Wireless network coding over finite rings

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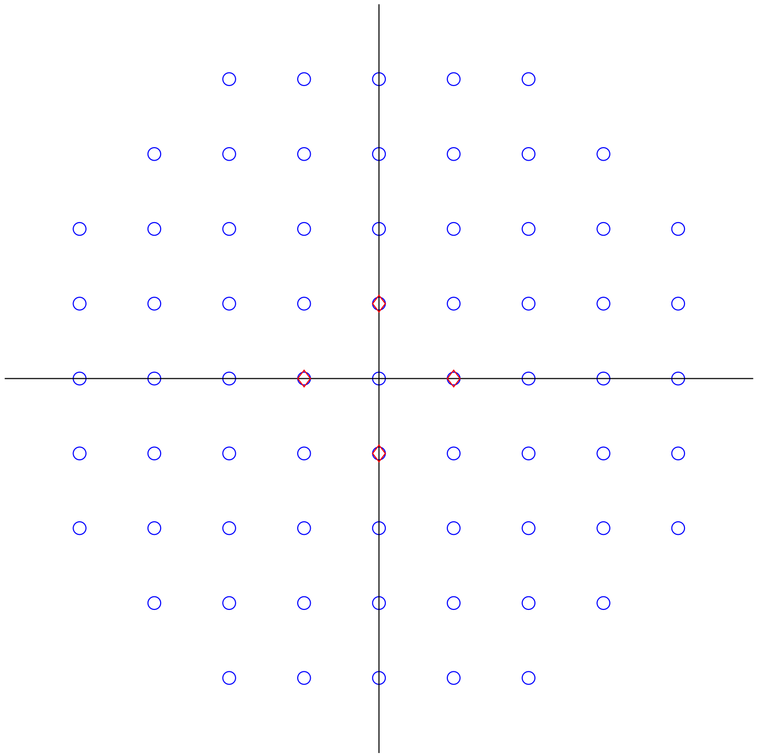
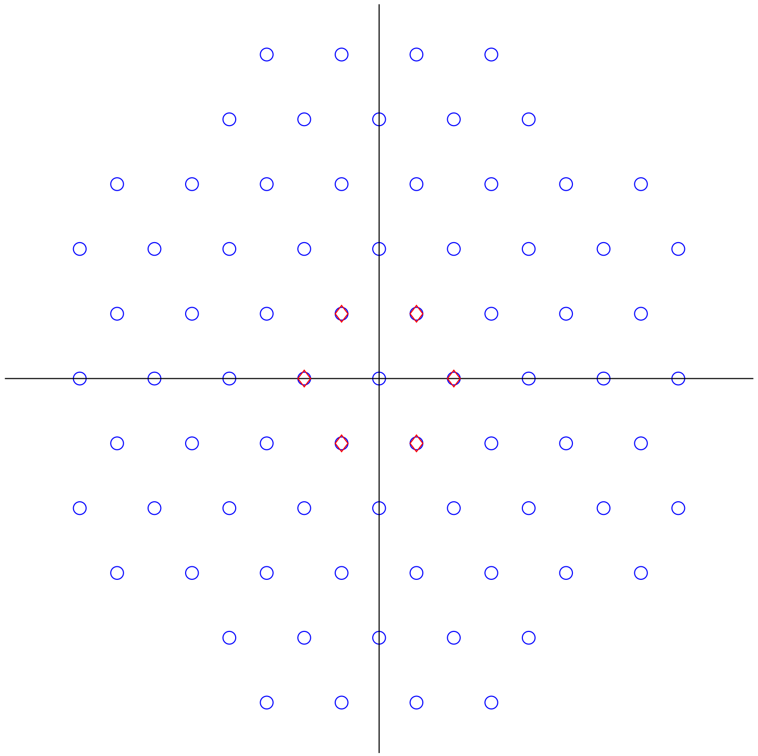
Motivation

- Mathematicians love prime numbers p and engineers love 2^m
- Bit labeling is a problem with p (loss of rate and additional complexity)
- Linear codes (n, k) over $\mathbf{F}_p$ can be mapped by Construction A to a lattice Λ and by working $\mod p$ to a subset of $(p\text{-PAM})^n$ finite constellation
- In lattice network coding $+$ and $\times \mod p$ operations provide the natural operations for p-PAM $\mod p$ constellations and we use the fact that the ring $\mathbf{Z}_p$ is equivalent to the field $\mathbf{F}_p$.
- Feng, Silva, and Kschischang, (2010-2011) have shown how to construct lattice network codes by concatenating linear codes over $\mathbf{F}_p$ with a finite 2D constellation with p points.
- Narayanan (2011) has shown how to improve the shaping of the p and p^2 point 2D constellations.

The infinite rings $\mathbf{Z}[i]$ *and* $\mathbf{Z}[\omega]$





- Basis: $\{1, \theta\}$ where $\theta = i = \sqrt{-1}$ or $\theta = \omega = e^{i2\pi/3}$
- Elements: $\{a + b\theta : a, b \in \mathbf{Z}\}$
- Units: $\{\pm 1, \pm i\}$ $\{\pm 1, \pm\omega, \pm\omega^*, \}$

Motivation Cont'd

- To have almost always invertible network equations we need large p
- An invertible matrix A over a ring R must have

$$\frac{1}{det(A)} \in R$$

- Often rings have few invertible elements (units of R) hence we have a very limited choice for the network equations.
- We need more freedom so we need to put in more units.
- This can be effectively achieved by working in finite rings where the integers are taken $\mathrm{mod}\ 2^m$

$$R = \mathbf{Z}_{2^m}[\theta] = \{a + \theta b | a, b \in \mathbf{Z}_{2^m}\}$$

The Gaussian integers $\mod 2^m$

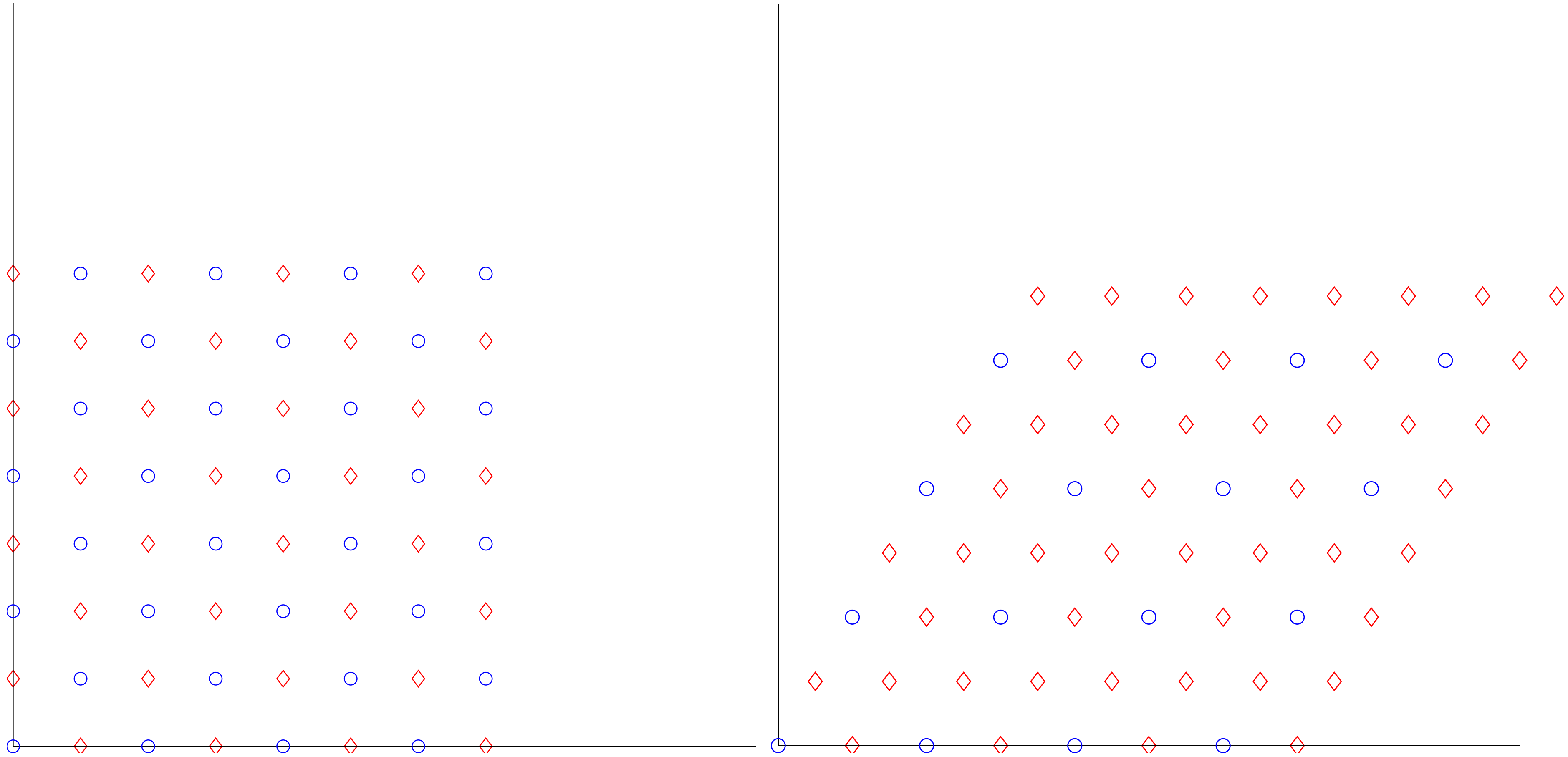
Problem: Build a large set of invertible matrices over the finite ring

$$R = \mathbf{Z}_{2^m}[i] = \{a + ib | a, b \in \mathbf{Z}_{2^m}\}$$

- Units of R: $R^* = \mathbf{Z}_{2^m}[i] = \{a + ib | a, b \in \mathbf{Z}_{2^m}, a + b = 1 \bmod 2^m\} = D_2 + (1, 0) \cap \mathcal{B}$
- Non units $\bar{R} = \mathbf{Z}_{2^m}[i] = \{a + ib | a, b \in \mathbf{Z}_{2^m}, a + b = 0 \bmod 2^m\} = D_2 \cap \mathcal{B}$
- Properties:
  - $\bar{R} + \bar{R} = \bar{R}$
  - $R^* + R^* = \bar{R}$
  - $R^* + \bar{R} = \bar{R}$
  - $\bar{R}\bar{R} = \bar{R}$
  - $R^* R^* = R^*$
  - $R^* \bar{R} = \bar{R}$
- A possible solution: the matrix $A = (a_{ij})$ is invertible if $a_{ii} \in R^*$ and $a_{ij} \in \bar{R}$.

What is this? can it be improved/generalized to Eisenstein integers or even quaternions?

The finite rings



- Red diamonds are the units R^*
- Blue circles are non-invertible elements $\bar{R}$

Commutative rings

A *commutative ring* R is a set closed under two binary operations, addition and multiplication such that

1. R is an Abelian group under addition
2. $ab = ba$ for all $a, b \in R$ (*commutativity*)
3. $a(bc) = (ab)c$ for all $a, b, c \in R$ (*associativity*)
4. there exists a element $1 \in R$ such that $1a = a$ for all $a \in R$ (*identity element*)
5. $a(b + c) = ab + ac$ for all $a, b, c \in R$ (*distributivity*)

- Examples of rings: $\mathbf{Z}$, $\mathbf{Z}[i]$
- These are not rings: $2\mathbf{Z} + 1$, $\mathbf{Z}^+$

Ideals

An *ideal* in a commutative ring R is a subset I such that for all $a, b \in R$

1. $0 \in I$;
2. if $a, b \in I$, then $a + b \in I$;
3. if $a \in I$ and $r \in R$, then $ra \in I$.

- Examples of ideals: $2\mathbf{Z}$, $(1 + i)\mathbf{Z}[i]$
- These are not ideals: $2\mathbf{Z} + 1$, $\mathbf{Z}^+$

Invertible elements in $\mathbf{Z}_{2^m}$

The group of units of $\mathbf{Z}_{2^m}$ is

$$\{1, 3, 5, \dots, 2^m - 1\}$$

Proof

- Let $a \in \mathbf{Z}_{2^m}$. It is enough consider the modular equation to find the inverse element x

$$ax = 1 \mod 2^m$$

- This has a solution, if and only if we can solve

$$ax - 2^m q = 1$$

  for some integers x and q.

- It is well known that the above equation can be solved using the extended Euclidean algorithm if and only if GCD$(a, 2^m) = 1$ which is the case for all odd $a = 2k + 1$.
- Note that an even $a = 2k$ will have GCD$(a, 2^m) \geq 2$.

Group of units of R

The group of units of R is given by

$$R^* = \{a + ib | a, b \in \mathbf{Z}_{2^m}, a + b = 1 \text{ mod } 2\}$$

and the non units form the maximal ideal

$$\bar{R} = \{a + ib | a, b \in \mathbf{Z}_{2^m}, a + b = 0 \text{ mod } 2\}$$

Proof

- Let $a + ib \in R$. It is enough to consider the inverse element

$$x = \frac{a - ib}{a^2 + b^2}$$

- This is in R iff $a^2 + b^2$ is invertible in $\mathbf{Z}_{2^m}$.
- This is true iff $a^2 + b^2 = 1 \mod 2$, which is equivalent to $a + b = 1 \mod 2$.
- To prove that $\bar{R}$ is an ideal we consider 3) property of ideals.
- Let $a + ib \in \bar{R}$ and $x + iy \in R$ then by adding real and imaginary part of the product we get

$$(ax - by) + (bx + ay) = (a + b)x + (a - b)y = 0 \mod 2$$

  since $a - b = 0 \mod 2$.
- Finally, since $R = R^* \cup \bar{R}$, $\bar{R}$ is a *maximal ideal* of R, i.e. is not contained in any larger non trivial ideal of R.

More definitions

- Given the two rings R and S, a *ring homomorphism* is a mapping $\varphi : R \to S$ such that for all $a, b \in R$
  1. $\varphi(a+b) = \varphi(a) + \varphi(b)$
  2. $\varphi(ab) = \varphi(a)\varphi(b)$
  3. $\varphi(1) = 1$

- Given the two sided ideal I we define the *quotient ring* R/I where addition $\oplus$ and multiplication $\otimes$ are defined as

$$(a+I) \oplus (b+I) = ((a+b)+I)$$
$$(a+I) \otimes (b+I) = (ab+I)$$

  where $a, b \in R$ and $'+'$ and $\cdot$ are the operations in the ring R

- We define the *natural map* $\phi : R \to R/I$ as the ring homomorphism defined by $a \mapsto a + I$.

Mapping to $\mathbf{F}_2$

The quotient ring $R/\bar{R}$ is isomorphic to the field $\mathbf{F}_2$.

Proof – The image of the natural map $\phi : R \to R/\bar{R}$ is composed of two element $\bar{R}$ and R^*. By mapping

$$\bar{R} \mapsto 0$$

$$R^* \mapsto 1$$

the above properties provide the explicit addition and multiplication tables of $\mathbf{F}_2 = \{0, 1\}$.

Alternatively, the proof is a direct application on the quotient of a commutative ring by a maximal ideal.

The invertible matrices

The matrices $A = (a_{ij})$ with $a_{ii} \in R^*$ and $a_{ij} \in \bar{R}$ $i \neq j$ are invertible in the ring of matrices $\mathcal{M}_n(R)$ with coefficients in R.

Proof – It is enough to show that det$(A) \in R^*$, i.e., it has an inverse in R.
Extending the natural map ϕ we define the the matrix ring homomorphism

$$\Phi : \mathcal{M}_n(R) \to \mathcal{M}_n(\mathbf{F}_2).$$

All the matrices A are mapped to the identity matrix I in $\mathcal{M}_n(\mathbf{F}_2)$, which is invertible in $\mathbf{F}_2$.
Using the properties of ring homomorphisms in the Leibniz formula for the determinant

$$\det(A) = \sum_{\pi \in S_n} sgn(\pi) \prod_{i=1}^{n} a_{i,\pi(i)}$$

we have

$$\phi(\det(A)) = \det(\Phi(A)) = \det(I) = 1$$

which implies that that det$(A) \in R^*$.

More invertible matrices

More invertible matrices can be obtained by applying the inverse map Φ^{-1} to any binary invertible matrix.

$$\begin{pmatrix} R^* & \bar{R} & \bar{R} \\ \bar{R} & R^* & \bar{R} \\ \bar{R} & \bar{R} & R^* \end{pmatrix} \quad \begin{pmatrix} \bar{R} & \bar{R} & R^* \\ R^* & \bar{R} & \bar{R} \\ \bar{R} & R^* & \bar{R} \end{pmatrix} \quad \begin{pmatrix} R^* & \bar{R} & \bar{R} \\ \bar{R} & \bar{R} & R^* \\ \bar{R} & R^* & \bar{R} \end{pmatrix} \quad \cdots$$

"Disquisitiones"

- We have made the engineers happy with 2^m.
- We can still generate many invertible network equations, which quantize the channel well.
- We do not rely on large field and randomness.
- In physical layer network coding we need a ring structure because of the multiplicative effect of the channel.
- The field structure is often used because we know a lot about codes over fields ...
- ... but the code over the field is not usually easy to match to a finite constellation: Hamming distance or Lee distance is not matched to Euclidean distance
- Using the ring structure we do not need to go through a linear code over a field and we are allowed to take any lattice that we like, as for BCM and set-partitioning.
- With ring codes we can work with channels that are ring homomorphisms transformations of the input ring.