Controllability for Schrödinger equations and applications

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Consider the Schrödinger equation:

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i\dot{z} &= -\Delta z + V(x)z + \beta^{\omega}(t)Q(x)z, \quad x \in D,\\
z|_{\partial D} &= 0,\\
z(0,x) &= z_0^{\omega}(x),
\end{aligned}
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where $D \Subset \mathbb{R}^d$, $\partial D \in C^{\infty}$, $d \geq 1$, $V, Q \in C^{\infty}(\overline{D}, \mathbb{R})$ are given functions, β is a random noise, z is the state.

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Definition

Probability measure μ is invariant if $\mathcal{D}(z_0^\omega) = \mu$ implies $\mathcal{D}(\mathcal{U}_t(z_0, \beta^\omega)) = \mu$ for any $t > 0$.

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Theorem (V.N.)

Finite-dimensional approximations of Schrödinger equation admit a unique invariant measure μ, and any solution converges exponentially to μ in total variational norm.

Control results

Beauchard, Coron, Laurent

Chambrion, Mason, Sigalotti, Boscain

Mirrahimi, Beauchard, V.N.

V.N., H. Nersisyan

Main result

Schrödinger equation

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is globally exactly controllable in infinite time in H^3 generically in V and Q.

For any $z_0, z_1 \in H^3$ there is a control $u \in H^s(\mathbb{R}_+, \mathbb{R})$ and a sequence $T_n \to +\infty$ such $\mathcal{U}_{T_n}(z_0, u) \rightharpoonup z_1$ in H^3.

Proof

Let $\mathcal{U}_\infty(z_0, u)$ be the H^3-weak ω-limit set of the trajectory corresponding to u and $z_0 \in H^3$:

$$\mathcal{U}_\infty(z_0, u) := \{z \in H^3 : \mathcal{U}_{T_n}(z_0, u) \rightharpoonup z \text{ in } H^3 \text{ for some } T_n \to +\infty\}.$$

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Lemma

For any $u \in H^s(\mathbb{R}_+, \mathbb{R})$ and $z_0 \in H^3$, the trajectory $\mathcal{U}_{T_n}(z_0, u)$ is bounded in H^3.

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Lemma

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Consider the multivalued function

$$\begin{aligned} \mathcal{U}_\infty(\cdot, \cdot) : S \cap H^3 \times H^s(\mathbb{R}_+, \mathbb{R}) &\to 2^{S \cap H^3}, \\ (z_0, u) &\to \mathcal{U}_\infty(z_0, u). \end{aligned}$$

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We apply the inverse function theorem for this multivalued mapping.

Proof

The linearized system is equivalent to the following moment problem

$$\int_0^T e^{i\omega_{mk}s}u(s)\mathrm{d}s = d_{mk}, \qquad\qquad d_{mk} \in \ell^2.$$

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Proposition

For any $d_{mk} \in \ell^2$ Problem (1) admits a solution $u \in H^s(\mathbb{R}_+, \mathbb{R})$.

Generalization

The proof works also for the defocusing nonlinear Schrödinger equation:

$$i\dot{z} = -\Delta z + V(x)z + |z|^{2p}z + u(t)Q(x)z, \qquad x \in \mathbb{T}^d,$$

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Theorem

The nonlinear Schrödinger equation is exactly controllable in infinite time near the stationary solutions.

Applications

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Open problem

Existence of an invariant measure.