Approximate controllability for a two trapped ions system

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Outline

- Two ions.
- Each ion is a two level system.
- Coupled to the same quantized harmonic oscillator with vibration quantum ω and annihilation operator
  $\mathcal{A} = \frac{1}{\sqrt{2}}(x + \frac{\partial}{\partial x})$.
- Controls : electromagnetic waves of complex amplitude u_1 and u_2
  Phases depending on spatial coordinate :
  $\Omega_j^L - k_j x$, $j = 1, 2$.
  $k_j x = \eta_j(\mathcal{A} + \mathcal{A}^\dagger)$
  η_j : Lamb-Dicke parameters (small)

When an ion absorbs a photon, its energy changes and its impulsion captures the photon impulsion and excite the (quantized) vibration mode (phonon) inside the trap.

Mathematical model

Pauli matrices :

$$\sigma_{1,z} = (|e><e| - |g><g|)_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_{2,z} = (|e><e| - |g><g|)_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_{1,x} = (|g><e| + |e><g|)_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\sigma_{2,x} = (|g><e| + |e><g|)_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Schrödinger system

State of the system : 4-d vector-wave function

$$|\psi >= \psi =^t (\psi_{gg}, \psi_{ge}, \psi_{eg}, \psi_{ee})$$

$$\begin{aligned} i\tfrac{\partial\psi}{\partial t} = \quad & \omega(\mathcal{A}^\dagger\mathcal{A} + \frac{1}{2})\psi + \frac{\Omega}{2}\sigma_{1,z}\psi + \frac{\Omega}{2}\sigma_{2,z}\psi \\ & +(u_1 e^{i(\Omega_1^L t - k_1 x)} + u_1^* e^{-i(\Omega_1^L t - k_1 x)})\sigma_{1,x}\psi \\ & +(u_2 e^{i(\Omega_2^L t - k_2 x)} + u_2^* e^{-i(\Omega_2^L t - k_2 x)})\sigma_{2,x}\psi, \\ \psi(0) = \psi^0 \quad & . \end{aligned}$$

Question : Given an initial configuration ψ^0 and a final configuration ψ^1, can we find control amplitudes u_1 and u_2 in order to drive the system at time T "close" to ψ^1?
Parameters :

$$\omega \;\text{ large and }\; \Omega \;\text{ very large},$$

$$|\Omega_1^L - \Omega| << \Omega, \quad |\Omega_2^L - \Omega| << \Omega, \quad \omega << \Omega,$$

$$|u_1| << \Omega, \quad |u_2| << \Omega, \quad |\frac{du_1}{dt}| << \Omega, \quad |\frac{du_2}{dt}| << \Omega.$$

Laser frame

Set

$$\psi = e^{-i\frac{\Omega_1^L}{2}t\sigma_{1,z}}.e^{-i\frac{\Omega_2^L}{2}t\sigma_{2,z}}\varphi$$

or

$$\varphi = e^{i\frac{\Omega_2^L}{2}t\sigma_{2,z}}.e^{-i\frac{\Omega_1^L}{2}t\sigma_{1,z}}\psi.$$

And

$$\Delta_1 = \frac{\Omega - \Omega_1^L}{2},\ \Delta_2 = \frac{\Omega - \Omega_2^L}{2},$$

$$k_1 x = \eta_1(\mathcal{A} + \mathcal{A}^\dagger),\ k_2 x = \eta_2(\mathcal{A} + \mathcal{A}^\dagger).$$

Interaction frame

$$A = \mathcal{A}^\dagger \mathcal{A} + \frac{1}{2},$$

$$S(t) = e^{-i\omega t A}.e^{-i\Delta_1 t \sigma_{1,z}}.e^{-i\Delta_2 t \sigma_{2,z}}$$

(A, $\sigma_{1,z}$ and $\sigma_{2,z}$ commute.)

$$\xi(t) = S(-t)\varphi(t) \qquad , \qquad \varphi(t) = S(t)\xi(t).$$

$$\begin{aligned} i\frac{\partial \xi}{\partial t} = S(-t)\Big(u_1 e^{2i\Omega_1^L t - i\eta_1(\mathcal{A}+\mathcal{A}^\dagger)} + u_1^* e^{i\eta_1(\mathcal{A}+\mathcal{A}^\dagger)}\Big)(|e><g|)_1 S(t)\xi \\ +S(-t)\Big(u_1 e^{-i\eta_1(\mathcal{A}+\mathcal{A}^\dagger)} + u_1^* e^{-2i\Omega_1^L t + i\eta_1(\mathcal{A}+\mathcal{A}^\dagger)}\Big)(|g><e|)_1 S(t)\xi \\ +S(-t)\Big(u_2 e^{2i\Omega_2^L t - i\eta_2(\mathcal{A}+\mathcal{A}^\dagger)} + u_2^* e^{i\eta_2(\mathcal{A}+\mathcal{A}^\dagger)}\Big)(|e><g|)_2 S(t)\xi \\ +S(-t)\Big(u_2 e^{-i\eta_2(\mathcal{A}+\mathcal{A}^\dagger)} + u_2^* e^{-2i\Omega_2^L t + i\eta_2(\mathcal{A}+\mathcal{A}^\dagger)}\Big)(|g><e|)_2 S(t)\xi \end{aligned}$$

Lamb-Dicke approximation

$$|\eta_1|, |\eta_2| << 1.$$

$$e^{i\eta_j(\mathcal{A}+\mathcal{A}^\dagger)} \sim \Big(Id + i\eta_j(\mathcal{A}+\mathcal{A}^\dagger)\Big),\ e^{-i\eta_j(\mathcal{A}+\mathcal{A}^\dagger)} \sim \Big(Id - i\eta_j(\mathcal{A}+\mathcal{A}^\dagger)\Big).$$

We then have (for example)

$$e^{i\omega tA}(e^{i\eta_1(\mathcal{A}+\mathcal{A}^\dagger)})e^{-i\omega tA} \sim Id + i\eta_1(\mathcal{A}e^{-i\omega t} + \mathcal{A}^\dagger e^{i\omega t}).$$

We obtain

$$\begin{aligned} i\frac{\partial\xi}{\partial t} = \ & \Big(u_1 e^{2i\Omega_1^L t}\Big(Id - i\eta_1(\mathcal{A}e^{-i\omega t} + \mathcal{A}^\dagger e^{i\omega t})\Big) \\ & + u_1^* \Big(Id + i\eta_1(\mathcal{A}e^{-i\omega t} + \mathcal{A}^\dagger e^{i\omega t})\Big)\Big)e^{2i\Delta_1 t}(|e >< g|)_1\xi \\ & + \Big(u_1\Big(Id - i\eta_1(\mathcal{A}e^{-i\omega t} + \mathcal{A}^\dagger e^{i\omega t})\Big) \\ & + u_1^* e^{-2i\Omega_1^L t}\Big(Id + i\eta_1(\mathcal{A}e^{-i\omega t} + \mathcal{A}^\dagger e^{i\omega t})\Big)\Big)e^{-2i\Delta_1 t}(|g >< e|)_1\xi \\ & + \cdots \end{aligned}$$

Averaging approximation

First of all we take each control u_j to be a superposition of 3 monochromatic waves, two of them having a pulsation shifted by $\pm$ a vibration quantum ω. In fact we take

$$u_1(t)e^{-2i\Delta_1 t} = v_0(t) + \tilde{v}_r(t)e^{-i\omega t} + \tilde{v}_b(t)e^{i\omega t}$$

$$u_2(t)e^{-2i\Delta_2 t} = w_0(t) + \tilde{w}_r(t)e^{-i\omega t} + \tilde{w}_b(t)e^{i\omega t}.$$

Then we neglect the rapidly oscillating terms as ω, Ω_1^L, Ω_2^L and Ω are very large.

Approximate model

Similar to Law-Eberly equations in the case of one qubit.

$$\begin{aligned} i\frac{\partial y}{\partial t} = \quad & (v_0 - i\eta_1 \tilde{v}_r \mathcal{A}^\dagger - i\eta_1 \tilde{v}_b \mathcal{A})(|g >< e|)_1 y \\ & (v_0^* + i\eta_1 \tilde{v}_r^* \mathcal{A} + i\eta_1 \tilde{v}_b^* \mathcal{A}^\dagger)(|e >< g|)_1 y \\ & (w_0 - i\eta_2 \tilde{w}_r \mathcal{A}^\dagger - i\eta_2 \tilde{w}_b \mathcal{A})(|g >< e|)_2 y \\ & (w_0^* + i\eta_2 \tilde{w}_r^* \mathcal{A} + i\eta_2 \tilde{w}_b^* \mathcal{A}^\dagger)(|e >< g|)_2 y. \end{aligned}$$

Writing

$$v_r = -i\eta_1 \tilde{v}_r, \; v_b = -i\eta_1 \tilde{v}_b,$$

$$w_r = -i\eta_1 \tilde{w}_r, \; w_b = -i\eta_1 \tilde{w}_b,$$

and

$$y =^t (y_{gg}, y_{ge}, y_{eg}, y_{ee}),$$

we obtain

Approximate model

$$
\begin{aligned}
& i\frac{\partial y_{gg}}{\partial t} = (v_0 + v_r\mathcal{A}^\dagger + v_b\mathcal{A})y_{eg} + (w_0 + w_r\mathcal{A}^\dagger + w_b\mathcal{A})y_{ge} \\
& i\frac{\partial y_{ge}}{\partial t} = (v_0 + v_r\mathcal{A}^\dagger + v_b\mathcal{A})y_{ee} + (w_0^* + w_r^*\mathcal{A} + w_b^*\mathcal{A}^\dagger)y_{gg} \\
& i\frac{\partial y_{eg}}{\partial t} = (v_0^* + v_r^*\mathcal{A} + v_b^*\mathcal{A}^\dagger)y_{gg} + (w_0 + w_r\mathcal{A}^\dagger + w_b\mathcal{A})y_{ee} \\
& i\frac{\partial y_{ee}}{\partial t} = (v_0^* + v_r^*\mathcal{A} + v_b^*\mathcal{A}^\dagger)y_{ge} + (w_0^* + w_r^*\mathcal{A} + w_b^*\mathcal{A}^\dagger)y_{eg} \\
& y(0) = y^0.
\end{aligned}
$$