Quantum Process Tomography via Compressive Sensing

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A brief summary of:

Efficient Measurement of Quantum Dynamics via Compressive Sensing

by

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Matthew Broome, Marcelo de Almeida, Alessandro Fedrizzi, Andrew White.

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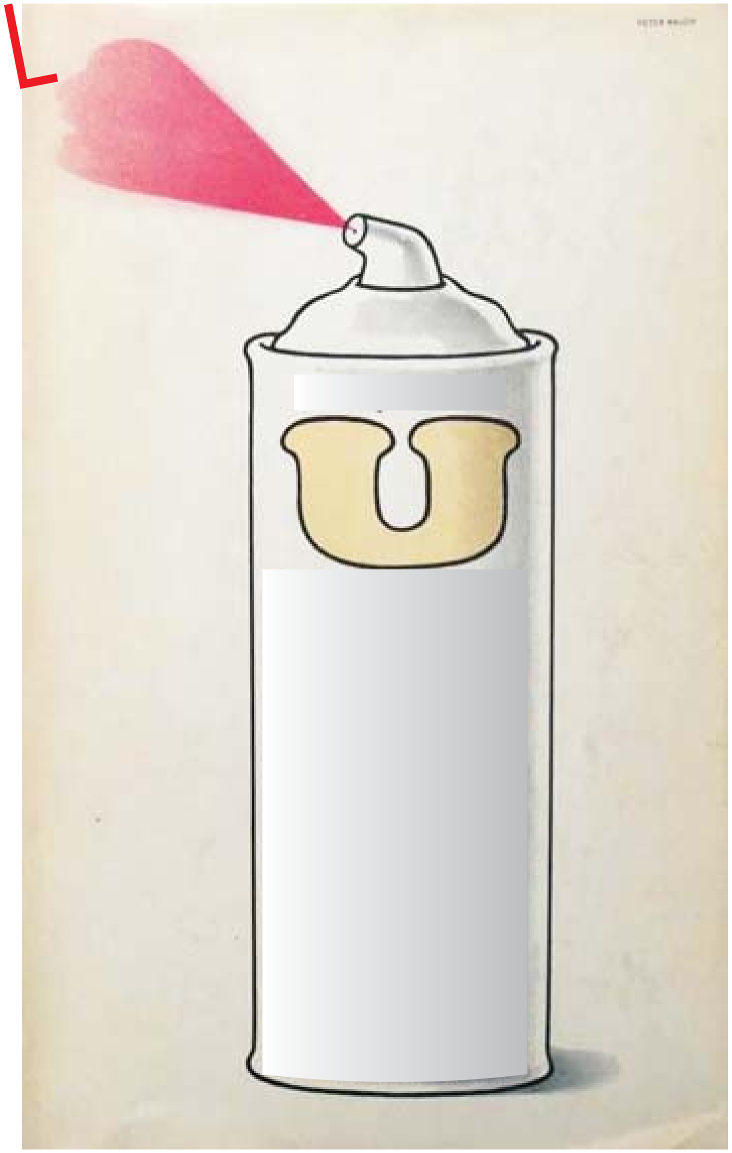
C O N T R O L

Ubiquitous
for
Quantum Information Processing

Control is required
Everywhere
there is a desired Unitary.



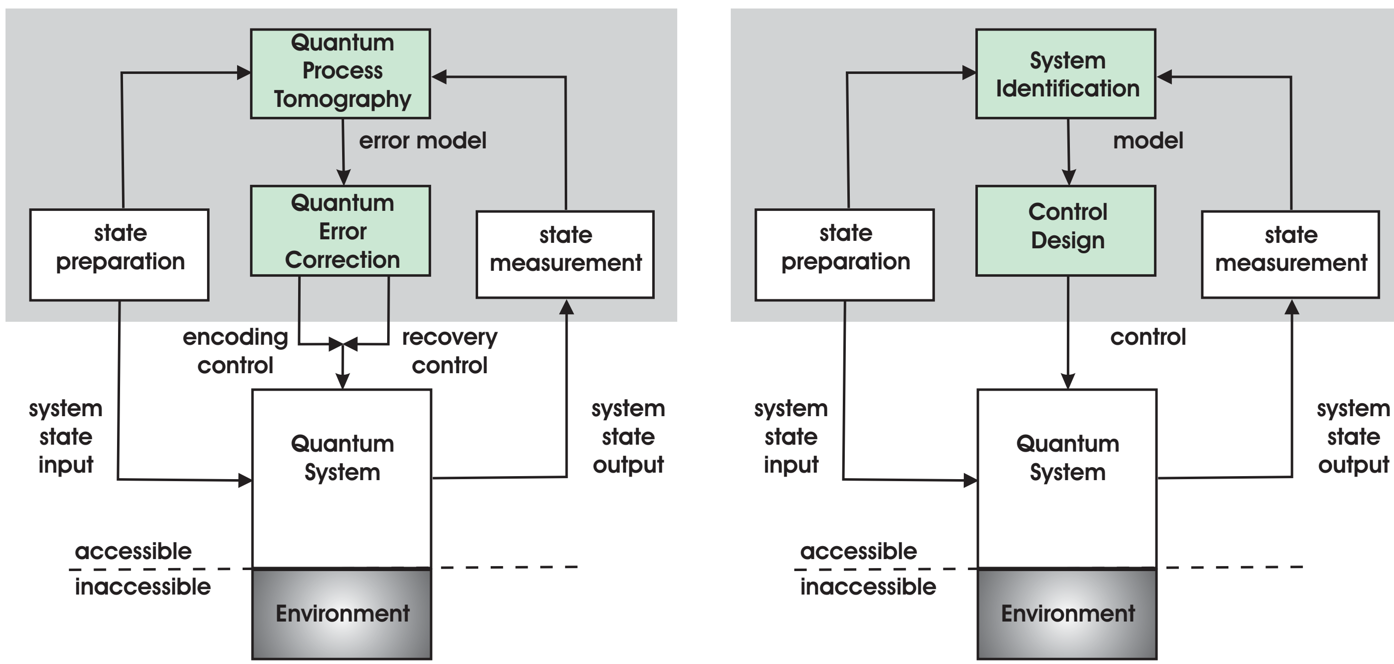
Source: 1969 Doubleday Book Cover
UBIK by Philip K. Dick

Ever present (probably long-term) goal

- **CQPT for Control design**

Control is ubiquitous in quantum information systems, *i.e.*, every required unitary logic gate is generated by control. Control is required to implement quantum error correction (QEC), dynamical decoupling (DD), or any Hamiltonian-based control design.

The efficiency of CQPT suggests an intriguing prospect: to implement a complete "on-line" quantum control scheme, that takes the results of CQPT as input, and iterates until it finds an optimal control.



Quantum estimation/system identification provides a means to

- characterize a quantum process
- verify the performance of a designed quantum device
- aid in the design or action of the device
- assist decoherence prevention/correction & control

Need protocols that

- require the minimum amount of resources
  (*e.g.*, measurements, inputs, computations, *etc.*.)
- provide an estimate of precision of the reconstruction
- establish the limit of performance

Convex Optimization can address some of these goals

The following can be cast as convex optimization problems

- quantum state & process tomography
- optimal experiment design
- quantum error correction
- quantum state detection

These are generally not convex optimization problems

- Hamiltonian parameter estimation
- quantum control design

Not every non-convex problem cannot be solved

Emerging Applications of Compressed Sensing for Quantum Estimation

- **Compressed Sensing** (CS) initially developed to exploit the general *nearly sparse* features of natural audio and video signals.
  - E. Candes, J. Romberg, T. Tao. Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information. *IEEE Trans. on Information Theory*, February 2006
  - D. Donoho, Compressed sensing. *IEEE Trans. on Information Theory*, April 2006

- **Applications** of compressed sensing have proliferated, though are just emerging for quantum system estimation:
  - extension of CS theory and numerical simulations for QPT
    - R.L. Kosut. Quantum process tomography via L1-norm minimization. *Arxiv preprint arXiv:0812.4323*, 2008.
    - A. Shabani, R. L. Kosut, and H. Rabitz. Compressed quantum process tomography. *arXiv.org:0910.5498[quant-ph]*, 2009.
  - the first experimental demonstration of CQPT (as discussed here).
    - A. Shabani, R. L. Kosut, H. Rabitz, A. Fedrizzi, M. Almeida, M. Broome, and A. White. Exponentially faster measurement of quantum dynamics via compressed sensing. *in preparation*, 2010.

more quantum system applications ...

– ghost-imaging

  * O. Katz, Y. Bromberg, and Y. Silberberg. Compressive ghost imaging. *Applied Physics Letters*, 95(13):1110, 2009.
  * W. Gong and S. Han. Super-resolution ghost imaging via compressive sampling reconstruction. *Arxiv preprint arXiv:0910.4823*, 2009.

– quantum state tomography (QST) for low rank density matrices

  * D. Gross, Y.K. Liu, S.T. Flammia, S. Becker, and J. Eisert. Quantum state tomography via compressed sensing. *Arxiv preprint arXiv:0909.3304*, 2009.

  approach is based on ideas from *Matrix Completion*

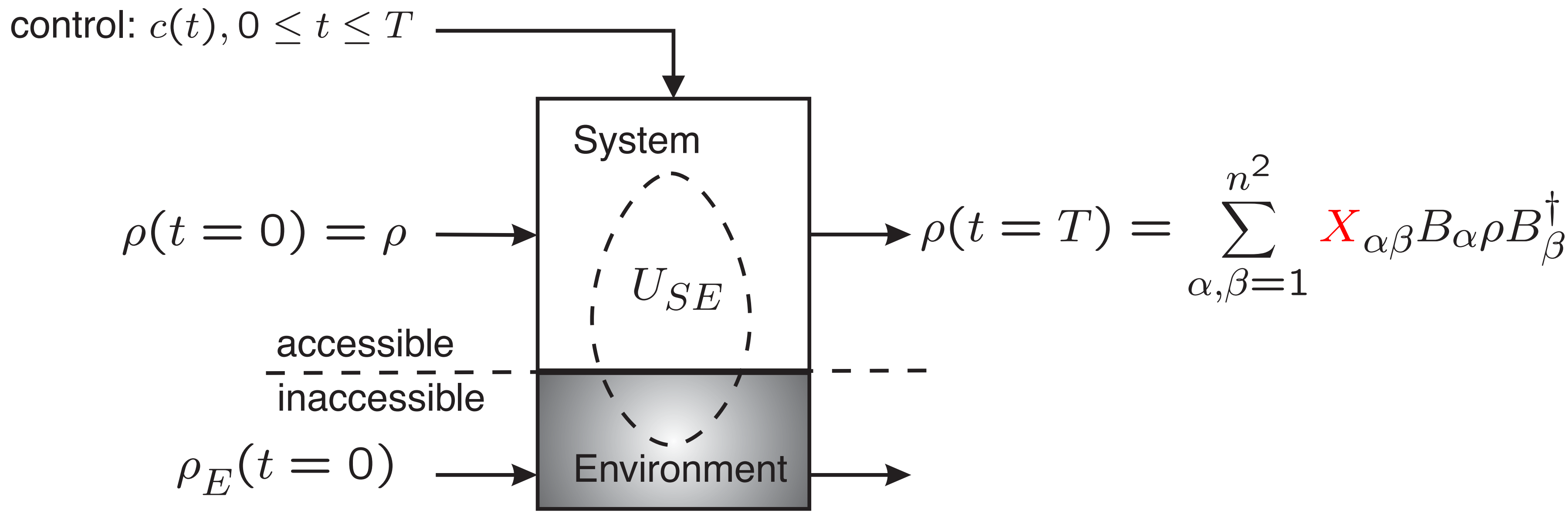
  * E. J. Candes and B. Recht. Exact matrix completion via convex optimization. *Found. of Comput. Math.*, pages 717–772, 2009.

Quantum Process Tomography

The characterization of dynamics of open quantum systems

- a fundamental problem in quantum information science and coherent control
  - for verifying the performance of an information-processing device
  - for design of decoherence prevention/correction methods

Process Matrix Representation



- $X \in \mathbf{C}^{n^2 \times n^2}$ with X positive semidefinite (written $X \ge 0$) characterizes state-to-state map of any *open* n-level system where $\rho, \ \rho_E$ are uncorrelated at $t=0$

- X is not unique, depends on matrix basis set $\{B_\alpha\}$, typically orthonormal: **Tr**$(B_\alpha^\dagger B_\beta) = \delta_{\alpha\beta}$.

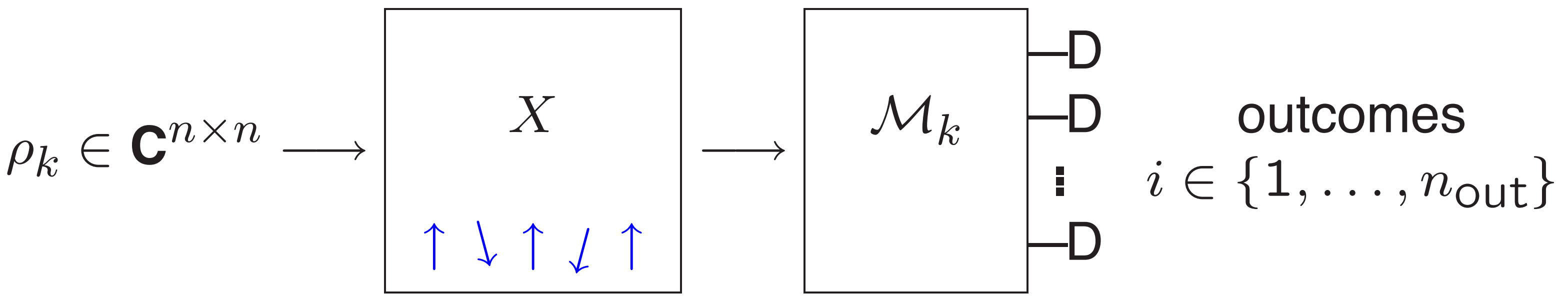
- trace-preserving if $\sum_{\alpha\beta} X_{\alpha\beta} B_\beta^\dagger B_\alpha = I_n$

- unitary system: $\mathcal{A}(\rho) = U \rho U^\dagger$ (single unitary OSR element)

- **Quantum Process Tomography (QPT)** – estimate X

Standard QPT

Repeat experiment N_k times in each configuration $k \in \{1, \ldots, n_{\text{cfg}}\}$



- form empirical estimate of probability outcomes: $p_{ik}^{\text{emp}} = N_{ik}/N_k$
  — N_{ik} is number of times outcome i occurred in configuration k

- form model probability outcomes: $p_{ik}(X) = \mathbf{Tr}\,(G_{ik}X)$
  — $(G_{ik})_{\alpha\beta} = \mathbf{Tr}(B_\alpha\, \rho_k\, B_\beta^\dagger\, M_{ik})$

- solve for least-squares estimate from:

$$\boxed{\begin{array}{ll} \text{minimize} & V(X) = \sum_{i,k} \left(p_{ik}^{\text{emp}} - p_{ik}(X) \right)^2 \\ \text{subject to} & \sum_{\alpha,\beta} X_{\alpha\beta} B_\beta^\dagger B_\alpha = I_n, \quad X \geq 0 \end{array}}$$

- this is a convex optimization in $X \in \mathbf{C}^{n^2\times n^2}$

Process matrix parameters

- the for process matrix $X \in \mathbf{C}^{n^2 \times n^2}$:

$$
\begin{array}{lcl}
X \geq 0 \text{ (positive semidefinite)} & \Rightarrow & n^4 \text{ real parameters} \\
\sum_{\alpha,\beta=1}^{n^2} X_{\alpha\beta} B_\beta^\dagger B_\alpha = I_n & \Rightarrow & n^2 \text{ linear constraints}
\end{array}
$$

- $\Rightarrow n^4 - n^2$ real parameters

- for q qubits $n = 2^q \Rightarrow$ scaling with parameters is exponential in the number of qubits, *e.g.*,

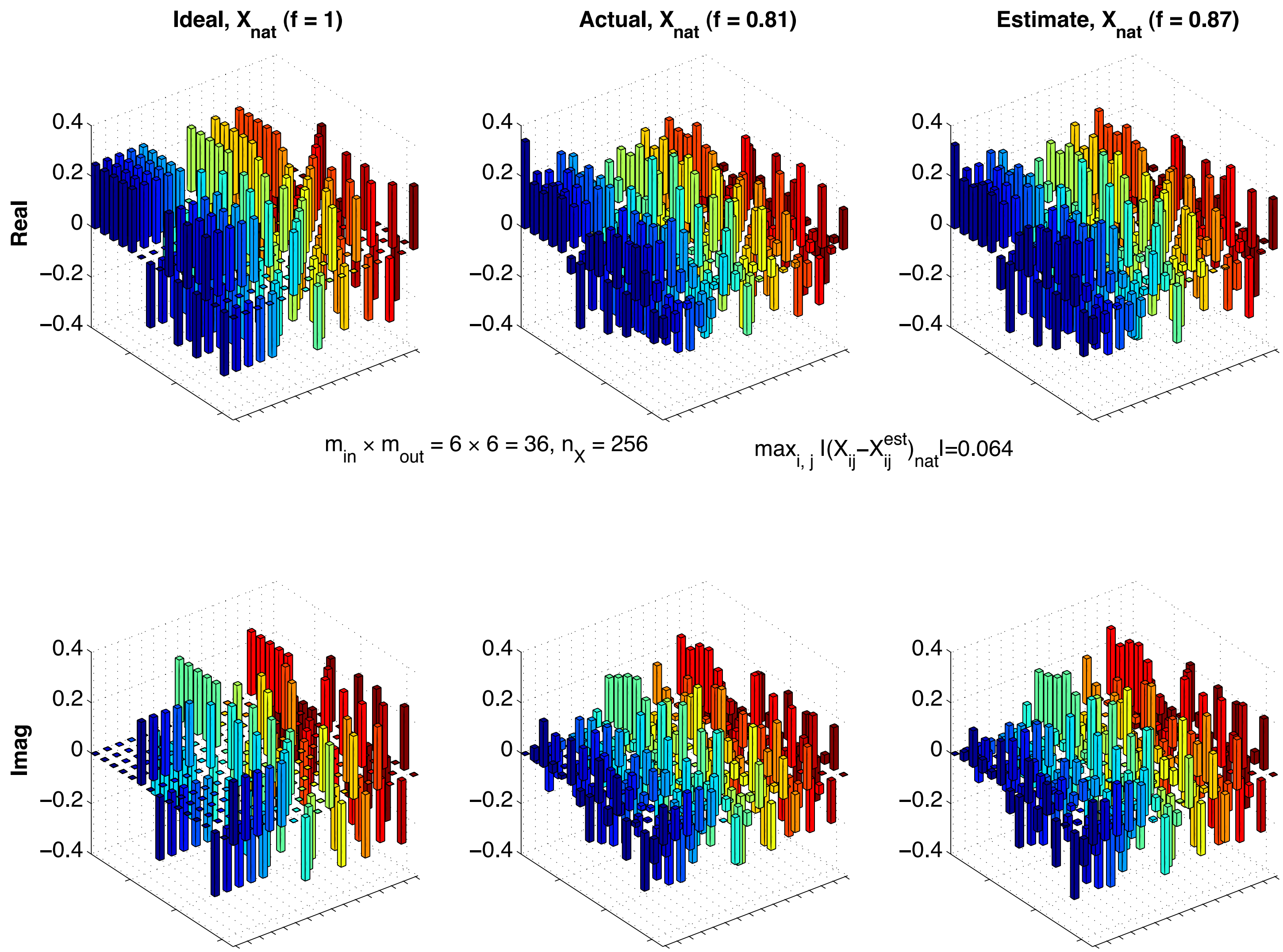
$$
q = [1,\ 2,\ 3,\ 4] \quad \Rightarrow \quad n^4 - n^2 = [12,\ 240,\ 4032,\ 65280]
$$

- demanding on laboratory resources and computation time.

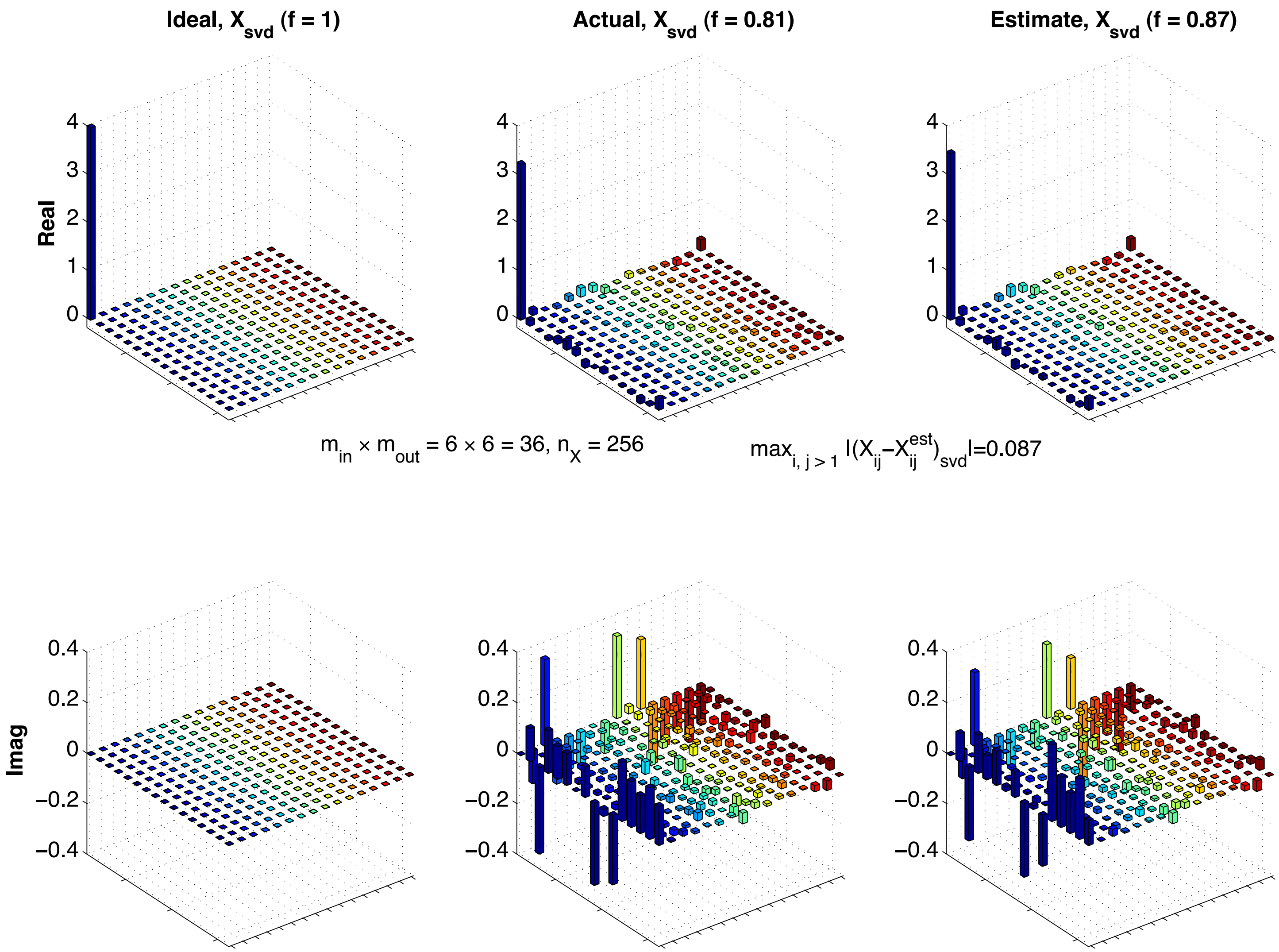
Sparsity

- Can the process matrix be characterized by a few parameters?
  - Many experimental results show a process matrix that is sparse or has a few significant elements, *i.e.*, *almost sparse*. This is seen, for example, in experiments with NMR, ion traps, and linear optics.
  - in these cases there are often a small number of sources of errors, or a few dominant system-environment interactions, or the system is weakly decohering.
  - for these reasons a "natural basis" selection is adequate although not as efficient as possible.
- Can this be generalized?
  - For an initially designed quantum system whose dynamics are close to a desired unitary (a primary goal in quantum information processing) the process matrix in a basis corresponding to the ideal unitary is almost sparse.
  - Ideally, with no environmental interactions, the process matrix in the ideal unitary basis has a single non-zero element in the 11-location equal to n, the dimension of the system.
  - Channel fidelity compares the actual channel X with the ideal unitary U_{ideal}. Expressing X in the basis of the ideal unitary, $f_{\text{chn}}(X, U_{\text{ideal}}) = X_{11}/n$

Numerical example: CQPT of noisy Quantum Fourier Transform (Natural-Basis)



Numerical example:
CQPT of noisy Quantum Fourier Transform (SVD-Basis)



Compressed Sensing

- Compressed Sensing methods (ℓ_1-norm minimization) have proven to be extremely effective in significantly reducing resources for estimating sparse or almost sparse signals.[a] [b]
- **Outline for this section**
  - review of standard QPT via least-squares
  - review of basic CS theory and algorithm
  - numerical example
  - experimental demonstration

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[a] E. Candes, J. Romberg, T. Tao. Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information. *IEEE Trans. on Information Theory*, February 2006

[b] D. Donoho, Compressed sensing. *IEEE Trans. on Information Theory*, April 2006

Review of Standard Linear Least-Squares Estimation (ℓ_2-norm minimization)

- standard problem: estimate $x_0 \in \mathbf{R}^N$ from noisy linear measurements $y \in \mathbf{R}^m$

$$y = Ax_0 + w, \qquad \|w\|_{\ell_2} \le \epsilon \quad (A \in \mathbf{R}^{m \times N})$$

  - *complete* measurements (**rank**$(A) \ge N$):

$$\text{minimize} \|Ax - y\|_{\ell_2} \quad \Longrightarrow \quad \|x^\star - x_0\|_{\ell_2} = \mathcal{O}(\epsilon)$$

  - *incomplete* measurements ($m < N$), result usually meaningless without a prior, *e.g.*, $\|P^{-1}(x_0 - \overline{x}_0)\|_{\ell_2} \le 1$

Compressed Sensing

- With incomplete measurements ($m < N$)

$$\boxed{\begin{array}{c}\text{minimize } \|x\|_{\ell_1} \text{ subject to } \|Ax - y\|_{\ell_2} \le \epsilon \\ \Downarrow \\ \|x^\star - x_0\|_{\ell_2} = \mathcal{O}(\|x_0(s) - x_0\|_{\ell_1}) + \mathcal{O}(\epsilon) \\ \\ (x_0(s) \text{ is the } \textit{best } s\textit{-sparse approximation} \text{ of } x_0)\end{array}}$$

- Known conditions:
  - the matrix A satisfies the *restricted isometry property* (RIP):

$$(1-\delta_s)\|x_s\|_{\ell_2}^2 \le \|Ax_s\|_{\ell_2}^2 \le (1+\delta_s)\|x_s\|_{\ell_2}^2, \text{ for all } s\text{-sparse } x_s$$

  - the *isometry constant* $\delta_{2s} < \sqrt{2} - 1$
  - the number of configurations $m \ge C_0 s \log(N/s)$

- if x_0 is s-sparse and noise-free measurements, $\epsilon = 0$, these conditions insure *perfect* signal recovery with high probability.

Compressed Quantum Process Tomography (CQPT) [a] [b]

- Estimate true $X_0 \in \mathbf{C}^{n^2 \times n^2}$ by solving the ℓ_1-norm minimization:

$$\boxed{\begin{array}{ll} \text{minimize} & \|\vec{X}\|_{\ell_1} \equiv \sum_{\alpha,\beta} |X_{\alpha\beta}| \\[2em] \text{subject to} & \|\vec{p}^{\,\text{emp}} - G\,\vec{X}\|_{\ell_2} \leq \epsilon \\ & X \geq 0, \quad \sum_{\alpha,\beta} X_{\alpha\beta} B_\beta^\dagger B_\alpha = I_n \end{array}}$$

- If $G \in \mathbf{C}^{m \times n^4}$ satisfies RIP, and $m = \mathcal{O}(s \log(n^4))$, then solution $X^\star$ satisfies:

$$\boxed{\begin{array}{c} \|\vec{X}^\star - \vec{X}_0\|_{\ell_2} = \mathcal{O}(\|\vec{X}_0(s) - \vec{X}_0\|_{\ell_1}) + \mathcal{O}(\epsilon) \\[2em] X_0(s) \text{ is the best } s\text{-sparse process matrix approximation of } X_0 \end{array}}$$

- for q-qubits, $n = 2^q \Rightarrow m = \mathcal{O}(sq)$.
  - Heralds a scaling of resources **linear** in number of qubits!

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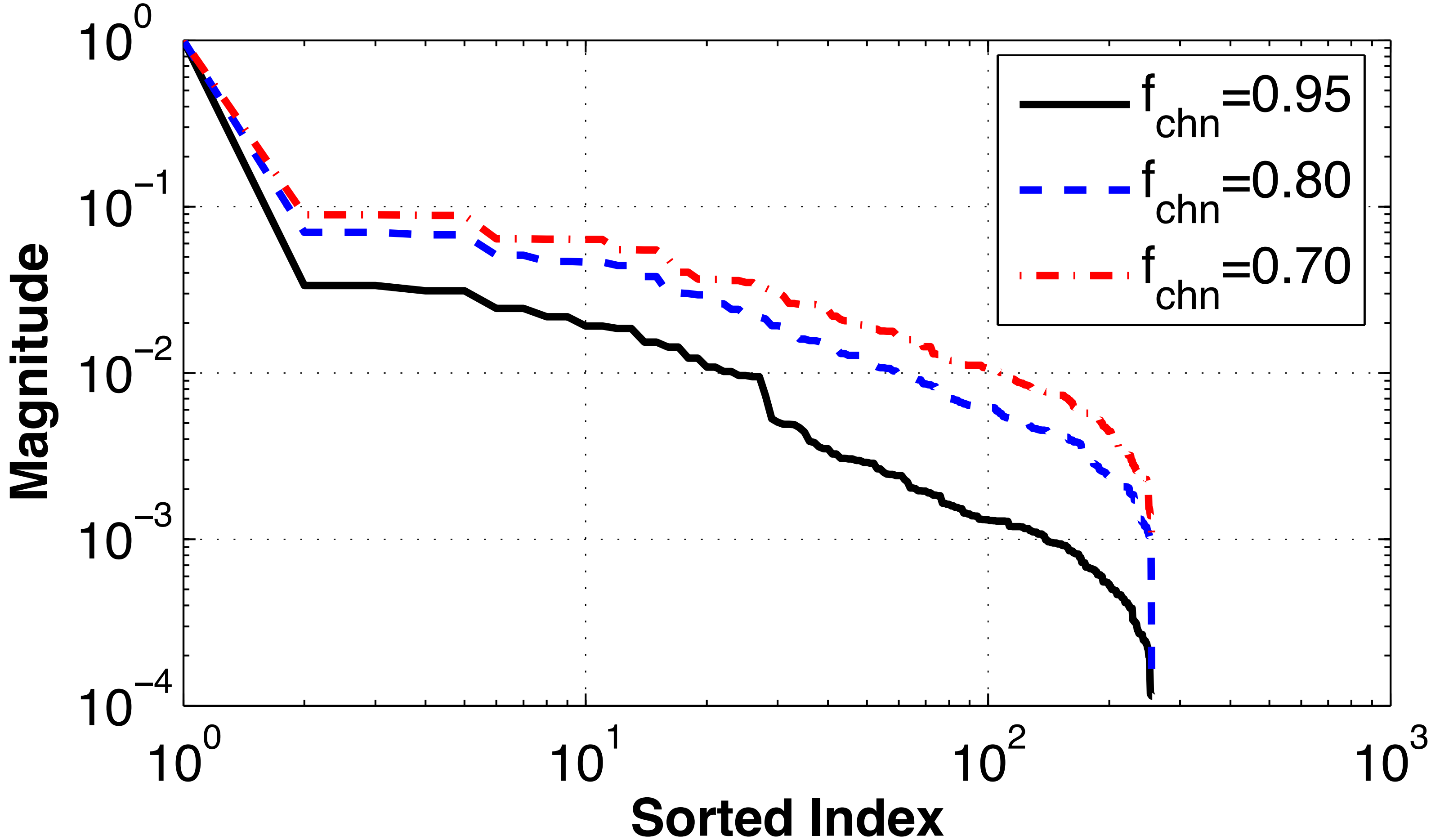
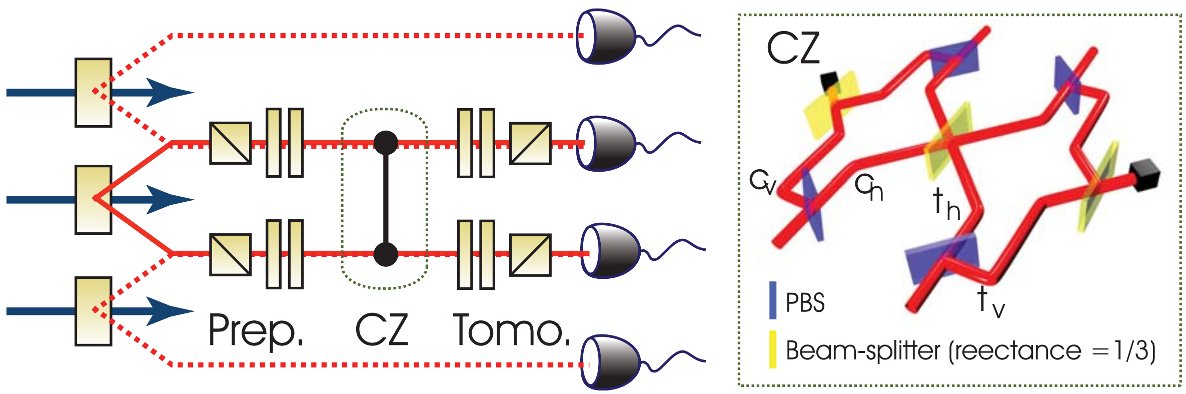
[a] A. Shabani, R. Kosut, H. Rabitz. Compressed Quantum Process Tomography. quant-ph/0910.5498v1, 2009

[b] R. Kosut. Quantum process tomography via ℓ_1-norm minimization. quant-ph/0812.4323, 2008



- Absolute values of the 256 process matrix elements of $X_{\mathsf{qft}}^{\mathsf{true}} \in \mathbf{C}^{16\times 16}$ sorted by relative magnitude (with respect to maximum) for $f_{\mathsf{chn}} \in \{0.95, 0.80, 0.70\}$.
- Sparsity approximation value (horizontal axis) where curves cross below 0.01 correlate well with resources to achieve high fidelity estimates.

Experimental set-up: CZ-Gate [a]



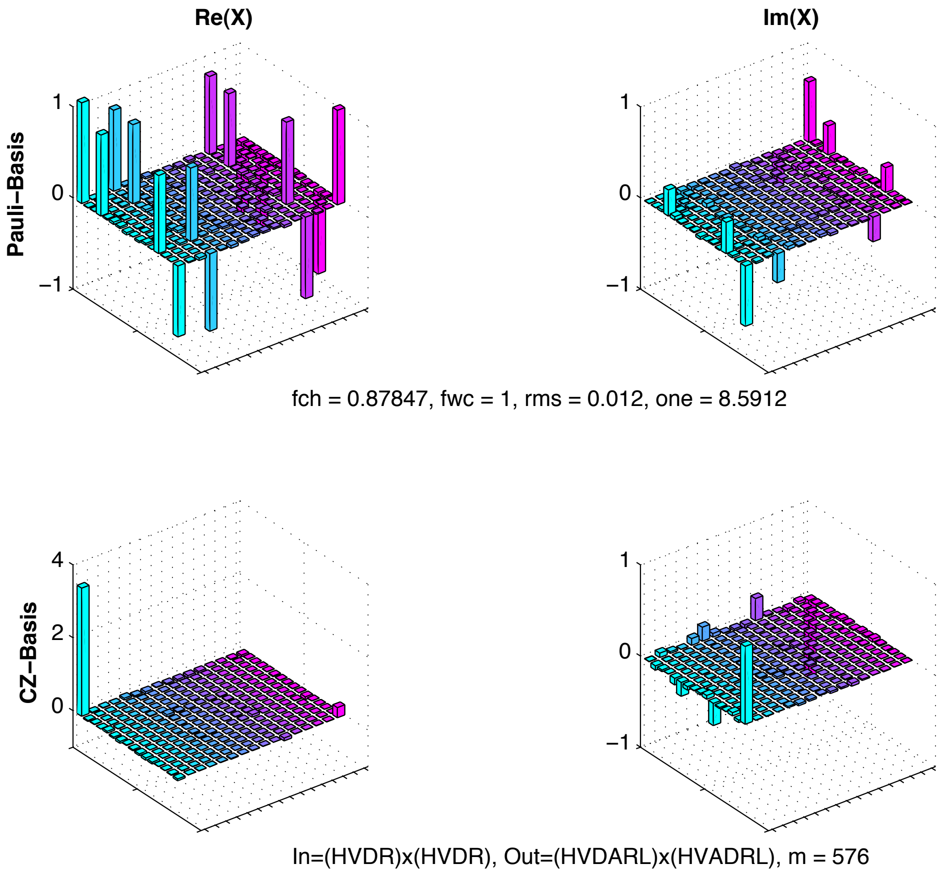
- Photon pairs are created via spontaneous parametric downconversion in a β-Barium-Borate crystal pumped with a 76 MHz fs laser at a wavelength of 410 nm. Polarizing beamsplitters, half- and quarter-wave plates are used to prepare the photons in a specific state. The photons are detected by single-photon avalanche photo detectors. The probabilistic CZ gate is based on two-photon interference at a partially polarizing beamsplitter. It was implemented between two polarization-encoded photons from the same creation event.

- CQPT was tested against standard QPT (via least-squares) by performing both on a range of decoherence levels engendered by varying the laser pump power.

[a] A. Fedrizzi, M. Almeida, M. Broome, and A. White, Center for Quantum Computer Technology, Department of Physics, The University of Queensland, Australia

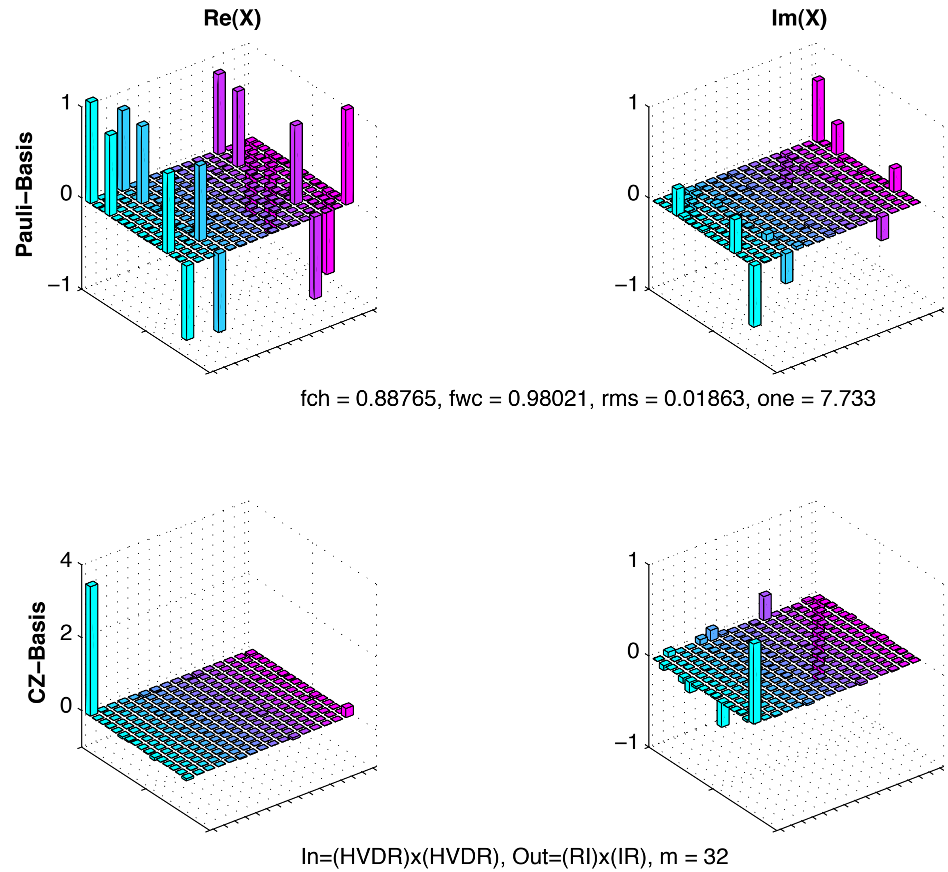
Experimental results: CQPT of CZ-Gate

All configurations
($m = 576$)



$$f_{\text{chn}} = 0.87847$$
$$f_{\text{wc}} = 1$$

Single-observable configurations
($m = 32$)

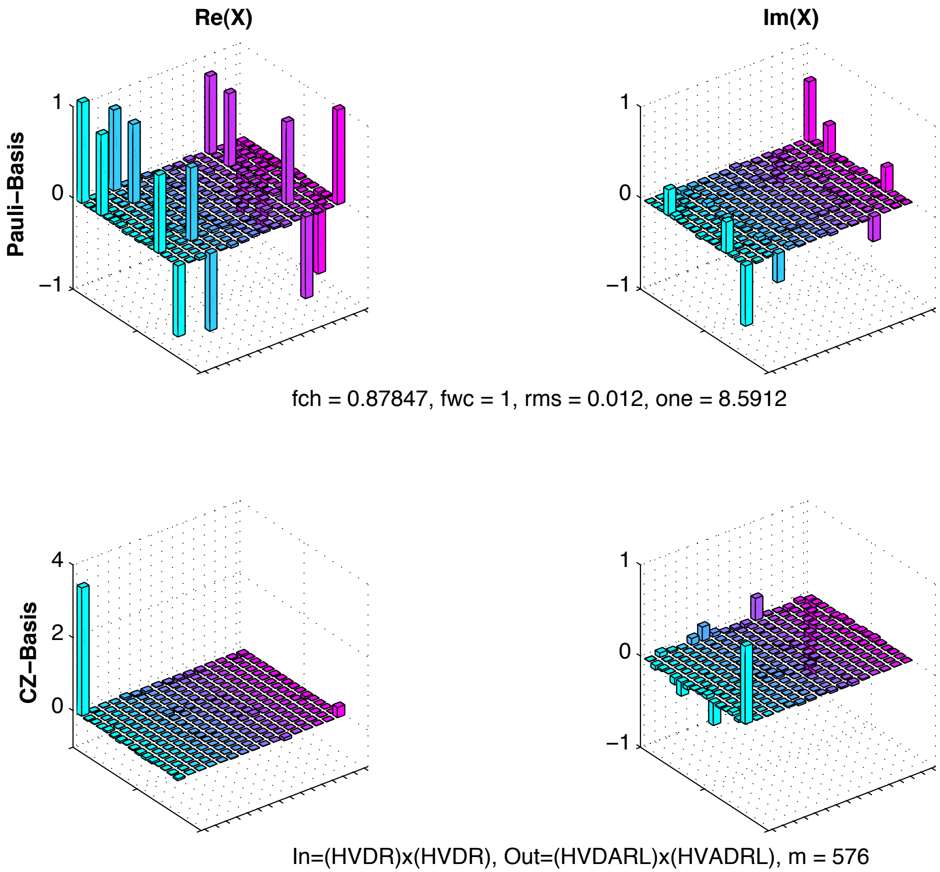


$$f_{\text{chn}} = 0.88765$$
$$f_{\text{wc}} = 0.98021$$

Experimental results: CQPT of CZ-Gate

All configurations

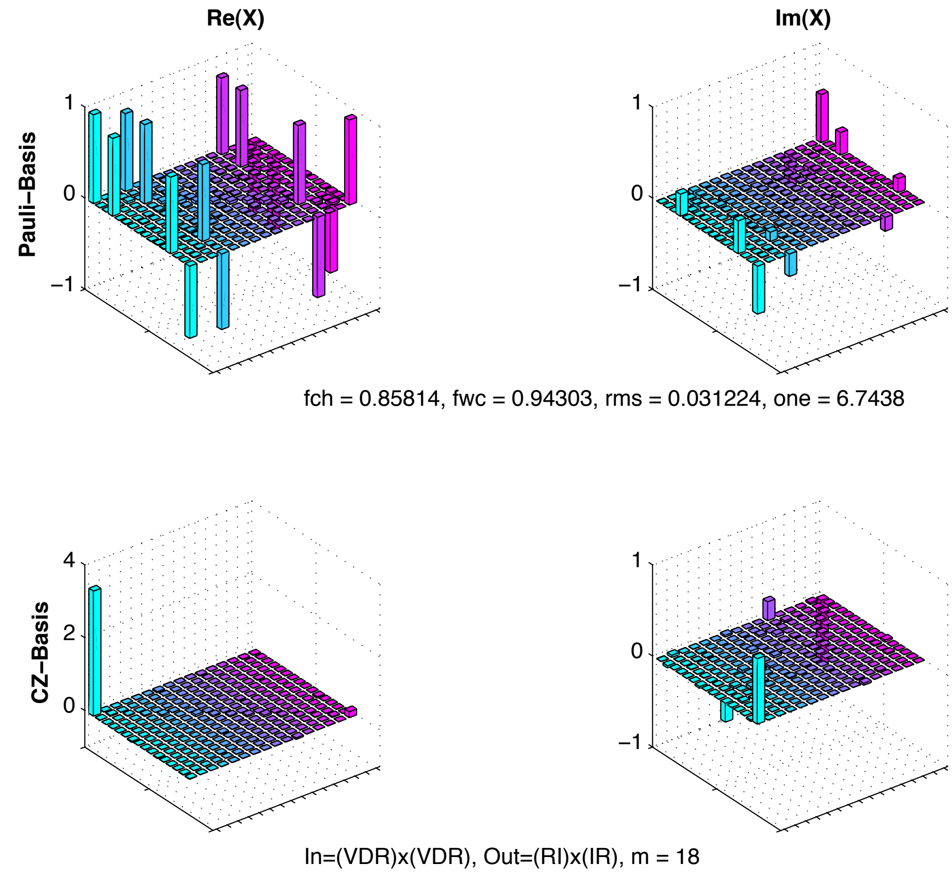
$(m = 576)$



$$f_{\text{chn}} = 0.87847$$
$$f_{\text{wc}} = 1$$

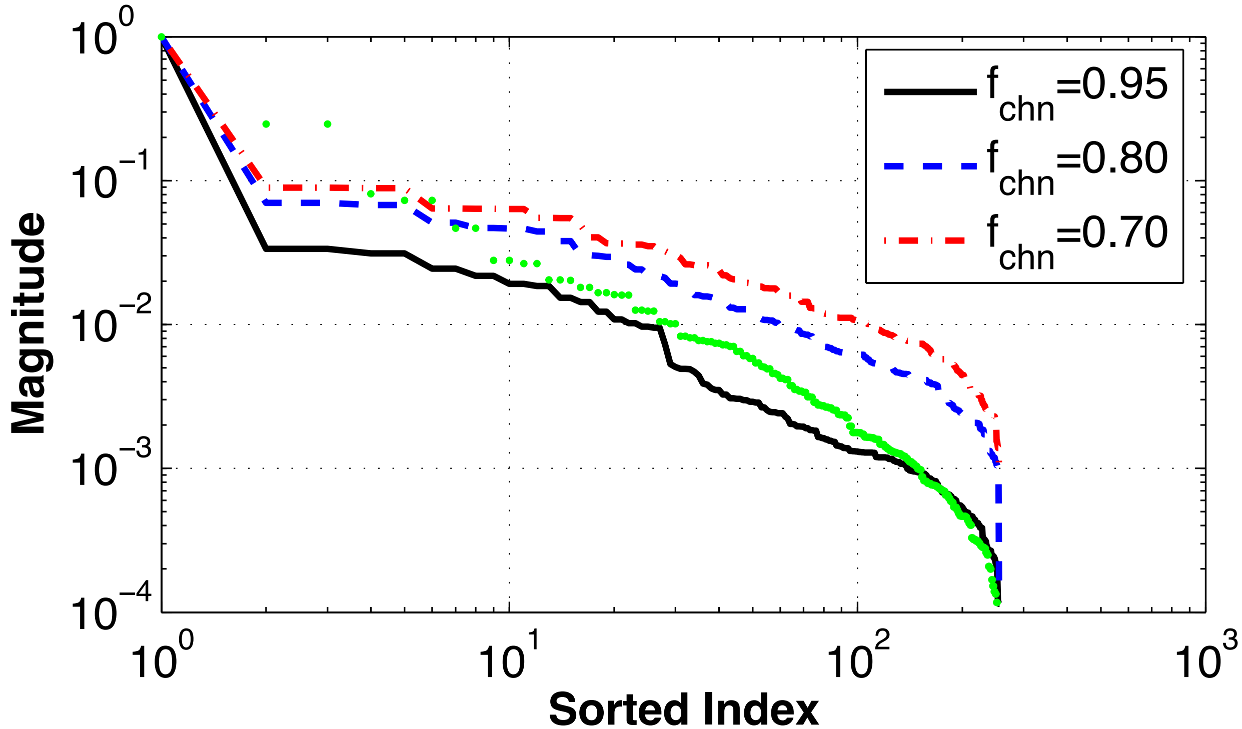
Single-observable configurations

$(m = 18)$



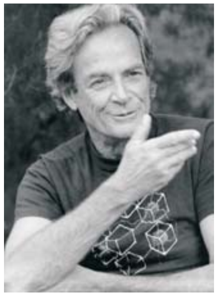
$$f_{\text{chn}} = 0.85814$$
$$f_{\text{wc}} = 0.94303$$

The high-fidelity estimates obtained by CQPT can be understood by examining the next figure which shows the absolute value, sorted by relative magnitude, of the full data estimated process matrix elements. If we take an error range of 0.01 to 0.02, then where the plot crosses that range is a reasonable guess as the s-sparse approximation levels indicated in the CS theory: $s \approx [20, 60]$ correlates well with the high value worst-case fidelities seen in the experiment and previous QFT simulations.



- Absolute values, sorted by relative magnitude, of the 256 process matrix elements of the process matrix in the SVD basis of ideal:
  - numerical example: noisy QFT for $f_{\text{chn}} \in \{0.95, 0.80, 0.70\}$.
  - experimental result: full (576) configuration estimate; $f_{\text{chn}} = 0.88$
- results are very similar for both simulation and experiment.

The Role of Estimation: Alleviate Havoc & Uncertainty



" ... and these imperfections may produce considerable havoc."

– Richard Feynman ("Quantum Mechanical Computers" *Optics News*, February 1985)

Oxford, 1907

"It ain't what you don't know
that gets you into trouble.
It's what you know for sure
that just ain't so."
– Mark Twain