,u)$. Assume that the observability properties (ii) and (iii) hold. Then, there exist $k \in \mathbb{N}^*$ such that, for every $T > 0$, there exist $\varepsilon > 0$, $u \in C^0(\mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^k; \mathbb{R}^m)$ and $v \in C^0(\mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^k; \mathbb{R}^k)$, of class C^∞ on $\mathbb{R} \times (\mathbb{R}^p \times \mathbb{R}^k \setminus \{0\})$, T-periodic with respect to time, vanishing on $\mathbb{R} \times \{0\}$ such that, $\forall s \in \mathbb{R}$,

$$\Big(\dot{x} = f(x, u(t,(h(x),z))),\ \dot{z} = v(t,(h(x),z)),\ (x(s),z(s)) = 0\Big) \Rightarrow \Big((x(\tau),z(\tau)) = 0,\ \forall \tau \geqslant s\Big),$$

$$\Big(\dot{x} = f(x, u(t,(h(x),z))),\ \dot{z} = v(t,(h(x),z)),\ |x(s)| + |z(s)| \leqslant \varepsilon\Big) \Rightarrow \Big((x(\tau),z(\tau)) = 0,\ \forall \tau \geqslant s+T\Big).$$

Time-varying feedback laws and measurement: An experiment

Material:
A jigsaw
A vise
Two Meccano© strips
(length $\simeq$ 30 and 4 cm)
A nut and 3 bolts
A plastic tube

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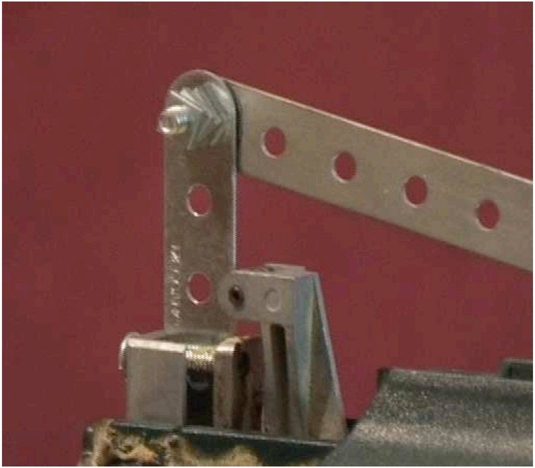
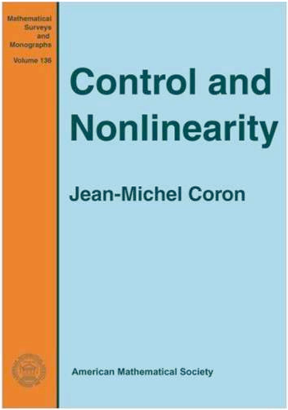


Design tools: Commercial break



JMC, Control and nonlinearity, Mathematical Surveys and Monographs, 136, 2007, 427 p. Pdf file freely available from my web page.

Control Lyapunov function

A basic tool to study the asymptotic stability of an equilibrium point is the Lyapunov function. In the case of a control system, the control is at our disposal, so there are more “chances” that a given function could be a Lyapunov function for a suitable choice of feedback laws. Hence Lyapunov functions are even more useful for the stabilization of control systems than for dynamical systems without control.

Definition

A function $V \in C^1(\mathbb{R}^n;\mathbb{R})$ is a control Lyapunov function for the control system (C) if

$$V(x) \to +\infty, \text{ as } |x| \to +\infty,$$
$$V(x) > 0, \forall x \in \mathbb{R}^n \setminus \{0\},$$
$$\forall x \in \mathbb{R}^n \setminus \{0\}, \exists u \in \mathbb{R}^m \text{ s.t. } f(x,u) \cdot \nabla V(x) < 0.$$

Moreover, V satisfies the *small control property* if, for every strictly positive real number ε, there exists a strictly positive real number η such that, for every $x \in \mathbb{R}^n$ with $0 < |x| < \eta$, there exists $u \in \mathbb{R}^m$ satisfying $|u| < \varepsilon$ and $f(x,u) \cdot \nabla V(x) < 0$.

Theorem

If the control system (C) is globally asymptotically stabilizable by means of continuous stationary feedback laws, then it admits a control Lyapunov function satisfying the small control property. If the control system (C) admits a control Lyapunov function satisfying the small control property, then it can be globally asymptotically stabilized by means of

1. *Continuous stationary feedback laws if the control system (C) is control affine ($f(x,u) = f_0(x) + \sum_{i=1}^{m} u_i f_i(x)$) (Z. Artstein (1983)),*
2. *Time-varying feedback laws for general f (JMC-L. Rosier (1994)).*

Damping

For mechanical systems at least, a natural candidate for a control Lyapunov function is given by the total energy, i.e., the sum of potential and kinetic energies.

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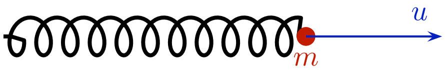
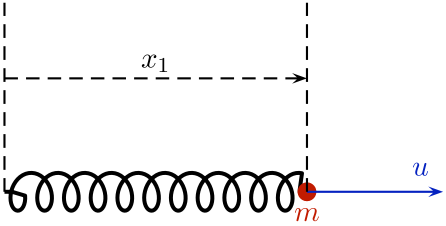
For mechanical systems at least, a natural candidate for a control Lyapunov function is given by the total energy, i.e., the sum of potential and kinetic energies. Consider the classical spring-mass control system.





The control system is

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -\frac{k}{m}x_1 + \frac{u}{m}, \qquad \text{(Spring-mass)}$$

where m is the mass of the point attached to the spring, x_1 is the displacement of the mass (on a line), x_2 is the speed of the mass, k is the spring constant, and u is the external force applied to the mass. The state is $(x_1, x_2)^{\mathrm{tr}} \in \mathbb{R}^2$ and the control is $u \in \mathbb{R}$.

The total energy E of the system is

$$E = \frac{1}{2}(kx_1^2 + mx_2^2).$$

One has

$$\dot{E} = ux_2.$$

Hence if $x_2 = 0$, one cannot have $\dot{E} < 0$. However it tempting to consider the following feedback laws

$$u := -\nu x_2,$$

where $\nu > 0$. Using the LaSalle invariance principle, one gets that these feedback laws globally asymptotically stabilize the spring-mass control system.

Application: Orbit transfer with low-thrust systems (JMC and L. Praly (1996))

Electric propulsion is characterized by a low-thrust acceleration level but a high specific impulse. They can be used for large amplitude orbit transfers if one is not in a hurry.
The state of the control system is the position of the satellite (here identified to a point: we are not considering the attitude of the satellite) and the speed of the satellite. Instead of using Cartesian coordinates, one prefers to use the "orbital" coordinates. The advantage of this set of coordinates is that, in this set, the first five coordinates remain unchanged if the thrust vanishes: these coordinates characterize the Keplerian elliptic orbit. When thrust is applied, they characterize the Keplerian elliptic osculating orbit of the satellite. The last component is an angle which gives the position of the satellite on the Keplerian elliptic osculating orbit of the satellite.

A usual set of orbital coordinates is

$$
\begin{aligned}
p &:= a(1-e^2),\\
e_x &:= e\cos\tilde{\omega}, \quad \text{with} \quad \tilde{\omega} = \omega + \Omega,\\
e_y &:= e\sin\tilde{\omega},\\
h_x &:= \tan\frac{i}{2}\cos\Omega,\\
h_y &:= \tan\frac{i}{2}\sin\Omega,\\
L &:= \tilde{\omega} + v,
\end{aligned}
$$

where a, e, ω, Ω, i characterize the Keplerian osculating orbit:

1. a is the semi-major axis,
2. e is the eccentricity,
3. i is the inclination with respect to the equator,
4. Ω is the right ascension of the ascending node,
5. ω is the angle between the ascending node and the perigee,

and where v is the true anomaly.

$$\dot{p} = 2\sqrt{\frac{p^3}{\mu}}\frac{1}{Z}S,$$

$$\dot{e}_x = \sqrt{\frac{p}{\mu}}\frac{1}{Z}\left[Z(\sin L)Q + AS - e_y(h_x \sin L - h_y \cos L)W\right],$$

$$\dot{e}_y = \sqrt{\frac{p}{\mu}}\frac{1}{Z}\left[-Z(\cos L)Q + BS - e_x(h_x \sin L - h_y \cos L)W\right],$$

$$\dot{h}_x = \frac{1}{2}\sqrt{\frac{p}{\mu}}\frac{X}{Z}(\cos L)W,\ \dot{h}_y = \frac{1}{2}\sqrt{\frac{p}{\mu}}\frac{X}{Z}(\sin L)W,$$

$$\dot{L} = \sqrt{\frac{\mu}{p^3}}Z^2 + \sqrt{\frac{p}{\mu}}\frac{1}{Z}\left(h_x \sin L - h_y \cos L\right)W,$$

where $\mu > 0$ is a gravitational coefficient depending on the central gravitational field, Q, S, W, are the radial, orthoradial, and normal components of the thrust and where

$$Z := 1 + e_x \cos L + e_y \sin L,\ A := e_x + (1 + Z)\cos L,$$
$$B := e_y + (1 + Z)\sin L,\ X := 1 + h_x^2 + h_y^2.$$

We study the case, useful in applications, where

$$Q = 0,$$

and, for some $\varepsilon > 0$,

$$|S| \leqslant \varepsilon \text{ and } |W| \leqslant \varepsilon.$$

Note that ε is small, since the thrust acceleration level is low.
The goal: give feedback laws, which (globally) asymptotically stabilize a given Keplerian elliptic orbit characterized by the coordinates $\bar{p}, \bar{e}_x, \bar{e}_y, \bar{h}_x, \bar{h}_y$.
In order to simplify the notations (this is not essential for the method), we restrict our attention to the case where the desired final orbit is geostationary, that is,

$$\bar{e}_x = \bar{e}_y = \bar{h}_x = \bar{h}_y = 0.$$

We start with a change of "time". One describes the evolution of (p, e_x, e_y, h_x, h_y) as a function of L instead of t. Then our system reads

$$\begin{cases} \dfrac{dp}{dL} = 2KpS, \\ \dfrac{de_x}{dL} = K[AS - e_y(h_x \sin L - h_y \cos L)W], \\ \dfrac{de_y}{dL} = K[BS - e_x(h_x \sin L - h_y \cos L)W], \\ \dfrac{dh_x}{dL} = \dfrac{K}{2} X(\cos L)W, \ \dfrac{dh_y}{dL} = \dfrac{K}{2} X(\sin L)W, \\ \dfrac{dt}{dL} = K\sqrt{\dfrac{\mu}{p}} Z, \end{cases}$$

with

$$K = \left[\frac{\mu}{p^2} Z^3 + (h_x \sin L - h_y \cos L)W \right]^{-1}. \tag{8}$$

Typically, one consider the following control Lyapunov function

$$V(p, e_x, e_y, h_x, h_y) = \frac{1}{2}\left(\frac{(p-\bar{p})^2}{p} + \frac{e^2}{1-e^2} + h^2\right),$$

with $e^2 = e_x^2 + e_y^2 < 1$ and $h^2 = h_x^2 + h_y^2$. The time derivative of V along a trajectory of our control system is is given by

$$\dot{V} = K(\alpha S + \beta W),$$

with

$$\begin{aligned} \alpha &:= 2p\frac{\partial V}{\partial p} + A\frac{\partial V}{\partial e_x} + B\frac{\partial V}{\partial e_y}, \\ \beta &:= (h_y \cos L - h_x \sin L)\left(e_y\frac{\partial V}{\partial e_x} + e_x\frac{\partial V}{\partial e_y}\right) \\ &\qquad + \frac{1}{2}X\left((\cos L)\frac{\partial V}{\partial h_x} + (\sin L)\frac{\partial V}{\partial h_y}\right). \end{aligned}$$

Following the damping method, one defines

$$\begin{array}{rcl} S & := & -\sigma_1(\alpha), \\ W & := & -\sigma_2(\beta)\sigma_3(p, e_x, e_y, h_x, h_y), \end{array}$$

where $\sigma_1 : \mathbb{R} \to \mathbb{R}, \sigma_2 : \mathbb{R} \to \mathbb{R}$ and $\sigma_3 : (0, +\infty) \times \mathcal{B}_1 \times \mathbb{R}^2 \to (0, 1]$ are such that

$$\sigma_1(s)s > 0,\ \sigma_2(s)s > 0, \quad \forall s \in \mathbb{R} \setminus \{0\},$$
$$\| \sigma_1 \|_{L^\infty(\mathbb{R})} < \varepsilon,\ \| \sigma_2 \|_{L^\infty(\mathbb{R})} < \varepsilon,$$
$$\sigma_3(p, e_x, e_y, h_x, h_y) \leqslant \frac{1}{1+\varepsilon}\frac{\mu}{p^2}\frac{(1-|e|)^3}{|h|}.$$

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It works!

Comparison with optimal control

It is interesting to compare the feedback constructed here to the open-loop optimal control for the minimal time problem (reach $(\bar{p}, 0, 0, 0, 0)$ in a minimal time with the constraint $|u(t)| \leqslant M$). Numerical experiments show that the use of the previous feedback laws (with suitable saturations σ_i, $i \in \{1, 2, 3\}$) gives trajectories which are nearly optimal if the state is not too close to $(\bar{p}, 0, 0, 0, 0)$. Note that our feedback laws are quite easy to compute compared to the optimal trajectory and provide already good robustness properties compared to the open-loop optimal trajectory (the optimal trajectory in a closed-loop form being, at least for the moment, out of reach numerically).

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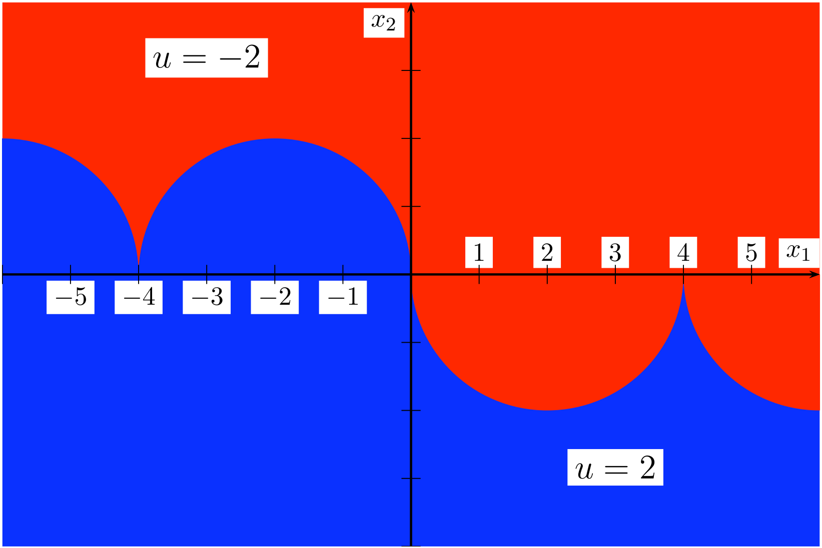
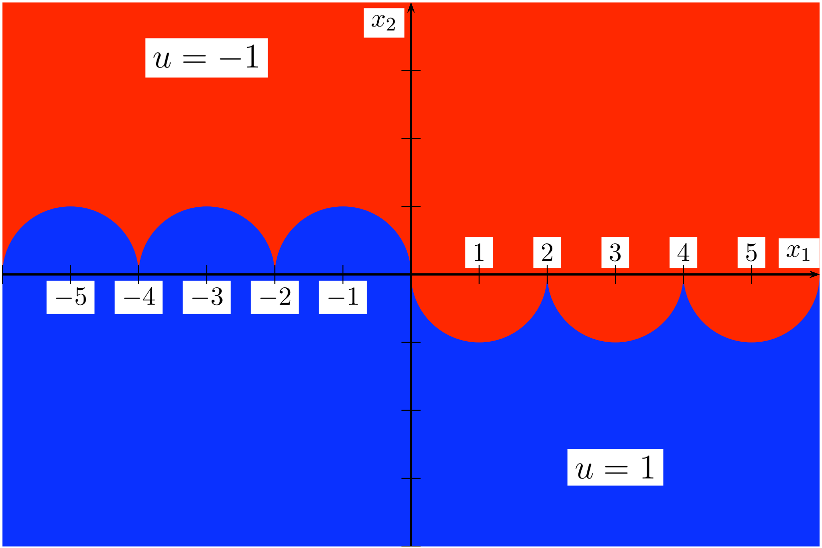
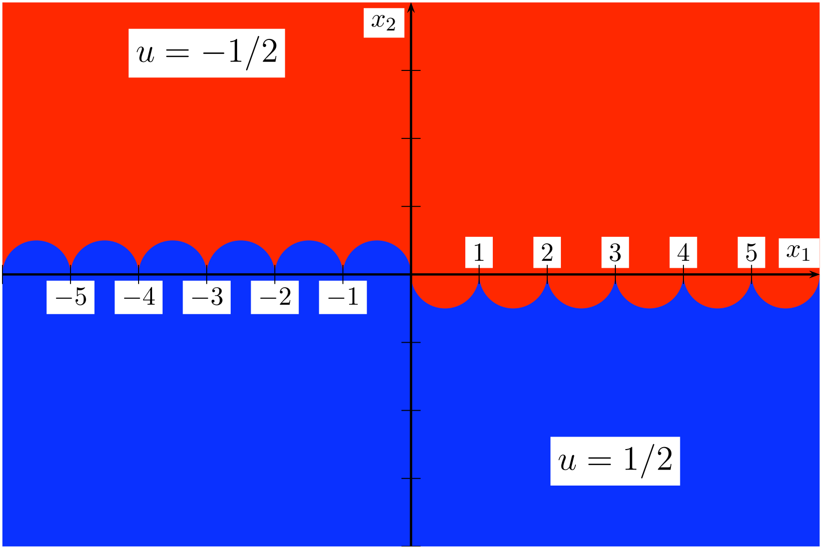
$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + u,\ |u| \leqslant 2$



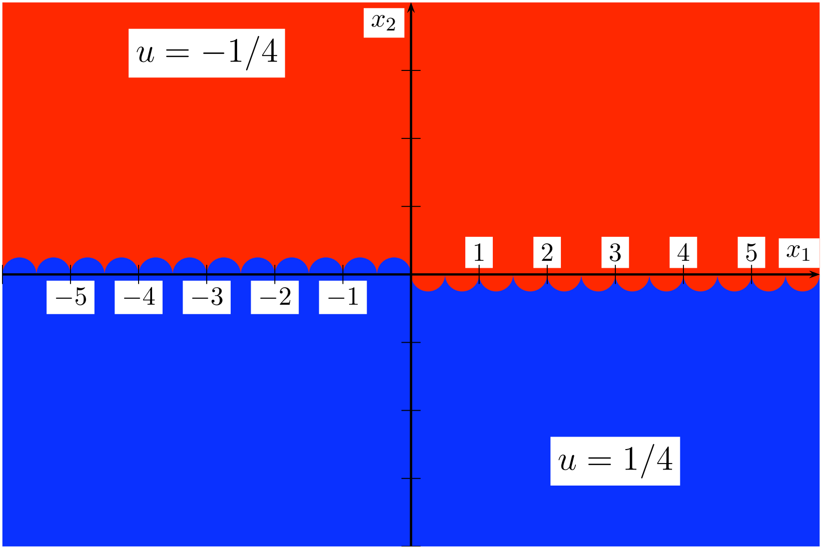
$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + u,\ |u| \leqslant 1$



$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + u,\ |u| \leqslant 1/2$



$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + u,\ |u| \leqslant 1/4$



An important limitation of the damping method

Let us come back to the spring-mass control system (with normalized physical constants)

$$\dot{x}_1 = x_2,\, \dot{x}_2 = -x_1 + u.$$

With the Lyapunov strategy used above, let $V\mathbb{R}^2 \to \mathbb{R})$ be defined by

$$V(x) = x_1^2 + x_2^2, \forall x = (x_1, x_2)^{\text{tr}} \in \mathbb{R}^2.$$

As we have sen above $\dot{V} = 2x_2u$. and it is tempting to take, at least if we remain in the class of linear feedback laws, $u := -\nu x_2$, where ν is some fixed positive real number. An a priori guess would be that, if we let ν be quite large, then we get a quite good convergence, as fast as we want.

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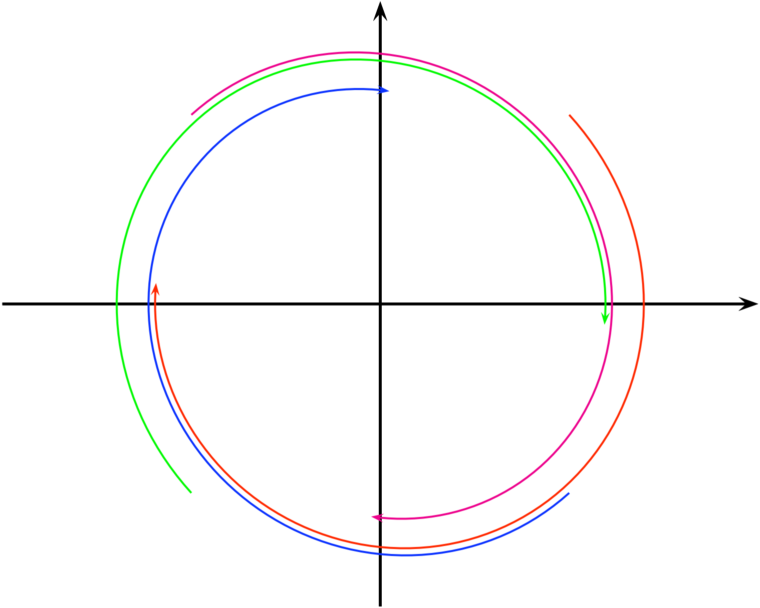
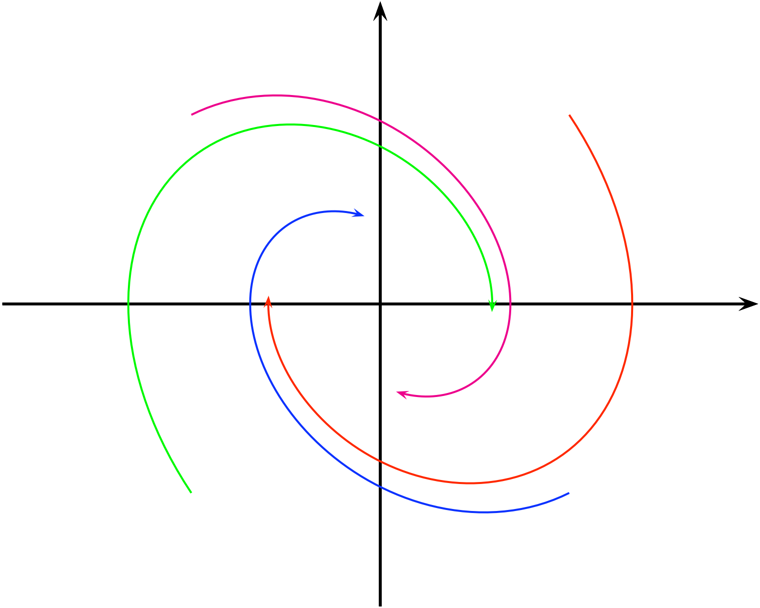
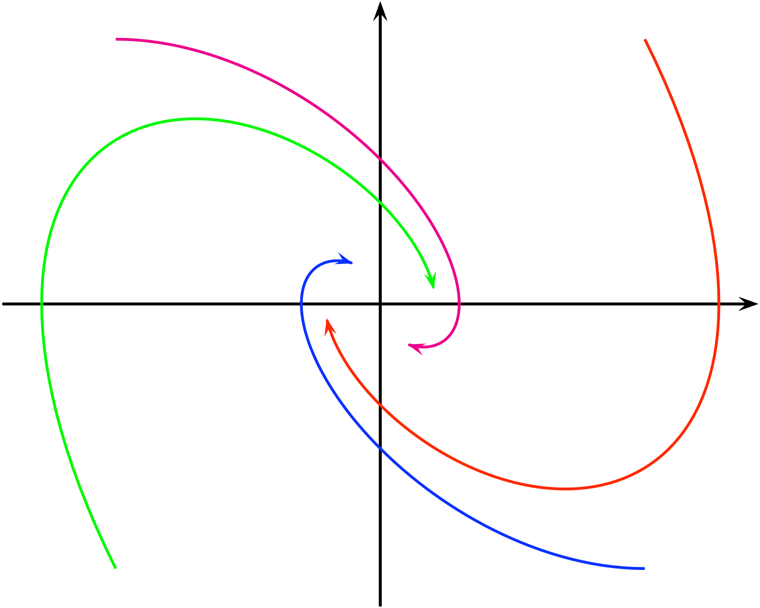
$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 - (1/10)x_2$



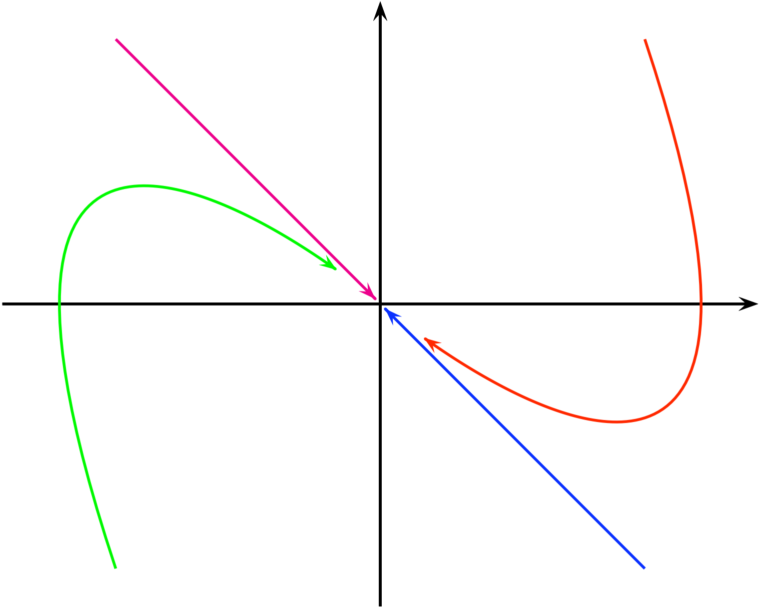
$\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 - (1/2)x_2$



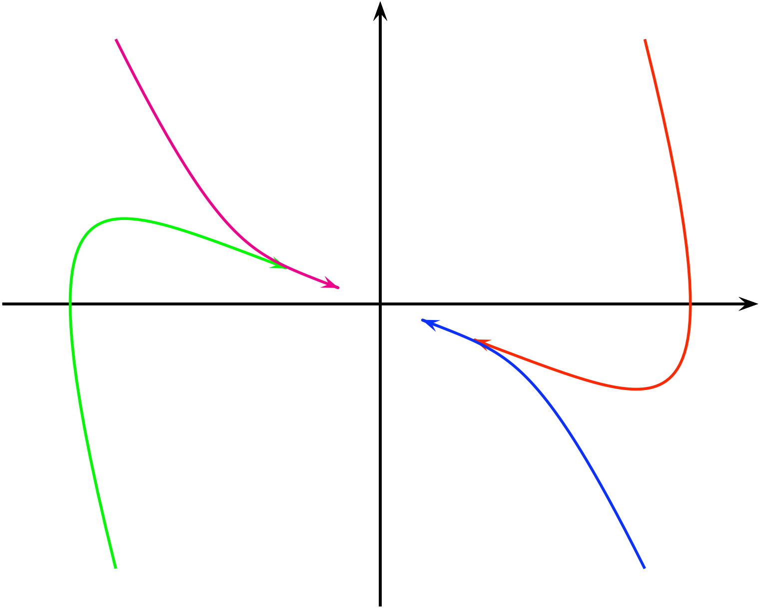
$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 - x_2$



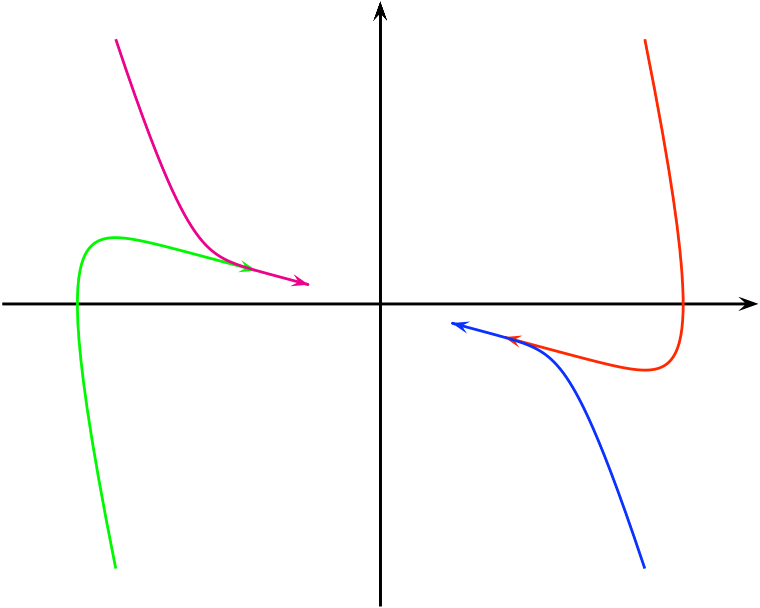
$\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 - 2x_2$



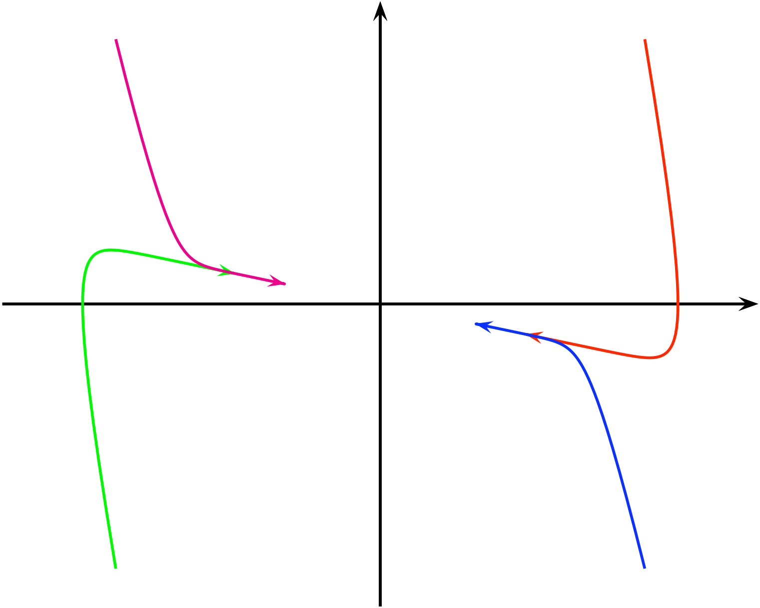
$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 - 3x_2$



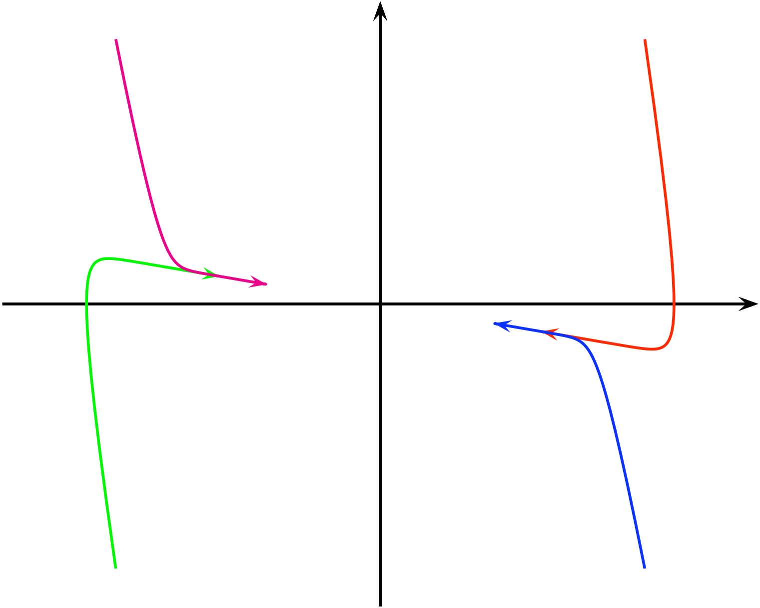
$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 - 4x_2$



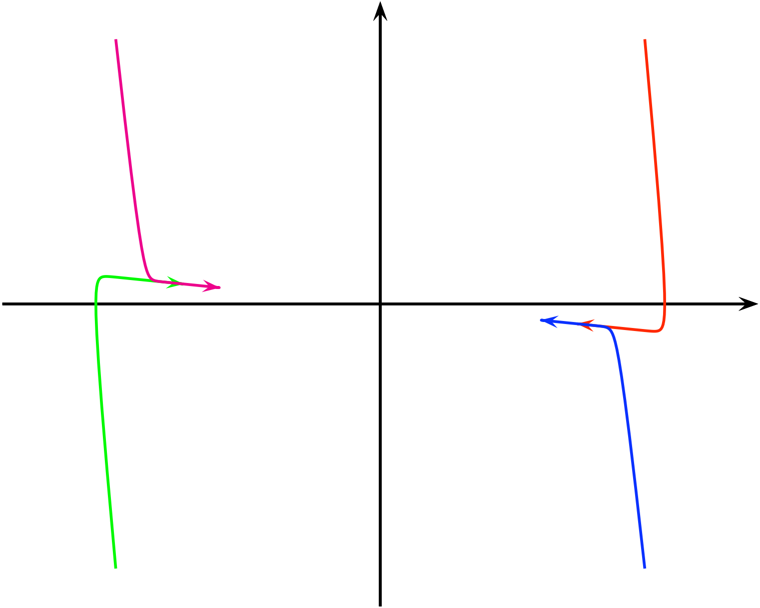
$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 - 5x_2$



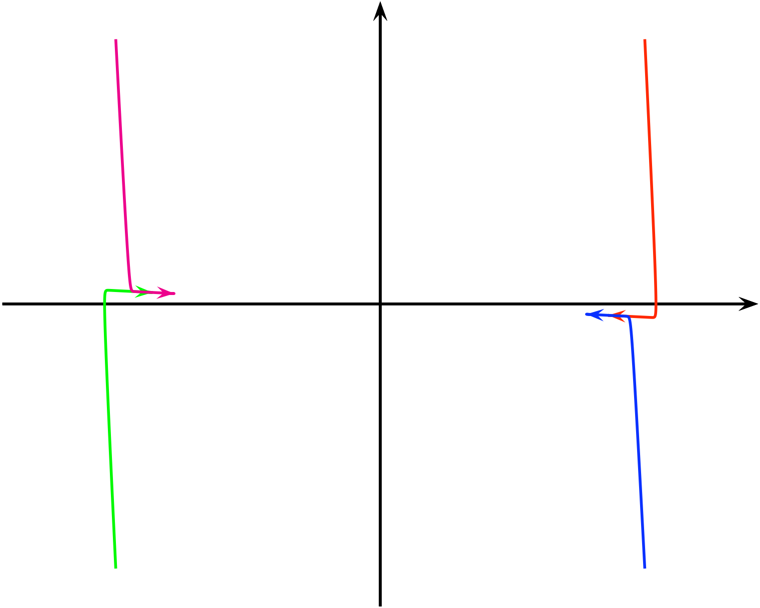
$\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 - 6x_2$



$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 - 10x_2$



$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 - 20x_2$



Damping and quantum control systems (M. Mirrahimi, P. Rouchon, G. Turinici (2005))

The quantum control system considered is

$$\iota\dot{\psi} = H_0\psi + uH_1\psi + \omega\psi,$$

where H_0 and H_1 are $N \times N$ Hermitian matrices. The state is $\psi \in \mathbb{S}^{2N-1}$, the unit sphere of $R^{2N} \simeq \mathbb{C}^N$, the control is $(u, \omega)^{\text{tr}} \in \mathbb{R}^2$. The control ω is a fictitious phase control which allows to take care of the fact that global phase of the state is physically meaning-less. Let $\psi_e \in \mathbb{S}^{2N-1}$ and $\lambda_e \in \mathbb{R}$ be such that $H_0\psi_e = \lambda\psi_e$. Replacing ψ by $\psi e^{-\iota\lambda_e}$ and ω by $\omega - \lambda_e$, we may assume that $\lambda_e = 0$. Then $(\psi, (u, \omega)) := (\psi_e, (0, 0)^{\text{tr}})$ is an equilibrium of our quantum control system. The goal is to stabilize asymptotically this equilibrium.

Remark

Since $\mathbb{S}^{2N-1}$ is not contractible, one cannot a global stabilizability. However one can try to get global stabilizability on $\mathbb{S}^{2N-1} \setminus \{-\psi_e\}$.

A natural control Lyapunov function to consider is

$$V := |\psi - \psi_e|^2 .$$

Indeed, the time-derivative of V along the trajectory of our control system is

$$\dot{V} = 2u\Im(\langle H_1\psi, \psi_e\rangle) + 2\omega\Im(\langle \psi, \psi_e\rangle).$$

This leads to choose the following feedback laws

$$u := -\nu_1\Im(\langle H_1\psi, \psi_e\rangle), \, \omega := -\nu_2\Im(\langle \psi, \psi_e\rangle),$$

where ν_1 and ν_2 are two strictly positive real numbers. With these feedback laws one has

$$\dot{V} = -2\nu_1\Im(\langle H_1\psi, \psi_e\rangle)^2 - 2\nu_2\omega\Im(\langle \psi, \psi_e\rangle)^2 \leqslant 0.$$

Theorem (M. Mirrahimi, P. Rouchon, G. Turinici (2005))

The above feedback law insures global asymptotic stabilization on $\mathbb{S}^{2N-1} \setminus \{-\psi_e\}$ *if and only the two following properties hold*

(i) *If* $(\alpha, \beta) \in \sigma(H_0)^2$ *and* $|\alpha| = |\beta|$, *then* $\alpha = \beta$,

(ii) *If* ϕ *in an eigenvector of* H_0 *which is not colinear to* Ψ_e, *then* $\langle \phi, H_1 \psi_e \rangle \neq 0$.

Remark

The properties (i) and (ii) hold together if and only if the linearized control system around $(\psi_e, 0) \in \mathbb{S}^{2N-1}$ *is controllable.*

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The properties (i) and (ii) hold together if and only if the linearized control system around $(\psi_e, 0) \in \mathbb{S}^{2N-1}$ is controllable.

Question: What to do if (i) or (ii) do not hold but the quantum control system is controllable? Partial solution: use the "phantom tracking" method.

Schrödinger pde control systems

We are now in infinite dimension. In order to apply LaSalle invariance principle one needs to have the precompactness of the trajectories. This is an open problem. However with clever and important modifications of the method one can get global approximate controllability results. Let us mention in particular

- M. Mirrahimi (2006),
- V. Nersesyan (2009,2010),
- K. Beauchard and M. Mirrahimi (2009).

LaSalle invariance principle/Strict Lyapunov function

Another possibility to overcome the problem of the precompactness of the trajectories is to try to modify the control Lyapunov function in order to get a strict Lyapunov function. Some recent examples of this possibility:

1. Partially dissipative hyperbolic systems (K. Beauchard, E. Zuazua, 2011):

$$y_t + \sum_{j=1}^{n} (F^j(y))_{x_j} = (0, B(y))^{\mathrm{tr}},\ x \in \mathbb{R}^n.$$

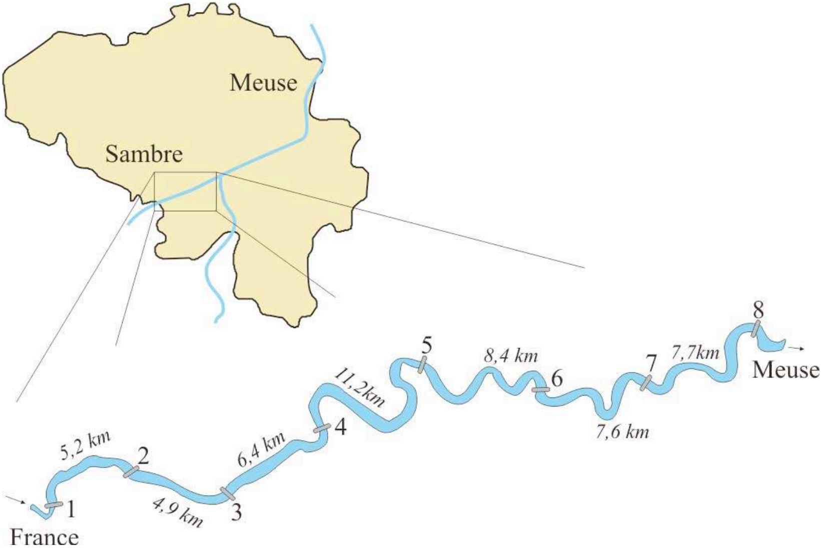
2. 1D quasilinear hyperbolic equations on a finite interval (dissipative boundary conditions): JMC, G. Bastin, B. d'Andréa-Novel (2007, 2008):

$$y_t + A(y)y_x = 0,\ x \in (0, L),$$

incoming Riemann invariants $= F$(outgoing Riemann invariants).

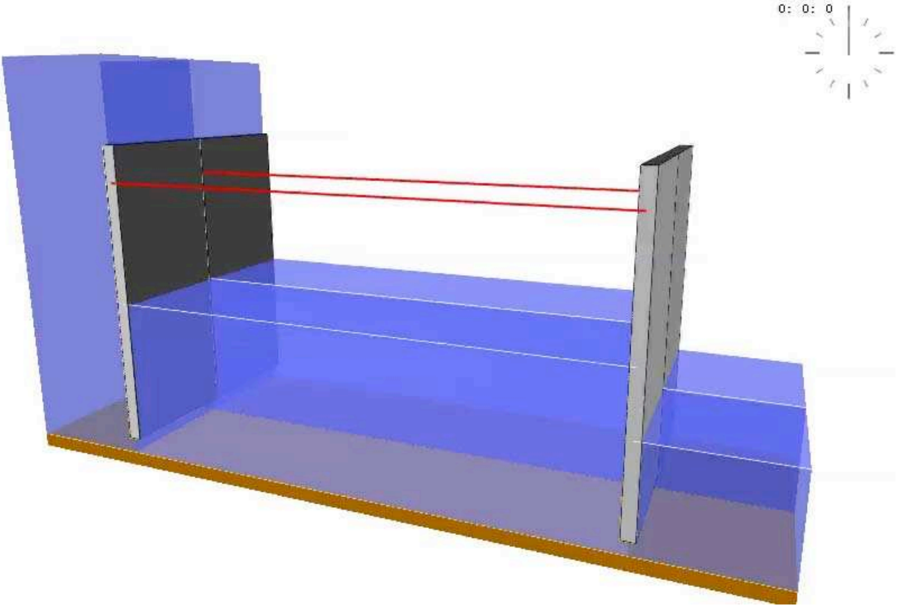
Application to the control of the river La Sambre

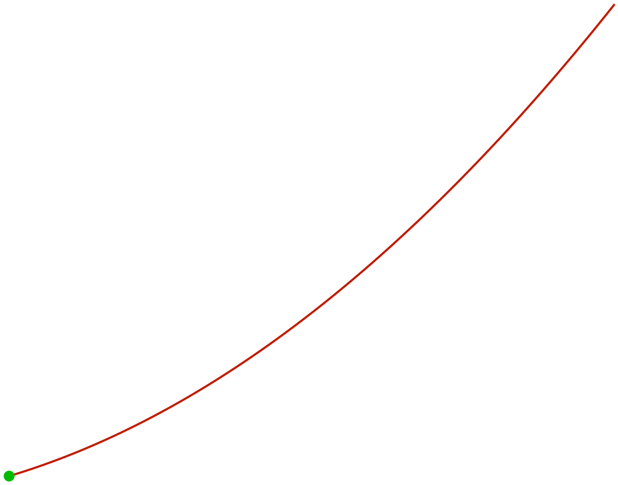
(B. d'Andréa-Novel, G. Bastin, JMC, J. de Lalleux, L. Moens, 2003-...)

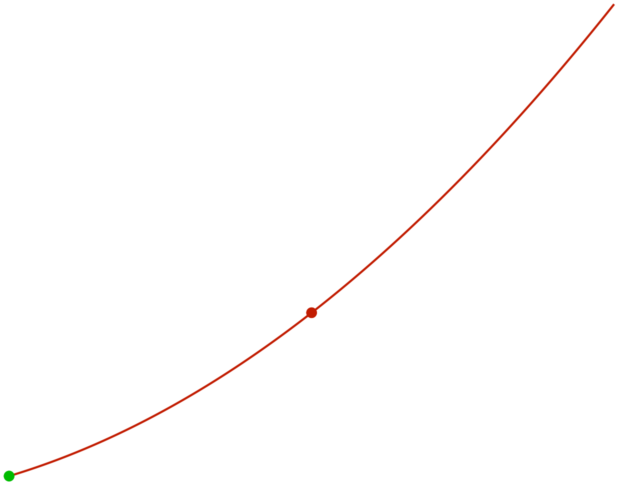

Meuse
Sambre
5,2 km
2
1
France
4,9 km
3
6,4 km
4
11,2km
5
8,4 km
6
7,6 km
7
7,7km
8
Meuse

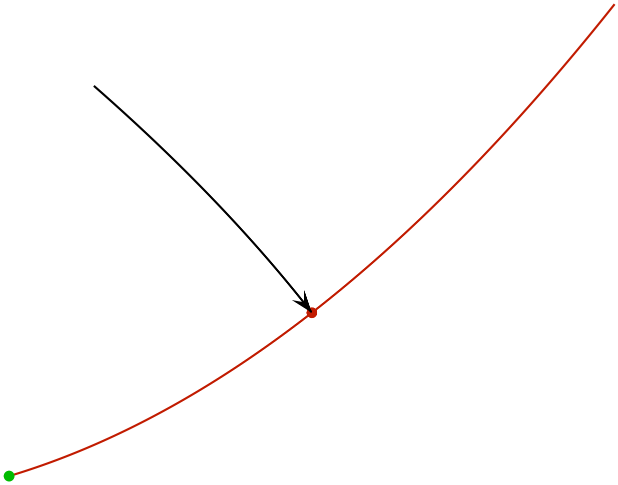
La Sambre: Hydraulic gates

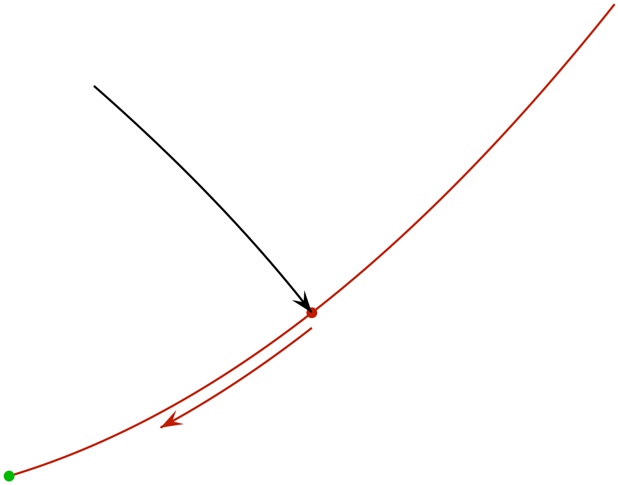
Closed loop versus open loop

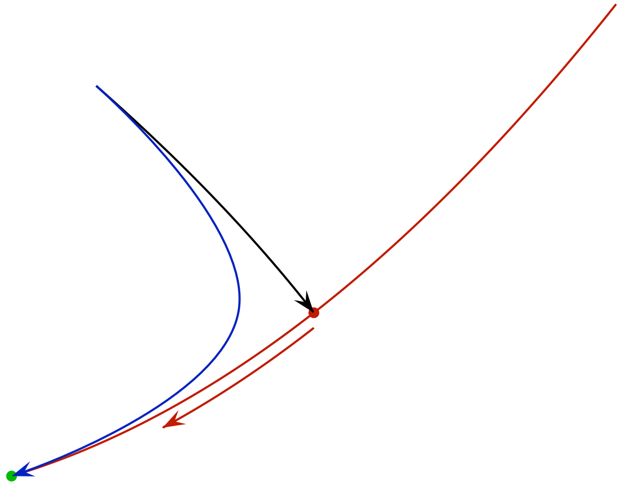












Phantom tracking method: An example

Let us consider the following control system.

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + u,\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3 + 2x_1x_2,$$

where the state is $(x_1, x_2, x_3, x_4)^{\mathrm{tr}} \in \mathbb{R}^4$ and the control is $u \in \mathbb{R}$.

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where the state is $(x_1, x_2, x_3, x_4)^{\mathrm{tr}} \in \mathbb{R}^4$ and the control is $u \in \mathbb{R}$.Roughly, we have two oscillators which are coupled by means of a quadratic term. The control is acting only on the first oscillator. The point $(x^\gamma, u_\gamma) := ((\gamma, 0, 0, 0)^{\mathrm{tr}}, \gamma)$ is an equilibrium of the control system. The linearized control system at this equilibrium is the linear control system

$$\dot{x}_1 = x_2,\, \dot{x}_2 = -x_1 + u,\, \dot{x}_3 = x_4,\, \dot{x}_4 = -x_3 + 2\gamma x_2,$$

where the state is $(x_1, x_2, x_3, x_4)^{\mathrm{tr}} \in \mathbb{R}^4$ and the control is $u \in \mathbb{R}$. This linear control system is controllable if (and only if) $\gamma \neq 0$. Therefore, if $\gamma \neq 0$ the equilibrium can be asymptotically stabilized for the nonlinear control system.

Stabilization of x^γ

One considers the following control Lyapunov function $V^\gamma : \mathbb{R}^4 \to \mathbb{R}$ defined by

$$V^\gamma(x) := (x_1 - \gamma)^2 + x_2^2 + x_3^2 + x_4^2, \ \forall x = (x_1, x_2, x_3, x_4)^{\mathrm{tr}} \in \mathbb{R}^4.$$

The time derivative of V^γ along the trajectory of our control system is

$$\dot{V}^\gamma = 2x_2(u - \gamma + 2x_1x_4).$$

Hence, in order to asymptotically stabilize x^γ for our control system, it is natural to consider the feedback law $u^\gamma : \mathbb{R}^4 \to \mathbb{R}$ defined by

$$u^\gamma := \gamma - 2x_1x_4 - x_2.$$

One gets $\dot{V}^\gamma = -2x_2^2$. Using the LaSalle invariance principle, one gets that this feedback law globally asymptotically stabilizes x^γ.

Let us now follow the phantom tracking strategy. In fact, instead of using $u^{\tilde{\gamma}}$ with a suitable $\tilde{\gamma} : \mathbb{R}^4 \to \mathbb{R}$ it is better to use directly a control Lyapunov of the type $V^{\tilde{\gamma}}$. Theoretically, the best way to choose $\tilde{\gamma}$ is to define it implicitly by proceeding in the following way. There exits an open neighborhood Ω of $0 \in \mathbb{R}^4$ and $V \in C^\infty(\Omega; \mathbb{R})$ such that

$$V(0) = 0, \forall x \in \Omega \setminus \{0\},$$
$$V(x) = (x_1 - V(x))^2 + x_2^2 + x_3^2 + x_4^2, \forall x = (x_1, x_2, x_3, x_4)^{\text{tr}} \in \Omega.$$

Therefore our choice of $\tilde{\gamma} = V(x)$, i.e. is such that $\tilde{\gamma}(x) = V^{\tilde{\gamma}(x)}$, $\tilde{\gamma}(0) = 0$. For the existence of V: use the implicit function theorem. In this simple case, V can be computed explicitly. One has $\dot{V} = 2(x_1 - V)(x_2 - \dot{V}) + 2x_2(-x_1 + u) + x_3x_4 + x_4(-x_3 + 2x_1x_2)$, i.e.,$(1 + 2x_1 - 2V)\dot{V} = 2x_2(u - V + x_1x_4)$. We define a feedback law $u : \Omega \to \mathbb{R}$ by $u := V - x_1x_4 - x_2$, which leads to $(1 + 2x_1 - 2V)\dot{V} = -2x_2^2 \leqslant 0$. One concludes that the feedback law u locally asymptotically our control system.

Two possible improvements:

(i) One can get global asymptotic stability. It suffices to modify V by requiring $V = V(x) = (x_1 - \theta(V(x)))^2 + x_2^2 + x_3^2 + x_4^2$, with a well chosen function $\theta : [0, +\infty) \to [0, +\infty)$.

(ii) One can get explicit feedback laws by using a dynamic extension: Replace the initial control system by the following one

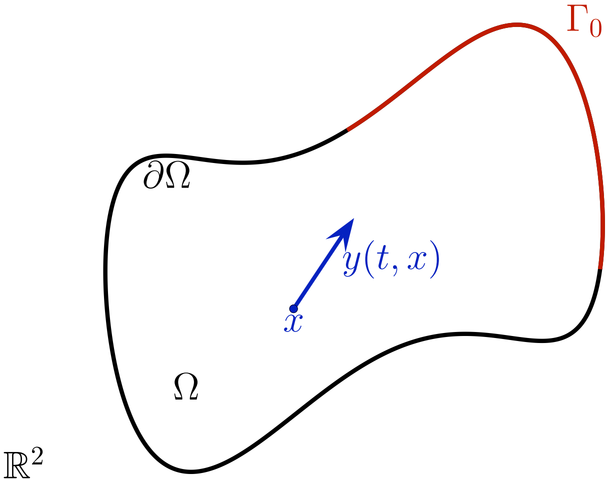
$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + u,\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3 + 2x_1x_2,\ \dot{\gamma} = v$$

where the state is $(x_1, x_2, x_3, x_4, \gamma)^{\text{tr}} \in \mathbb{R}^5$ and the control is $(u, v)^{\text{tr}} \in \mathbb{R}^2$. For $z = (x_1, x_2, x_3, x_4, \gamma)^{\text{tr}} \in \mathbb{R}^5$, one defines

$$\varphi(z) := (x_1 - \gamma)^2 + x_2^2 + x_3^2 + x_4^2,$$
$$W(z) := \varphi(z) + (\gamma - \varphi(z))^2.$$

Compute $\dot{W}$ etc.

Phantom tracking and stabilization of the Euler equations of incompressible fluids



Ω is assumed to be connected and simply connected.

The Euler control system

We denote by $\nu : \partial\Omega \to \mathbb{R}^2$ the outward unit normal vector field to Ω. Euler equations :

$$y_t + (y \cdot \nabla)y + \nabla p = 0, \text{ div } y = 0,$$
$$y \cdot \nu = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Gamma_0).$$

This system is under-determined. In order to have a determined system, one has to specify what is the control. There are at least two natural possibilities:

(a) The control is $y(t, x) \cdot n(x)$ on Γ_0 and the time derivative $\partial\omega/\partial t(t, x)$ of the vorticity at the points x of Γ_0 where $y(t, x) \cdot n(x) < 0$, i.e., at the points where the fluid enters into the domain Ω.

(b) The control is $y(t, x) \cdot n(x)$ on Γ_0 and the vorticity ω at the points x of Γ_0 where $y(t, x) \cdot n(x) < 0$.

To fix ideas, we deal only with the first case.

Failure of the linearization technique

The linearized control is

$$y_t + \nabla p = 0, \text{div } y = 0, \; y \cdot \nu = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Gamma_0).$$

Taking the curl of the first equation, on gets, with $\omega := \text{curl } y$

$$\omega_t = 0.$$

One cannot change ω! This linear control system is not stabilizable.

Definition of y^γ

Take $\theta : \overline{\Omega} \to \mathbb{R}$ such that

$$\Delta\theta = 0 \text{ in } \Omega,\ \frac{\partial\theta}{\partial\nu} = 0 \text{ on } \partial\Omega \setminus \Gamma_0.$$

Let us define define $(y^\gamma, p^\gamma) : [0,T] \times \Omega \to \mathbb{R}^2 \times \mathbb{R}$ by

$$y^\gamma(x) = \gamma\nabla\theta(x),\ p^\gamma(x) = -\frac{\gamma^2}{2}|\nabla\theta(x)|^2.$$

Then (y^γ, p^γ) is an equilibrium point of our Euler control system. The corresponding control is $\gamma\partial\theta/\partial n$ on Γ_0 and 0 for the vorticity at the points x of Γ_0 where $\partial\theta/\partial n < 0$.

Stabilization of the linearized control system around y^γ

The linearized control system around y^γ is

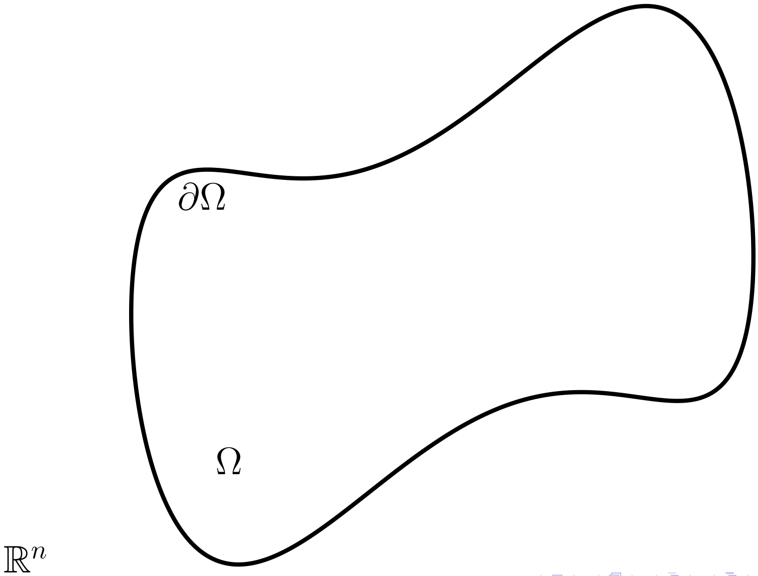
$$\begin{cases} y_t + (y^\gamma \cdot \nabla)y + (y \cdot \nabla)y^\gamma + \nabla p = 0, \ \text{div } y = 0 \text{ in } [0,T] \times \Omega, \\ y \cdot \nu = 0 \text{ on } [0,T] \times (\partial\Omega \setminus \Gamma_0). \end{cases}$$

Taking once more the curl of the first equation, one gets

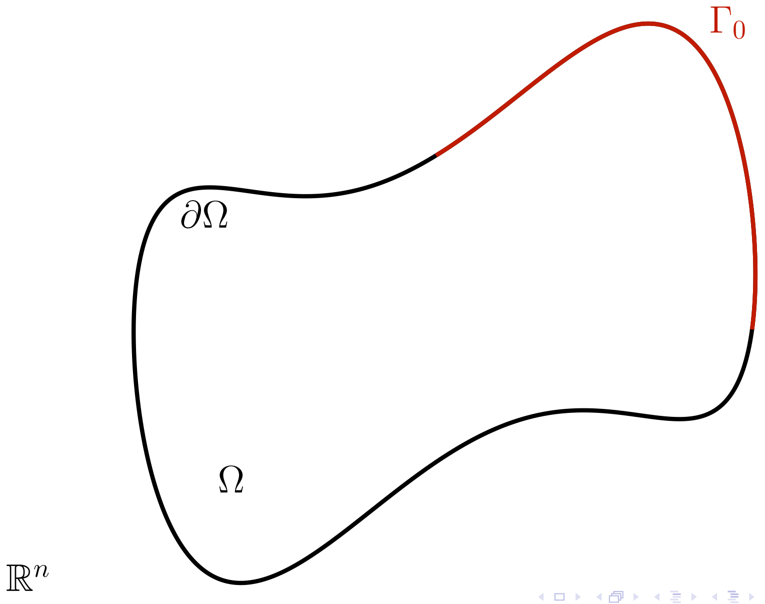
$$\omega_t + (y^\gamma \cdot \nabla)\omega = 0. \tag{*}$$

This is a simple transport equation on ω. If there exists $a \in \overline{\Omega}$ such that $\nabla\theta(a) = 0$, then $y^\gamma(a) = 0$ and $\omega_t(t,a) = 0$ showing that (*) is not stabilizable. This is the only obstruction: If $\nabla\theta$ does not vanish in $\overline{\Omega}$, one can easily stabilize (*): just use the control $\omega(t,x) = 0$ on the set $\{x \in \Gamma_0; \partial\theta/\partial\nu(x) < 0\}$.

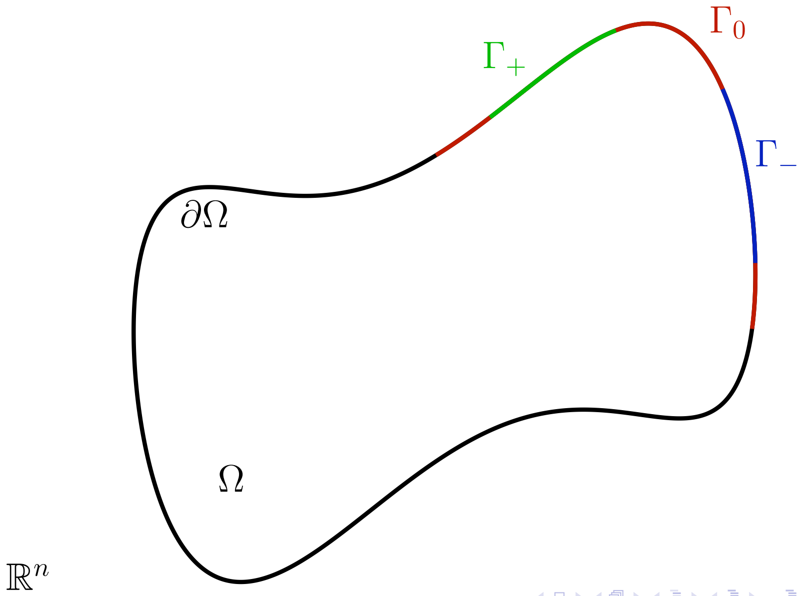
Construction of a good θ



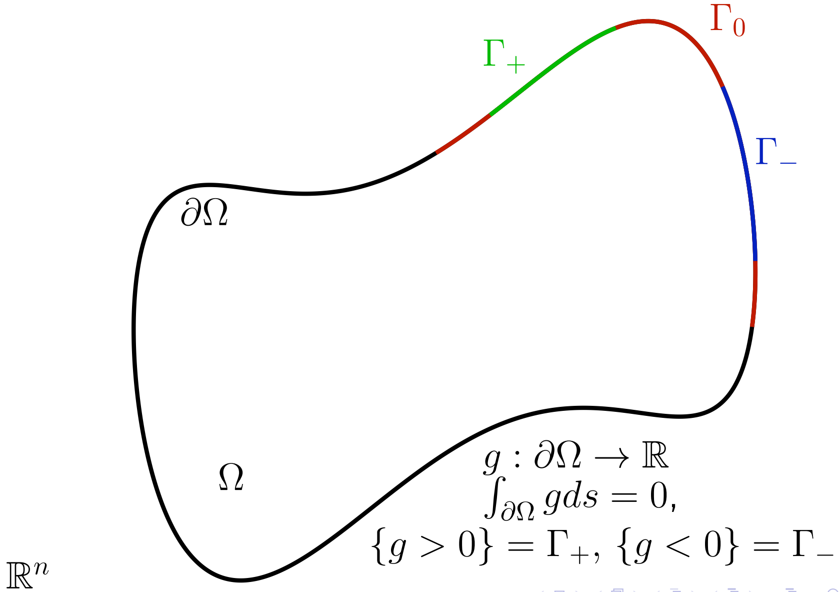
Construction of a good θ



Construction of a good θ



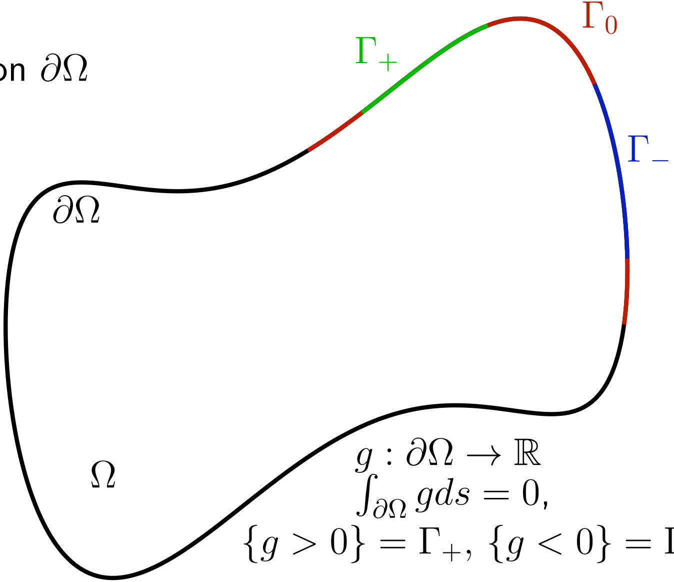
Construction of a good θ



Construction of a good θ

$$\Delta\theta = 0,$$

$$\frac{\partial\theta}{\partial\nu} = g \text{ on } \partial\Omega$$



$$g : \partial\Omega \to \mathbb{R}$$

$$\int_{\partial\Omega} g\,ds = 0,$$

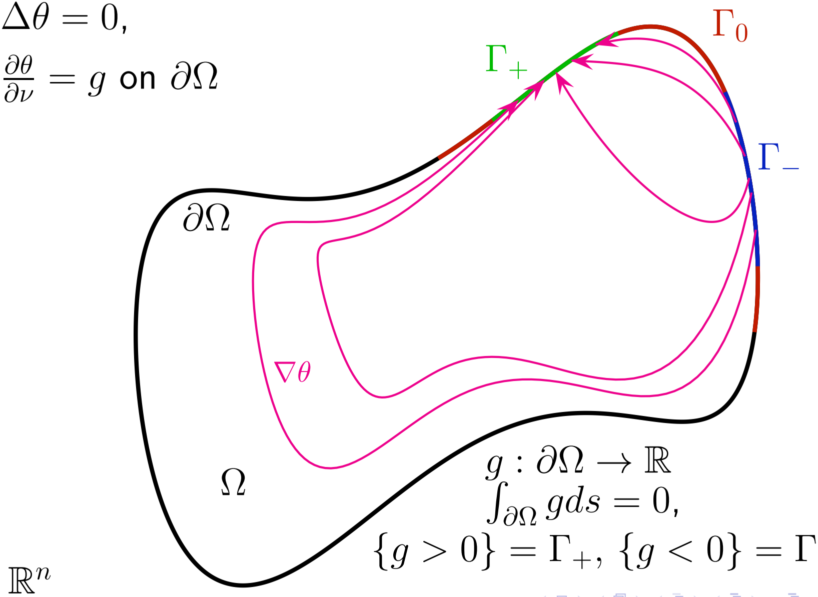
$$\{g > 0\} = \Gamma_+,\ \{g < 0\} = \Gamma_-$$

$\mathbb{R}^n$

Construction of a good θ

$$\Delta\theta = 0,$$

$$\frac{\partial\theta}{\partial\nu} = g \text{ on } \partial\Omega$$



$$g : \partial\Omega \to \mathbb{R}$$

$$\int_{\partial\Omega} g\,ds = 0,$$

$$\{g > 0\} = \Gamma_+,\ \{g < 0\} = \Gamma_-$$

Asymptotic stabilization of the Euler equations

Our stabilizing feedback law is

$$y \cdot \nu := M\,|\omega|_0 \,\frac{\partial \theta}{\partial \nu} \text{ on } \Gamma_-, \ \frac{\partial \omega}{\partial t} := -M\,|\omega|_{C^0(\overline{\Omega})}\,\omega \text{ on } \Gamma_-.$$

Theorem (JMC (1999))

There exists a positive constant M_0 such that, for every $\varepsilon \in (0,1]$, every $M \geqslant M_0/\varepsilon$ and every solution ω of the closed loop system,

$$|\omega(t)|_0 \leqslant \ \mathit{Min}\ \left\{ |\omega(0)|_{C^0(\overline{\Omega})}, \frac{\varepsilon}{t} \right\},\ \forall t > 0.$$

Remark

O. Glass (2005) has shown how to asymptotically stabilize the Euler equations if Ω is not simply connected by adding an integrator on ω.

Application to quantum control systems

1. K. Beauchard, JMC, M. Mirrahimi, P. Rouchon (2007) for

$$\iota\dot{\psi} = H_0 + uH_1\psi,\ \psi(t,\cdot) \in \mathbb{S}^{2N-1}.$$

2. K. Beauchard and M. Mirrahimi (2009) for a quantum particle in a one-dimensional infinite square potential well:

$$\begin{cases} \iota\psi_t = -\psi_{xx} + u(t)x\psi,\ x \in (0,1),\ \psi \in L^2((0,1);\mathbb{C}) \\ \int_0^1 |\psi(t,x)|^2\, dx = 1. \end{cases}$$

   This control is in infinite dimension and there is a problem to use the LaSalle invariance principle. This leads to important difficulties.

3. JMC, A. Grigoriu (in progress) for

$$\iota\dot{\psi} = H_0 + uH_1\psi + u^2 H_2\psi,\ \psi(t,\cdot) \in \mathbb{S}^{2N-1}.$$

Averaging (JMC, A. Grigoriu, C. Lefter and G. Turinici (2009))

The control system is

$$\iota\dot{\psi} = H_0\psi + uH_1\psi + u^2 H_2 + \omega\psi,$$

where H_0, H_1 and H_2 are $N \times N$ Hermitian matrices. The state is $\psi \in \mathbb{S}^{2N-1}$, the unit sphere of $\mathbb{R}^{2N} \simeq \mathbb{C}^N$, the control is $(u, \omega)^{\text{tr}} \in \mathbb{R}^2$. Again we assume that 0 is an eigenvalue of H_0 and consider a corresponding eigenvector $\psi_e \in \mathbb{S}^{2N-1}$ We consider the following time dependent feedback:

$$u(t, \psi) = \alpha(\psi) + \beta(\psi)\sin(t/\varepsilon).$$

The closed loop system is

$$(C_\varepsilon)\left\{\begin{array}{rcl} \iota\dot{\psi} & = & \Big(H_0 + \alpha(\psi)H_1 + \beta(\psi)\sin(t/\varepsilon)H_1 \\ & & +\alpha^2(\psi)H_2 + 2\alpha(\psi)\beta(\psi)\sin(t/\varepsilon)H_2 \\ & & +\beta^2(\psi)\sin^2(t/\varepsilon)H_2 + \omega(t)\Big)\psi(t). \end{array}\right.$$

Averaged system

For a differential system $\dot{x} = f(t, x)$, with f a T-periodic function: $f(T + t, x) = f(t, x)$, the averaged system is defined by $\dot{x}_{av} = f_{av}(x)$ where $f_{av}(x) = \frac{1}{T}\int_0^T f(t, x)dt$.
In our case the averaged system corresponding to the closed loop system is:

$$(C_{av}) \quad i\dot{\psi} = \left(H_0 + \alpha H_1 + \left(\alpha^2 + \frac{1}{2}\beta^2\right) H_2 + \omega\right)\psi.$$

In some sense we have now three independent controls, namely α, β and ω, instead of two, namely u and ω. Moreover, one knows that, if $\varepsilon > 0$ is small enough, the trajectories of (C_ε) are close to the trajectory of (C_{av}). The strategy is now simple. Using the damping method, one gets feedback laws $\psi \mapsto (\alpha(\psi); \beta(\psi); \omega(\psi))$ leading to global stabilizability of ψ_e on $\mathbb{S}^{2N-1} \setminus \{-\psi_e\}$ for the averaged system. Then, taking $\varepsilon > 0$ small enough, one gets a "practical" global stabilizability of ψ_e on $\mathbb{S}^{2N-1} \setminus \mathcal{V}$ for the closed loop system (C_ε), where $\mathcal{V}$ is a given neighborhood of $-\psi_e$.

Backstepping

For the backstepping method, we are interested in a control system (C) having the following structure:

$$\dot{x}_1 = f_1(x_1, x_2),\ \dot{x}_2 = u, \qquad \Sigma$$

where the state is $x = (x_1; x_2) := (x_1^{\mathsf{tr}}, x_2^{\mathsf{tr}})^{\mathsf{tr}} \in \mathbb{R}^{n_1+m} = \mathbb{R}^n$ with $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^m$ and the control is $u \in \mathbb{R}^m$. The key theorem for backstepping is the following one.

Theorem

Assume that $f_1 \in C^1(\mathbb{R}^{n_1} \times \mathbb{R}^m; \mathbb{R}^{n_1})$ and that the control system

$$\dot{x}_1 = f_1(x_1, v), \qquad \Sigma_1$$

where the state is $x_1 \in \mathbb{R}^{n_1}$ and the control $v \in \mathbb{R}^m$, can be globally asymptotically stabilized by means of a feedback law of class C^1. Then Σ can be globally asymptotically stabilized by means of a continuous feedback law.

References for the backstepping theorem

- Local version: very old, precise father(s)/mother(s) unknown,
- Global version:
  - D. Koditschek (1987),
  - C. Byrnes and A. Isidori (1989),
  - J. Tsinias (1989),
  - L. Praly, B. d'Andréa-Novel and JMC (1991) for low regularity for the feedback stabilizing $\dot{x}_1 = f_1(x_1, v)$.

Proof of the theorem

Let $v \in C^1(\mathbb{R}^{n_1}; \mathbb{R}^m)$ be a feedback law which globally asymptotically stabilizes $0 \in \mathbb{R}^{n_1}$ for the control system Σ_1. Then, by the converse of the second Lyapunov theorem, there exists a Lyapunov function of class C^∞ for the closed-loop system $\dot{x}_1 = f_1(x_1, v(x_1))$, that is, there exists a function $V \in C^\infty(\mathbb{R}^{n_1}; \mathbb{R})$ such that

$$f_1(x_1, v(x_1)) \cdot \nabla V(x_1) < 0, \, \forall x_1 \in \mathbb{R}^{n_1} \setminus \{0\},$$
$$V(x_1) \to +\infty \text{ as } |x_1| \to +\infty,$$
$$V(x_1) > V(0), \, \forall x_1 \in \mathbb{R}^{n_1} \setminus \{0\}.$$

A natural candidate for a control Lyapunov function for Σ is

$$W(x_1; x_2) := V(x_1) + \frac{1}{2}|x_2 - v(x_1)|^2, \, \forall (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^m.$$

Indeed, one has, for such a W,

$$W(x_1; x_2) \to +\infty \text{ as } |x_1| + |x_2| \to +\infty,$$
$$W(x_1; x_2) > W(0, 0), \, \forall (x_1, x_2) \in (\mathbb{R}^{n_1} \times \mathbb{R}^m) \setminus \{(0, 0)\}.$$

The time-derivative $\dot{W}$ of W along the trajectories of Σ is

$$\dot{W} = f_1(x_1, x_2) \cdot \nabla V(x_1) - (x_2 - v(x_1)) \cdot (v'(x_1) f_1(x_1, x_2) - u).$$

There exists $G \in C^0(\mathbb{R}^{n_1} \times \mathbb{R}^m \times \mathbb{R}^m; \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{n_1}))$ such that $f_1(x_1, x_2) - f_1(x_1, y) = G(x_1, x_2, y)(x_2 - y)$. Therefore

$$\dot{W} = f_1(x_1, v(x_1)) \cdot \nabla V(x_1)$$
$$+ \left[u^{\text{tr}} - (v'(x_1) f_1(x_1, x_2))^{\text{tr}} + (\nabla V(x_1))^{\text{tr}} G(x_1, x_2, v(x_1))\right](x_2 - v(x_1)).$$

Hence, if one takes as a feedback law for the control system Σ

$$u := v'(x_1) f_1(x_1, x_2) - G(x_1, x_2, v(x_1))^{\text{tr}} \nabla V(x_1) - (x_2 - v(x_1)),$$

one gets $\dot{W} = f_1(x_1, v(x_1)) \cdot \nabla V(x_1) - |x_2 - v(x_1)|^2$. Hence

$$\dot{W}(x_1; x_2) < 0, \, \forall (x_1, x_2) \in (\mathbb{R}^{n_1} \times \mathbb{R}^m) \setminus \{(0, 0)\}.$$

In conclusion, u globally asymptotically stabilizes Σ.

Various generalizations

Various (straightforward) adaptations are possible. For example

1. Adaptation to the framework of time-varying feedback laws.
2. It is possible to add an "integrator" to only part of the components of v.
3. In the above construction, instead of using a strict Lyapunov function one can use a Lyapunov function satisfying the assumptions of the LaSalle invariance principle.

In the next slides we show how to use these three adaptations together in order to asymptotically stabilize the baby stroller control system.

Stabilization of the baby stroller

The baby stroller control system is

$$\dot{x}_1 = u_1 \cos x_3,\ \dot{x}_2 = u_1 \sin x_3,\ \dot{x}_3 = u_2, \qquad \Sigma$$

where the state is $x := (x_1; x_2; x_3) \in \mathbb{R}^3$ and the control is $(u_1; u_2) \in \mathbb{R}^2$. Following the backstepping approach, we first deal with the stabilization of the control system

$$\dot{x}_1 = u_1 \cos x_3,\ \dot{x}_2 = u_1 \sin x_3, \qquad \Sigma_1$$

where the state is $(x_1; x_2) \in \mathbb{R}^2$ and the control is $(u_1; x_3) \in \mathbb{R}^2$ and then we add an "integration x_3". As a potential control Lyapunov for Σ_1, we consider

$$V(x_1; x_2) := \frac{1}{2}\left(x_1^2 + x_2^2\right),\ \forall (x_1; x_2) \in \mathbb{R}^2.$$

Note that

$$V(x') > V(0),\ \forall x' \in \mathbb{R}^2,\ \lim_{|x'| \to +\infty} V(x') = +\infty.$$

Along the trajectory of Σ_1, we have

$$\dot{V} = (x_1 \cos x_3 + x_2 \sin x_3) u_1.$$

Let $T > 0$. Let $f \in C^1(\mathbb{R})$ be a T-periodic function which is not constant. We define $\bar{x}_3 : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ and $\bar{u}_1 : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ by

$$\bar{x}_3(t, x') := f(t)|x'|^2,$$
$$\bar{u}_1(t, x') := -(x_1 \cos(\bar{x}_3(t, x')) + x_2 \sin(\bar{x}_3(t, x'))).$$

The functions $\bar{x}_3$ and $\bar{u}_1$ are T-periodic to time and vanish on $\mathbb{R} \times \{0\}$. One has $\dot{V} = -\bar{u}_1^2 \leqslant 0$. Using the LaSalle invariance principle one checks that $0 \in \mathbb{R}^2$ is globally asymptotically stable for the closed loop system

$$\dot{x}_1 = \bar{u}_1 \cos \bar{x}_3,\ \dot{x}_2 = \bar{u}_1 \sin \bar{x}_3.$$

Following the backstepping strategy, we consider the potential time-varying control Lyapunov function W for the baby stroller control system Σ (which is obtained by "adding an integration" on the variable x_3 to the control system Σ_1)

$$W(t,x) := V(x') + \frac{1}{2}(x_3 - \bar{x}_3(t,x'))^2, \ \forall t \in \mathbb{R}, \ \forall x = (x_1; x_2; x_3) \in \mathbb{R}^3,$$

with $x' := (x_1; x_2)$. Note that W is T-periodic with respect to time and that

$$W(t,x) > W(t,0) = 0, \ \forall t \in \mathbb{R}, \ \forall x \in \mathbb{R}^3 \setminus \{0\},$$
$$\lim_{|x| \to +\infty} \min\{W(t,x); \ t \in [0,T]\} = +\infty.$$

The time derivative of W along the trajectory of Σ is

$$\dot{W} = (x_1 \cos x_3 + x_2 \sin x_3)u_1 + (x_3 - \bar{x}_3)(u_2 - \xi),$$

with $\xi(t,(x;u_1)) := f'(t)|x'|^2 + 2f(t)u_1(x_1 \cos x_3 + x_2 \sin x_3)$.

We define our time-varying stabilizing feedback law by

$$\begin{aligned} u_1(t,(x_1;x_2;x_3)) &:= -(x_1\cos x_3 + x_2 \sin x_3), \\ u_2(t,(x_1;x_2;x_3)) &:= \xi(t,(x;-(x_1\cos x_3 + x_2\sin x_3))) \\ &\quad -(x_3 - \bar{x}_3(t,x')), \end{aligned}$$

so that

$$\dot{W} = -(x_1\cos x_3 + x_2\sin x_3)^2 + (x_3 - \bar{x}_3)^2 \leqslant 0.$$

The functions u_1 and u_2 are T-periodic to time and vanish on $\mathbb{R}\times\{0\}$. Using once more the LaSalle invariance principle, one checks that $0\in\mathbb{R}^3$ is globally asymptotically stable for the closed loop system

$$\dot{x}_1 = u_1\cos x_3,\ \dot{x}_2 = u_1\sin x_3,\ \dot{x}_3 = u_2.$$