A Tutorial on Discontinuous Galerkin Methods

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RKDG Methods for 1d Scalar Conservation Laws

Consider

$$\begin{cases} u_t + f(u)_x = 0, & 0 < x < 1, \;\; t > 0 \\ u(x,0) = u_0(x), & 0 < x < 1 \end{cases}$$

with the periodic boundary condition.

RKDG methods:

- Discontinuous Galerkin (DG) discretizations in space
- explicit Runge-Kutta methods in time

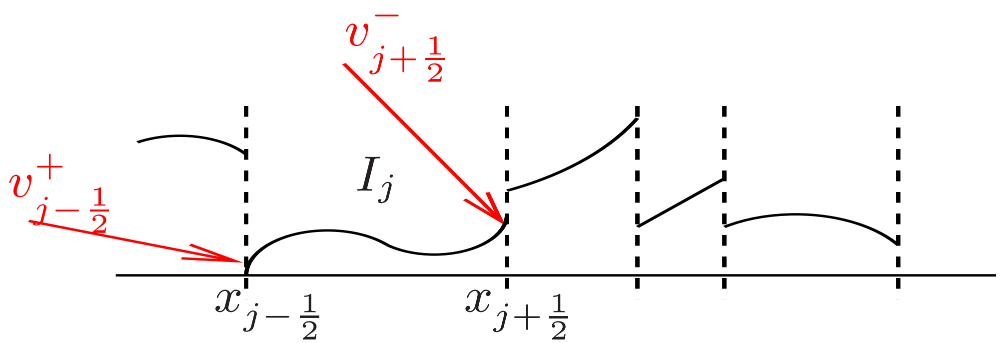
Outline

- Semi-discrete DG formulation
- Time discretization
- Cell entropy inequality and L^2-stability
- Limiter and total variation stability
- Error estimate
- Further discussion

An incomplete history of the DG method for conservation laws

- 1973: First discontinuous Galerkin (DG) method for steady state linear scalar conservation laws (Reed and Hill).
- 1974: First error estimate (for tensor product mesh) of the DG method of Reed and Hill (LeSaint and Raviart).
- 1986: Error estimates for the DG method of Reed and Hill (Johnson and Pitkaranta).
- 1989-1998: Runge-Kutta DG methods for nonlinear conservation laws (Cockburn, Shu, ...).
- 1994: Cell entropy inequality for DG methods for nonlinear conservation laws in general multidimensional triangulations (Jiang and Shu).

Semi-discrete DG formulation



Discrete space:

$$V_h = V_h^k = \{v : v \in P^k(I_j),\ \forall j\}$$

Some notations:

$$I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], \quad \Delta x_j = |I_j|, \quad \Delta x \text{ or } h = \max_j |I_j|$$

$$v^-_{j+\frac{1}{2}} = \lim_{\epsilon \to 0^-} v(x_{j+\frac{1}{2}} + \epsilon), \qquad v^+_{j-\frac{1}{2}} = \lim_{\epsilon \to 0^+} v(x_{j-\frac{1}{2}} + \epsilon)$$

$$[v]_{j-\frac{1}{2}} = v^+_{j-\frac{1}{2}} - v^-_{j-\frac{1}{2}}$$

$$\int_{I_j} (u_t + f(u)_x)\, v dx = 0$$

$$\int_{I_j} u_t v - f(u) v_x dx + f(u(x_{j+\frac{1}{2}}, t)) v(x_{j+\frac{1}{2}}) - f(u(x_{j-\frac{1}{2}}, t)) v(x_{j-\frac{1}{2}}) = 0$$

Semi-discrete DG discretization: look for $u_h \in V_h$, such that $\forall v_h \in V_h$ and $\forall j$, there is

$$\int_{I_j} u_{h,t} v_h dx - \int_{I_j} f(u_h) v_{h,x} dx + \hat{f}_{j+\frac{1}{2}} v^-_{h,j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} v^+_{h,j-\frac{1}{2}} = 0,$$

$$\int_{I_j} u_h(x,0) v_h dx = \int_{I_j} u_0(x) v_h(x) dx, \qquad (L^2\text{-projection})$$

where $\hat{f}_{j+\frac{1}{2}} = \hat{f}(u^-_{h,j+\frac{1}{2}}, u^+_{h,j+\frac{1}{2}})$.

$\hat{f}(u^-, u^+)$ is a monotone numerical flux:

- Consistency: $\hat{f}(u,u) = f(u)$
- Lipschitz continuity
- Monotonicity: $\hat{f}(\uparrow, \downarrow)$

Some examples:

- Lax-Friedrichs flux:

$$\hat{f}^{\mathrm{LF}}(u^-, u^+) = \frac{1}{2}(f(u^-) + f(u^+) - \alpha(u^+ - u^-)), \qquad \alpha = \max_u |f'(u)|$$

- Godunov flux:

$$\hat{f}^{\mathrm{G}}(u^-, u^+) = \begin{cases} \min_{u^- \le u \le u^+} f(u), & \text{if } u^- < u^+ \\ \max_{u^+ \le u \le u^-} f(u), & \text{if } u^- \ge u^+ \end{cases}$$

- Engquist-Osher flux:

$$\hat{f}^{\mathrm{EO}}(u^-, u^+) = \int_0^{u^-} \max(f'(u), 0)du + \int_0^{u^+} \min(f'(u), 0)du + f(0)$$

Remarks:

- local conservation: set $v_h|_{I_j} = 1$

$$\frac{d}{dt}\int_{I_j} u_h dx + \hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} = 0.$$

- for $k = 0$, $u_h|_{I_j} = u_j(t)$, the method is the same as the finite volume method.

$$\frac{du_j}{dt} + \frac{1}{\Delta x_j}\Big(\hat{f}(u_j, u_{j+1}) - \hat{f}(u_{j-1}, u_j)\Big) = 0$$

- for $f(u) = au$, all monotone fluxes become the upwinding flux

$$\hat{f}(u^-, u^+) = \begin{cases} au^- & \text{if } a > 0 \\ au^+ & \text{if } a < 0 \end{cases}$$

* Implementation

- basis of $V_h^k|_{I_j} = P^k(I_j)$: $\{\phi_j^m(x)\}_{m=0}^k$

(1) $1, \xi_j, \cdots, \xi_j^k$ with $\xi_j = \frac{(x-x_j)}{\Delta x_j}$

(2) Legendre (orthogonal): $1, \xi_j, \frac{1}{2}(3\xi_j^2 - 1), \cdots,$ with
$\xi_j = \frac{(x-x_j)}{\Delta x_j/2}$

- semi-discrete DG method: look for u_h with $u_h|_{I_i} = \sum_{l=0}^{k} u_i^l(t)\phi_i^l(x)$, such that $\forall v_h = \phi_j^m(x)$ on I_j, $m = 0, \cdots, k$ and $\forall j$

$$\int_{I_j} \frac{d}{dt}\left(\sum_{l=0}^{k} u_j^l(t)\phi_j^l(x)\right)\phi_j^m(x)dx = \int_{I_j} f(u_h)\frac{d}{dx}\phi_j^m(x)dx$$

$$-\hat{f}_{j+\frac{1}{2}}(u_h|_{I_j}, u_h|_{I_{j+1}})\phi_j^m(x_{j+\frac{1}{2}}) + \hat{f}_{j-\frac{1}{2}}(u_h|_{I_{j-1}}, u_h|_{I_j})\phi_j^m(x_{j-\frac{1}{2}})$$

$$\int_{I_j}\left(\sum_{l=0}^{k} u_j^l(0)\phi_j^l(x)\right)\phi_j^m(x)dx = \int_{I_j} u_0(x)\phi_j^m(x)dx$$

- matrix form: Set

$$\mathbf{U} = [\mathbf{U}_1, \cdots, \mathbf{U}_N]^T, \text{ with } \mathbf{U}_j = [u_j^0, \cdots, u_j^k]^T,$$

and

$$\mathbf{M} = \text{diag}(\mathbf{M}_1, \cdots, \mathbf{M}_N)$$

where $\mathbf{M}_j \in \mathbb{R}^{(k+1)\times(k+1)}$ with the (m,l)-th entry $\int_{I_j} \phi_j^l(x)\phi_j^m(x)dx$. Then the semi-discrete DG method becomes

$$\mathbf{M}\mathbf{U}_t = \tilde{L}(\mathbf{U}), \qquad \mathbf{M}\mathbf{U}(0) = \tilde{\mathbf{U}}_0$$

or

$$\begin{cases} \mathbf{U}_t & = \mathbf{M}^{-1}\tilde{L}(\mathbf{U}) = L(\mathbf{U}) \\ \mathbf{U}(0) & = \mathbf{M}^{-1}\tilde{\mathbf{U}}_0 = \mathbf{U}_0. \end{cases}$$

Note: - the mass matrix $\mathbf{M}$ is block-diagonal.

- numerical quadratures might be needed.

Time discretization

- Forward Euler method: $\mathbf{U}^{n+1} = \mathbf{U}^n + \Delta t L(\mathbf{U}^n)$
- TVD-RK methods
  - 2nd order

$$
\begin{aligned}
w^{(1)} &= \mathbf{U}^n + \Delta t L(\mathbf{U}^n) \\
\mathbf{U}^{n+1} &= \frac{1}{2}\mathbf{U}^n + \frac{1}{2}\left(w^{(1)} + \Delta t L(w^{(1)})\right)
\end{aligned}
$$

  - 3rd order

$$
\begin{aligned}
w^{(1)} &= \mathbf{U}^n + \Delta t L(\mathbf{U}^n) \\
w^{(2)} &= \frac{3}{4}\mathbf{U}^n + \frac{1}{4}\left(w^{(1)} + \Delta t L(w^{(1)})\right) \\
\mathbf{U}^{n+1} &= \frac{1}{3}\mathbf{U}^n + \frac{2}{3}\left(w^{(2)} + \Delta t L(w^{(2)})\right)
\end{aligned}
$$

The CFL number (C_{CFL}) for the linear L^2-stability when polynomials of degree k and Runge-Kutta methods of the order ν are used.

k	0	1	2	3	4	5	6	7	8
$\nu = 1$	1.000	★	★	★	★	★	★	★	★
$\nu = 2$	1.000	0.333	★	★	★	★	★	★	★
$\nu = 3$	1.256	0.409	0.209	0.130	0.089	0.066	0.051	0.040	0.033
$\nu = 4$	1.392	0.464	0.235	0.145	0.100	0.073	0.056	0.045	0.037
$\nu = 5$	1.608	0.534	0.271	0.167	0.115	0.085	0.065	0.052	0.042
$\nu = 6$	1.776	0.592	0.300	0.185	0.127	0.093	0.072	0.057	0.047
$\nu = 7$	1.977	0.659	0.333	0.206	0.142	0.104	0.080	0.064	0.052
$\nu = 8$	2.156	0.718	0.364	0.225	0.154	0.114	0.087	0.070	0.057
$\nu = 9$	2.350	0.783	0.396	0.245	0.168	0.124	0.095	0.076	0.062
$\nu = 10$	2.534	0.844	0.428	0.264	0.182	0.134	0.103	0.082	0.067
$\nu = 11$	2.725	0.908	0.460	0.284	0.195	0.144	0.111	0.088	0.072
$\nu = 12$	2.911	0.970	0.491	0.303	0.209	0.153	0.118	0.094	0.077

Note: $\alpha \frac{\Delta t}{\Delta x} \le C_{\mathrm{CFL}}$ with $\alpha = \max_w |f'(w)|$.

More discussions: These are examples of strong stability preserving (SSP) time discretizations. That is, when the first-order forward Euler time discretization is strongly stable under a certain norm or semi-norm (such as maximum norm, total variation norm) when the time step Δt is suitably restricted, then SSP time discretizations maintain strong stability for the same norm, perhaps under a different time step. (Shu, Osher, Gottlieb)

Use TVD-RK3 as an example.

$$w^{(1)} = \mathbf{U}^n + \Delta t L(\mathbf{U}^n)$$

$$w^{(2)} = \frac{3}{4}\mathbf{U}^n + \frac{1}{4}\left(w^{(1)} + \Delta t L(w^{(1)})\right)$$

$$\mathbf{U}^{n+1} = \frac{1}{3}\mathbf{U}^n + \frac{2}{3}\left(w^{(2)} + \Delta t L(w^{(2)})\right)$$

Suppose for $w = v + \Delta t L(v)$, there is $|w|_\star \leq |v|_\star$ under $|\Delta t| \leq \alpha$. Then with the same restriction on Δt, $|\mathbf{U}^{n+1}|_\star \leq |\mathbf{U}^n|_\star$ holds.

$$|w^{(1)}|_\star \leq |\mathbf{U}^n|_\star$$

$$\begin{aligned} |w^{(2)}|_\star &\leq \frac{3}{4}|\mathbf{U}^n|_\star + \frac{1}{4}|w^{(1)} + \Delta t L(w^{(1)})|_\star \\ &\leq \frac{3}{4}|\mathbf{U}^n|_\star + \frac{1}{4}|w^{(1)}|_\star \leq |\mathbf{U}^n|_\star \\ |\mathbf{U}^{n+1}|_\star &\leq \frac{1}{3}|\mathbf{U}^n|_\star + \frac{2}{3}|w^{(2)} + \Delta t L(w^{(2)})|_\star \\ &\leq \frac{1}{3}|\mathbf{U}^n|_\star + \frac{2}{3}|w^{(2)}|_\star \leq |\mathbf{U}^n|_\star \end{aligned}$$

Use TVD-RK3 as an example.

$$
\begin{aligned}
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\mathbf{U}^{n+1} &= \frac{1}{3}\mathbf{U}^n + \frac{2}{3}\left(w^{(2)} + \Delta t L(w^{(2)})\right)
\end{aligned}
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$$
\begin{aligned}
|w^{(1)}|_\star &\le |\mathbf{U}^n|_\star \\
|w^{(2)}|_\star &\le \frac{3}{4}|\mathbf{U}^n|_\star + \frac{1}{4}|w^{(1)} + \Delta t L(w^{(1)})|_\star \\
&\le \frac{3}{4}|\mathbf{U}^n|_\star + \frac{1}{4}|w^{(1)}|_\star \le |\mathbf{U}^n|_\star \\
|\mathbf{U}^{n+1}|_\star &\le \frac{1}{3}|\mathbf{U}^n|_\star + \frac{2}{3}|w^{(2)} + \Delta t L(w^{(2)})|_\star \\
&\le \frac{1}{3}|\mathbf{U}^n|_\star + \frac{2}{3}|w^{(2)}|_\star \le |\mathbf{U}^n|_\star
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\begin{aligned}
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&\le \frac{3}{4}|\mathbf{U}^n|_\star + \frac{1}{4}|w^{(1)}|_\star \le |\mathbf{U}^n|_\star \\
|\mathbf{U}^{n+1}|_\star &\le \frac{1}{3}|\mathbf{U}^n|_\star + \frac{2}{3}|w^{(2)} + \Delta t L(w^{(2)})|_\star \\
&\le \frac{1}{3}|\mathbf{U}^n|_\star + \frac{2}{3}|w^{(2)}|_\star \le |\mathbf{U}^n|_\star
\end{aligned}
$$

Cell entropy inequality and L^2-stability

The physically relevant weak solution (called entropy solution, or viscosity solution) satisfies the *entropy condition:*

$$U(u)_t + F(u)_x \leq 0$$

in distribution sense for any convex entropy $U(u)$ and the corresponding entropy flux $F(u) = \int^u U'(u)f'(u)du$.

Note: For $U(u) = u^2/2$, the entropy flux is

$$F(u) = uf(u) - \int^u f(u)du.$$

Proposition (Jiang and Shu 1994): *(1) The solution u_h to the semi-discrete DG scheme satisfies the following cell entropy inequality*

$$\frac{d}{dt}\int_{I_j} U(u_h)dx + \hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}} \leq 0$$

for the square entropy $U(u) = \frac{u^2}{2}$ and for some consistent entropy flux

$$\hat{F}_{j+\frac{1}{2}} = \hat{F}(u^-_{h,j+\frac{1}{2}}, u^+_{h,j+\frac{1}{2}})$$

with $\hat{F}(u,u) = F(u) = uf(u) - \int^u f(u)du$.

(2) Furthermore, u_h satisfies the following L^2-stability

$$\frac{d}{dt}\int_0^1 (u_h)^2 dx \leq 0,$$

or, with $||\cdot||_0$ as the L^2 norm,

$$||u_h(\cdot,t)||_0 \leq ||u_h(\cdot,0)||_0 \leq ||u_0||_0.$$

Proof: The semi-discrete DG method: find $u_h \in V_h$, s.t. $\forall v_h \in V_h$, $\forall j$

$$B_j(u_h, v_h) = 0,$$

here

$$B_j(w,v) = \int_{I_j} w_t v dx - \int_{I_j} f(w) v_x dx + \hat{f}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}.$$

Step 1:

$$B_j(u_h, u_h) = 0, \qquad \forall j$$

Step 2: To show $\forall w \in V_h$, with $U(w) = w^2/2$ and $H(w) = \int^w f(w)dw$

$$B_j(w,w) = \frac{d}{dt}\int_{I_j} U(w)dx + \hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}} + \Theta_{j-\frac{1}{2}}$$

with $\Theta_{j-\frac{1}{2}} \geq 0$, and $\hat{F}(w^-, w^+) = w^- \hat{f} - H(w^-)$ which is consistent to the entropy flux $F(w) = wf(w) - H(w)$.

With $U(w) = w^2/2$ and $H(w) = \int^w f(w)dw$,

$$
\begin{aligned}
B_j(w,w) =& \int_{I_j} w_t w dx - \int_{I_j} f(w) w_x dx + \hat{f}_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} \\
=& \frac{d}{dt}\int_{I_j} U(w)dx - H(w^-_{j+\frac{1}{2}}) + H(w^+_{j-\frac{1}{2}}) + \hat{f}_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} \\
=& \frac{d}{dt}\int_{I_j} U(w)dx + \underbrace{\left(\hat{f}_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} - H(w^-_{j+\frac{1}{2}})\right)}_{\hat{F}_{j+\frac{1}{2}}} \\
& - \underbrace{\left(\hat{f}_{j-\frac{1}{2}} w^-_{j-\frac{1}{2}} - H(w^-_{j-\frac{1}{2}})\right)}_{\hat{F}_{j-\frac{1}{2}}} + \underbrace{\left([H(w)]_{j-\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} [w]_{j-\frac{1}{2}}\right)}_{\Theta_{j-\frac{1}{2}}}
\end{aligned}
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=& \frac{d}{dt}\int_{I_j} U(w)dx - H(w^-_{j+\frac{1}{2}}) + H(w^+_{j-\frac{1}{2}}) + \hat{f}_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} \\
=& \frac{d}{dt}\int_{I_j} U(w)dx + \underbrace{\left(\hat{f}_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} - H(w^-_{j+\frac{1}{2}})\right)}_{\hat{F}_{j+\frac{1}{2}}} \\
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B_j(w,w) &= \int_{I_j} w_t w dx - \int_{I_j} f(w) w_x dx + \hat{f}_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} \\
&= \frac{d}{dt}\int_{I_j} U(w)dx - H(w^-_{j+\frac{1}{2}}) + H(w^+_{j-\frac{1}{2}}) + \hat{f}_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} \\
&= \frac{d}{dt}\int_{I_j} U(w)dx + \underbrace{\left(\hat{f}_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} - H(w^-_{j+\frac{1}{2}})\right)}_{\hat{F}_{j+\frac{1}{2}}} \\
&\quad - \underbrace{\left(\hat{f}_{j-\frac{1}{2}} w^-_{j-\frac{1}{2}} - H(w^-_{j-\frac{1}{2}})\right)}_{\hat{F}_{j-\frac{1}{2}}} + \underbrace{\left([H(w)]_{j-\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} [w]_{j-\frac{1}{2}}\right)}_{\Theta_{j-\frac{1}{2}}}
\end{aligned}
$$

What remains is to show $\Theta \geq 0$. Note that $H(w) = \int^{w} f(w)dw$

$$
\begin{aligned}
\Theta &= [H(w)] - \hat{f}[w] \\
&= \int_{w^-}^{w^+} f(w)dw - \hat{f}(w^-, w^+)(w^+ - w^-) \\
&= \int_{w^-}^{w^+} \hat{f}(w, w)dw - \int_{w^-}^{w^+} \hat{f}(w^-, w^+)dw \\
&= \int_{w^-}^{w^+} \hat{f}(w, w) - \hat{f}(w^-, w)dw + \int_{w^-}^{w^+} \hat{f}(w^-, w) - \hat{f}(w^-, w^+)dw \\
&\geq 0
\end{aligned}
$$

The last inequality is due to the monotonicity of $\hat{f}(\cdot, \cdot)$.

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&= \int_{w^-}^{w^+} \hat{f}(w, w)dw - \int_{w^-}^{w^+} \hat{f}(w^-, w^+)dw \\
&= \int_{w^-}^{w^+} \hat{f}(w, w) - \hat{f}(w^-, w)dw + \int_{w^-}^{w^+} \hat{f}(w^-, w) - \hat{f}(w^-, w^+)dw \\
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&= \int_{w^-}^{w^+} \hat{f}(w, w)dw - \int_{w^-}^{w^+} \hat{f}(w^-, w^+)dw \\
&= \int_{w^-}^{w^+} \hat{f}(w, w) - \hat{f}(w^-, w)dw + \int_{w^-}^{w^+} \hat{f}(w^-, w) - \hat{f}(w^-, w^+)dw \\
&\geq 0
\end{aligned}
$$

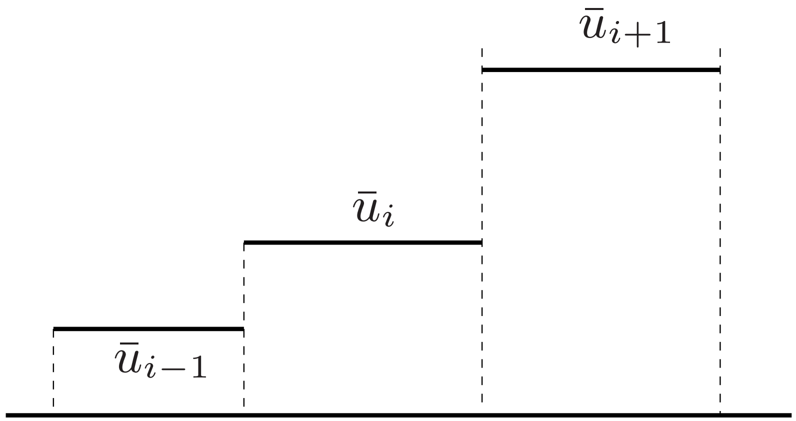
The last inequality is due to the monotonicity of $\hat{f}(\cdot, \cdot)$.

Limiter and total variation stability

For discontinuous solutions, the cell entropy inequality and the L2 -stability, although helpful, are not enough to control spurious numerical oscillations near discontinuities. In practice, especially for problems containing strong discontinuities, we often need to apply nonlinear limiters to control these oscillations and to obtain provable total variation stability.

Requirement on the limiter:

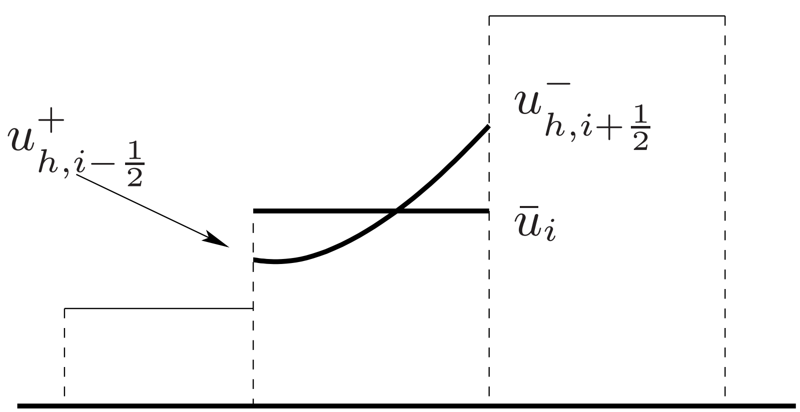
- Maintain the local conservation: keep the cell average
- Do not degrade the accuracy of the scheme



$$\overline{u}_i = \frac{1}{\Delta x_i} \int_{I_i} u_h dx$$

$$\Delta_+ \overline{u}_i = \overline{u}_{i+1} - \overline{u}_i$$

$$\Delta_- \overline{u}_i = \overline{u}_i - \overline{u}_{i-1}$$



$$\tilde{u}_i = u^-_{h,i+\frac{1}{2}} - \overline{u}_i$$

$$\tilde{\tilde{u}}_i = \overline{u}_i - u^+_{h,i-\frac{1}{2}}$$

Minmod limiter:

$$\tilde{u}_i^{(\mathrm{mod})} = m(\tilde{u}_i, \Delta_+ \overline{u}_i, \Delta_- \overline{u}_i), \qquad \tilde{\tilde{u}}_i^{(\mathrm{mod})} = m(\tilde{\tilde{u}}_i, \Delta_+ \overline{u}_i, \Delta_- \overline{u}_i)$$

with

$$m(a_1, \cdots, a_l) = \begin{cases} s \ \min(|a_1|, \cdots, |a_l|), & \text{if } s = \mathrm{sign}(a_1) = \cdots \mathrm{sign}(a_l) \\ 0, & \text{otherwise} \end{cases}$$

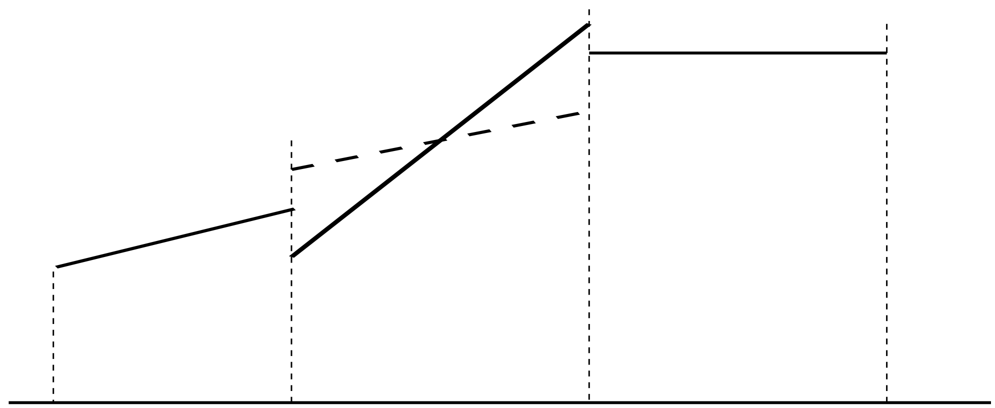
Then set

$$u_h^{(\mathrm{mod})}(x^-_{i+\frac{1}{2}}) = \overline{u}_i + \tilde{u}_i^{(\mathrm{mod})}, \qquad u_h^{(\mathrm{mod})}(x^+_{i-\frac{1}{2}}) = \overline{u}_i - \tilde{\tilde{u}}_i^{(\mathrm{mod})}$$

and a new u_h is reconstructed based on $\bar{u}_i$, $u_h^{(\mathrm{mod})}(x^-_{i+\frac{1}{2}})$, $u_h^{(\mathrm{mod})}(x^+_{i-\frac{1}{2}})$.

Note: The reconstruction is uniquely determined for $k = 1$ ($\tilde{u}_i^{(\mathrm{mod})} = \tilde{\tilde{u}}_i^{(\mathrm{mod})}$) and for $k = 2$.

The solution before (solid line) and after (dashed line) using the limiter:



Lemma (Harten 1997): *If a scheme can be written in the form*

$$u_i^{n+1} = u_i^n + C_{i+\frac{1}{2}} \Delta_+ u_i^n - D_{i-\frac{1}{2}} \Delta_- u_i^n$$

with periodic or compacted supported boundary conditions, where $C_{i+\frac{1}{2}}$ *and* $D_{i-\frac{1}{2}}$ *may be nonlinear functions of the grid values* u_j^n *for* $j = i-p, \cdots, i+q$ *with some* $p, q \geq 0$*, satisfying*

$$C_{i+\frac{1}{2}} \geq 0, \quad D_{i+\frac{1}{2}} \geq 0, \quad C_{i+\frac{1}{2}} + D_{i+\frac{1}{2}} \leq 1, \quad \forall i,$$

then the scheme is TVD, namely

$$TV(u^{n+1}) \leq TV(u^n),$$

where the total variation seminorm is defined by

$$TV(u) = \sum_i |\Delta_+ u_i|.$$

Proposition: *The solution u_h^n of the DG scheme with the forward Euler time discretization using the limiter discussed above, is total variation diminishing in the means (TVDM), that is*

$$TVM(u_h^{n+1}) \leq TVM(u_h^n).$$

with the semi-norm defined as

$$TVM(u_h) = \sum_i |\Delta_+ \bar{u}_i|.$$

Similar result can be extended to high order SSP time discretizations.

How about accuracy?

(1) In the smooth, monotone region: assume u_h is an approximation to a locally smooth function u, then

$$\tilde{u}_i = \frac{1}{2} u_x(x_i)\Delta x_i + O(h^2), \qquad \tilde{\tilde{u}}_i = \frac{1}{2} u_x(x_i)\Delta x_i + O(h^2)$$

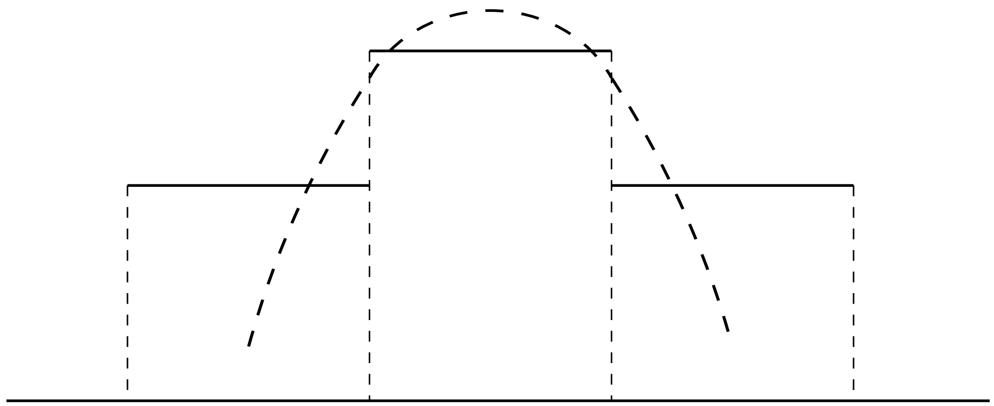
while

$$\Delta_+ \overline{u}_i = \frac{1}{2} u_x(x_i)(\Delta x_i + \Delta x_{i+1}) + O(h^2),$$
$$\Delta_- \overline{u}_i = \frac{1}{2} u_x(x_i)(\Delta x_i + \Delta x_{i-1}) + O(h^2).$$

and then

$$\tilde{u}_i^{(\mathrm{mod})} = m(\tilde{u}_i, \Delta_+ \overline{u}_i, \Delta_- \overline{u}_i) \approx \tilde{u}_i$$

(2) At the smooth extrema:



$(\Delta_+\overline{u}_i)(\Delta_-\overline{u}_i) < 0$ and

$$\tilde{u}_i^{(\mathrm{mod})} = m(\tilde{u}_i, \Delta_+\overline{u}_i, \Delta_-\overline{u}_i) = 0.$$

Accuracy loss!. *TVD schemes are at most first order accurate at smooth extrema.*

TVB limiter:

$$\tilde{m}(a_1, \cdots, a_l) = \begin{cases} a_1, & \text{if } |a_1| \leq Mh^2, \\ m(a_1, \cdots, a_l), & \text{otherwise.} \end{cases}$$

Note:

- With *suitably* chosen M, DG methods with this limiter is TVBM and can achieve uniform high order accuracy.
- Other limiters: moment limiters, WENO limiters

Error estimate

Proposition: *Let u be the smooth exact solution to the conservation law $u_t + u_x = 0$, and let u_h be the numerical solution to the semi-discrete DG method, then*

$$||u - u_h||_0 \leq Ch^{k+1}$$

here the constant C depends on the exact solution and it is independent of h, and $||\cdot||_0$ is the L^2 norm.

Note: - the monotone flux is upwinding $\hat{f}(u^-, u^+) = u^-$.

- similar result can be established for smooth solutions of general nonlinear conservation laws and for fully discretized RKDG methods (Zhang and Shu 2004).

Proof:

Step 1: A projection. $\forall w \in H^{k+1}(0,1)$, define $Pw \in V_h^k$, such that

$$\int_{I_i} (Pw(x) - w(x))v(x)dx = 0, \qquad \forall v \in P^{k-1}(I_i),$$

and $Pw(x^-_{i+\frac{1}{2}}) = w(x_{i+\frac{1}{2}})$. The following approximation result holds:

$$||w - Pw||_0 \leq C||w||_{H^{k+1}} h^{k+1}.$$

Denote $e_h = Pu - u_h \in V_h^k$, $\varepsilon_h = u - Pu$, then $u - u_h = e_h + \varepsilon_h$, and

$$||u - u_h||_0 = ||e_h + \varepsilon_h||_0 \leq ||e_h||_0 + Ch^{k+1}$$

Proof:

Step 1: A projection. $\forall w \in H^{k+1}(0,1)$, define $Pw \in V_h^k$, such that

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$$||u - u_h||_0 = ||e_h + \varepsilon_h||_0 \leq ||e_h||_0 + Ch^{k+1}$$

Step 2: Error equation / Galerkin orthogonality. Given i, from $B_i(u_h, v) = 0$ (scheme) and $B_i(u, v) = 0$ (consistency) for $v \in V_h^k$, there is

$$B_i(u - u_h, v) = 0, \qquad \forall v \in V_h^k,$$
$$\Rightarrow B_i(e_h + \varepsilon_h, v) = 0, \qquad \forall v \in V_h^k. \tag{1}$$

Step 3: to show $||e_h||_0 \leq Ch^{k+1}$. Take $v = e_h$ in (1),

$$B_i(e_h, e_h) = -B_i(\varepsilon_h, e_h)$$

The cell entropy inequality gives

$$B_i(e_h, e_h) = \frac{1}{2}\frac{d}{dt}\int_{I_i}(e_h)^2 dx + \hat{F}_{i+\frac{1}{2}} - \hat{F}_{i-\frac{1}{2}} + \Theta_{i-\frac{1}{2}}$$

with $\Theta_{i-\frac{1}{2}} \geq 0$. Therefore

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (e_h)^2 dx \leq \sum_i B_i(e_h, e_h).$$

Step 2: Error equation / Galerkin orthogonality. Given i, from $B_i(u_h, v) = 0$ (scheme) and $B_i(u, v) = 0$ (consistency) for $v \in V_h^k$, there is

$$B_i(u - u_h, v) = 0, \qquad \forall v \in V_h^k,$$
$$\Rightarrow B_i(e_h + \varepsilon_h, v) = 0, \qquad \forall v \in V_h^k. \tag{2}$$

Step 3: to show $||e_h||_0 \leq Ch^{k+1}$. Take $v = e_h$ in (2),

$$B_i(e_h, e_h) = -B_i(\varepsilon_h, e_h)$$

The cell entropy inequality gives

$$B_i(e_h, e_h) = \frac{1}{2}\frac{d}{dt}\int_{I_i}(e_h)^2 dx + \hat{F}_{i+\frac{1}{2}} - \hat{F}_{i-\frac{1}{2}} + \Theta_{i-\frac{1}{2}}$$

with $\Theta_{i-\frac{1}{2}} \geq 0$. Therefore

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (e_h)^2 dx \leq \sum_i B_i(e_h, e_h).$$

$$-B_i(\varepsilon_h, e_h) = -\int_{I_i} (\varepsilon_{h,t} e_h - \varepsilon_h e_{h,x})\, dx - (\varepsilon_h)^-_{i+\frac{1}{2}} (e_h)^-_{i+\frac{1}{2}} + (\varepsilon_h)^-_{i-\frac{1}{2}} (e_h)^+_{i-\frac{1}{2}}$$

$$= -\int_{I_i} \varepsilon_{h,t} e_h dx, \qquad \text{(definition of projection††)}$$

$$\leq \frac{1}{2}\left(\int_{I_i} (\varepsilon_{h,t})^2 dx + \int_{I_i} (e_h)^2 dx\right)$$

††: due to $e_{h,x} \in P^{k-1}(I_j)$,

$$\int_{I_i} \varepsilon_h e_{h,x} dx = \int_{I_i} (u - Pu) e_{h,x} dx = 0,$$

and

$$(\varepsilon_h)^-_{i+\frac{1}{2}} = u_{i+\frac{1}{2}} - (Pu)^-_{i+\frac{1}{2}} = 0$$

.

Sum up over i,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (e_h)^2 dx \leq \sum_i B_i(e_h, e_h) = -\sum_i B_i(\varepsilon_h, e_h)$$
$$\leq \frac{1}{2}\left(\int_0^1 (\varepsilon_{h,t})^2 dx + \int_0^1 (e_h)^2 dx\right)$$
$$\leq Ch^{2k+2} + \frac{1}{2}\int_0^1 (e_h)^2 dx$$

The last inequality uses

$$||\varepsilon_{h,t}||_0 = ||u_t - (Pu)_t||_0 \leq ||u_t - P(u_t)||_0 \leq Ch^{k+1}.$$

The final result follows from a Gronwall's inequality and the initial error $||u_0(\cdot) - u_h(\cdot, 0)||_0 \leq Ch^{k+1}$.

Further discussion

(1) DG methods can be formulated for multi-dimensional scalar or system of equations.

$$u_t + f(u)_x + g(u)_y = 0$$

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0, \qquad \mathbf{U}_t + \nabla \cdot \mathbf{F}(\mathbf{U}) = 0.$$

(2)

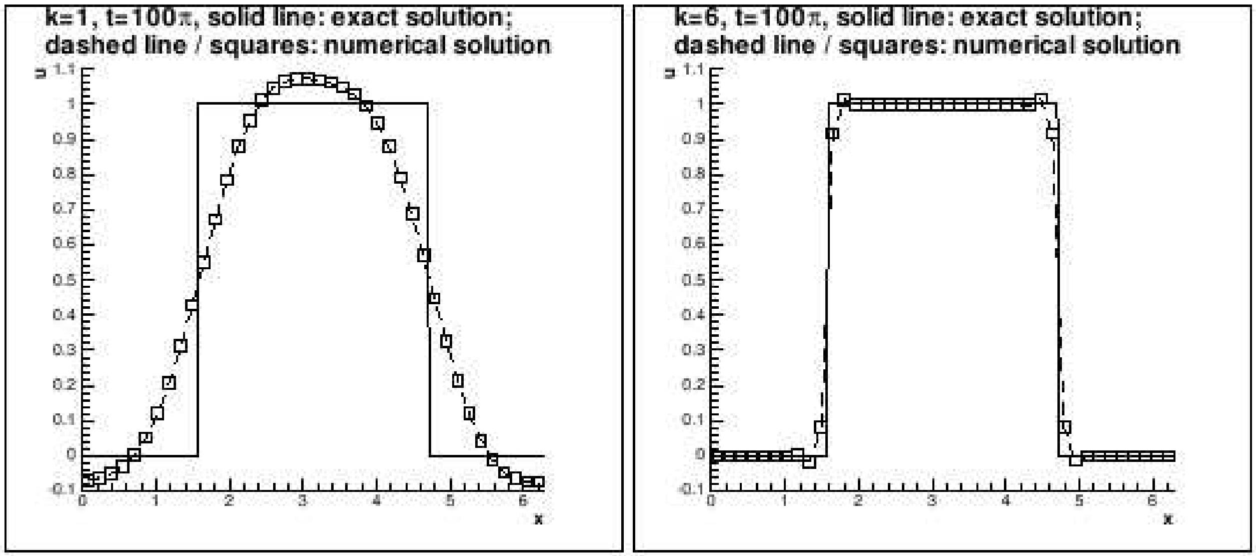


Figure 1: Transport equation: Comparison of the exact and the RKDG solutions at $T = 100\pi$ with second order (P^1, left) and seventh order (P^6, right) RKDG methods. One dimensional results with 40 cells, exact solution (solid line) and numerical solution (dashed line and symbols, one point per cell)

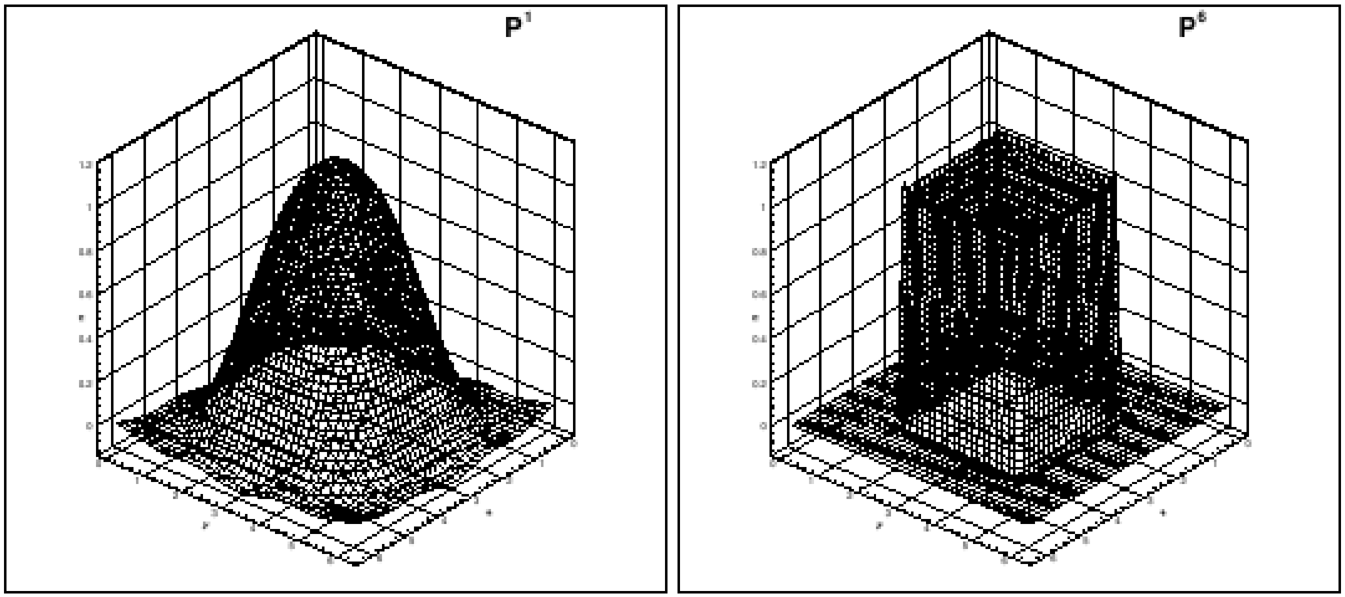
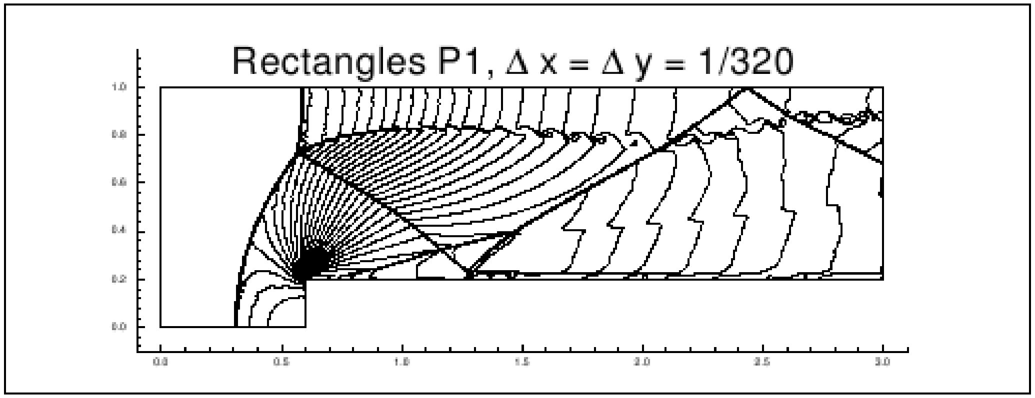
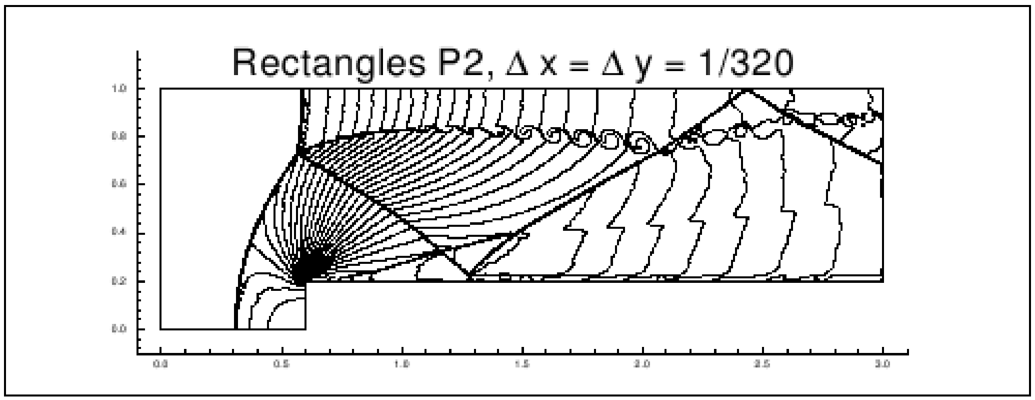


Figure 2: Transport equation: Comparison of the exact and the RKDG solutions at $T = 100\pi$ with second order (P^1, left) and seventh order (P^6, right) RKDG methods. Two dimensional results with 40×40 cells.





The flow passes a forward-facing step problem for the 2d compressible Euler equation

(3) Advantages of DG methods

- Easy handling of complicated geometry and boundary conditions (common to all finite element methods). Allowing hanging nodes in the mesh (unique to DG).
- Compact. Communication only with immediate neighbors, regardless of the order of the scheme.
- Explicit. Because of the discontinuous basis, the mass matrix is local to the cell, resulting in explicit time stepping (no systems to solve).
- Parallel efficiency. Achieve 99% parallel efficiency for static mesh and over 80% parallel efficiency for dynamic load balancing with adaptive meshes (Flaherty et al.).

- Provable cell entropy inequality and L^2 stability, for arbitrary scalar equations in any spatial dimension and any triangulation, for any order of accuracy, without limiters.
- At least $(k+\frac{1}{2})$-th order accurate, and often $(k+1)$-st order accurate for smooth solutions when piecewise polynomials of degree k are used, regardless of the structure of the meshes.
- Easy h-p adaptivity.
- Flexibility in using various local approximating functions (divergence-free, curl-free, harmonic polynomials, planewave functions, ...)
- Local conservation for conservation laws.

DG Methods for Convection-Diffusion Equations

Outline

- An 'inconsistent' discontinuous Galerkin (DG) method
- Local DG (LDG) methods for convection-diffusion equations
  - Semi-discrete formulation
  - 'Cell entropy inequality' and L^2-stability
  - Error estimate
  - Time discretization
- Further discussion

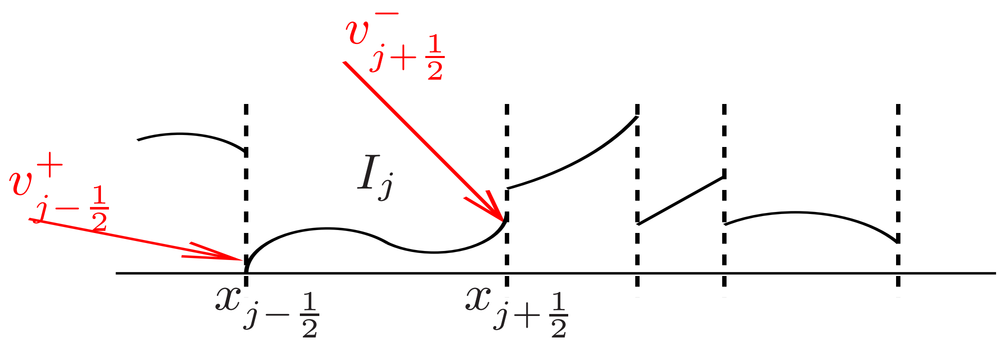
An 'inconsistent' DG method

Consider the heat equation

$$\begin{cases} u_t = u_{xx}, & 0 < x < 1, \quad t > 0 \\ u(x,0) = u_0(x), & 0 < x < 1 \end{cases}$$

with the periodic boundary condition.

A straightforward generalization of the DG method from the conservation law $u_t + f(u)_x = 0$ is to write down the same scheme and replace $f(u)$ by $-u_x$.



Discrete space:

$$V_h = V_h^k = \{v : v \in P^k(I_j),\ \forall j\}$$

Some notations:

$$I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], \quad \Delta x_j = |I_j|, \quad \Delta x \text{ or } h = \max_j |I_j|$$

$$v^-_{j+\frac{1}{2}} = \lim_{\epsilon \longrightarrow 0^-} v(x_{j+\frac{1}{2}} + \epsilon), \qquad v^+_{j-\frac{1}{2}} = \lim_{\epsilon \longrightarrow 0^+} v(x_{j-\frac{1}{2}} + \epsilon)$$

$$[v]_{j-\frac{1}{2}} = v^+_{j-\frac{1}{2}} - v^-_{j-\frac{1}{2}}, \qquad \{v\}_{j-\frac{1}{2}} = \frac{1}{2}(v^+_{j-\frac{1}{2}} + v^-_{j-\frac{1}{2}})$$

Recall the semi-discrete DG discretization for $u_t + f(u)_x = 0$: look for $u_h \in V_h$, such that $\forall v \in V_h$,

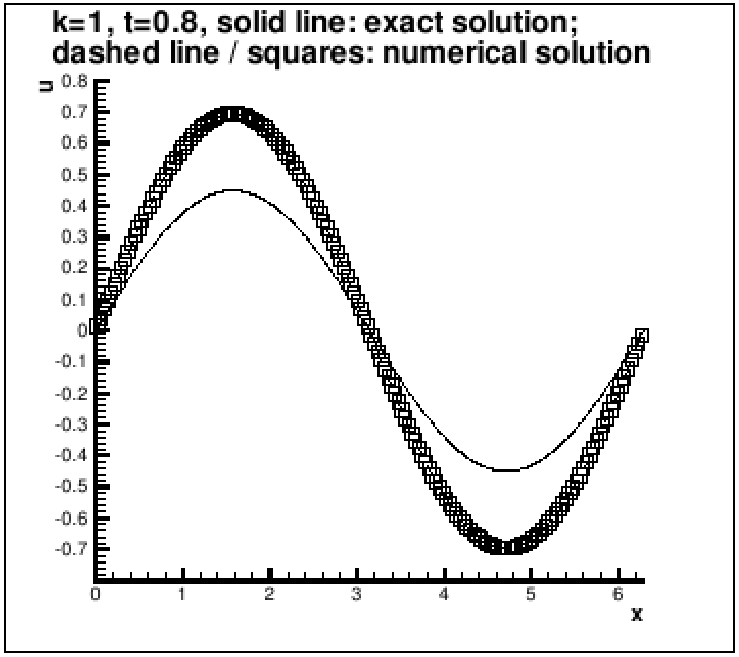
$$\int_{I_j} u_{h,t} v dx - \int_{I_j} f(u_h) v_x dx + \hat{f}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0$$

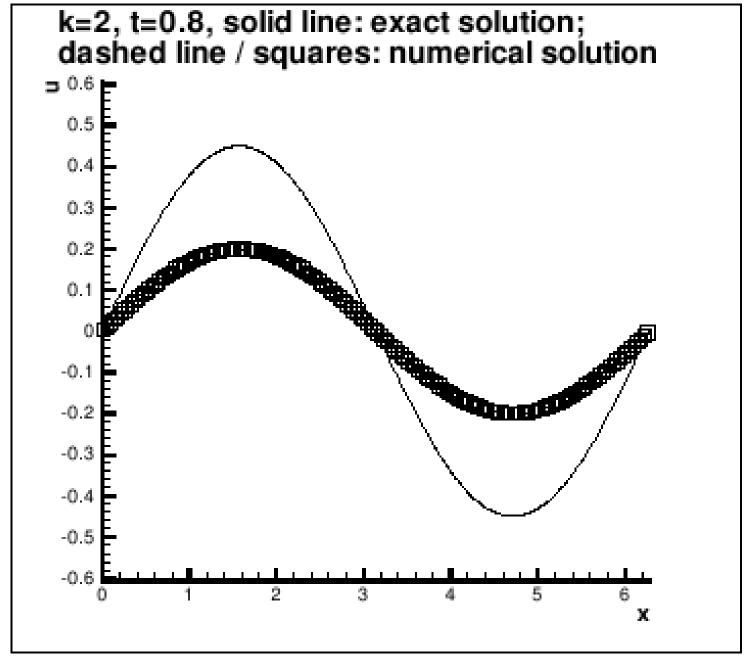
A generalization to the heat equation $u_t - u_{xx} = u_t + (-u_x)_x = 0$: look for $u_h \in V_h$ such that $\forall v \in V_h$,

$$\int_{I_j} u_{h,t} v dx + \int_{I_j} u_{h,x} v_x dx - (\widehat{u_{h,x}})_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + (\widehat{u_{h,x}})_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0,$$

and the central flux is a natural choice:

$$\widehat{u_{h,x}}_{j+\frac{1}{2}} = \{u_{h,x}\}_{j+\frac{1}{2}}$$





- The solution of the 'inconsistent' DG method for the heat equation on $(0, 2\pi)$ with $u_0(x) = \sin(x)$. 3rd order Runge-Kutta method is used in time with 160 mesh elements.

- The method is consistent to the heat equation yet it is (very weakly) unstable (Zhang and Shu 2003).

LDG methods for convection-diffusion equations

(Bassi and Rebay 1997, Cockburn and Shu 1998)

* **Semi-discrete formulation**: rewrite the heat equation

$$u_t = u_{xx}$$

into a system

$$\begin{cases} u_t = q_x \\ q = u_x \end{cases}$$

Find $u_h,\ q_h \in V_h$ such that $\forall\ v, w \in V_h$

$$\int_{I_j} u_{h,t} v dx + \int_{I_j} q_h v_x dx - (\hat{q}_h)_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + (\hat{q}_h)_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0,$$

$$\int_{I_j} q_h p dx + \int_{I_j} u_h p_x dx - (\hat{u}_h)_{j+\frac{1}{2}} p^-_{j+\frac{1}{2}} + (\hat{u}_h)_{j-\frac{1}{2}} p^+_{j-\frac{1}{2}} = 0.$$

A natural choice: (*central*)

$$\hat{u}_h = \{u_h\}, \qquad \hat{q}_h = \{q_h\}$$

A better choice: (*alternating*)

$$\hat{u}_h = u_h^-,\ \hat{q}_h = q_h^+, \qquad \text{or} \ \ \hat{u}_h = u_h^+,\ \hat{q}_h = q_h^-$$

Note: - the discrete space is the same for u_h and q_h.

- q_h can be solved locally in terms of u_h in each mesh element, this gives the name of the method.

Consider the convection-diffusion equation

$$\begin{cases} u_t + f(u)_x = (a(u)u_x)_x, & 0 < x < 1, \quad t > 0 \\ u(x,0) = u_0(x), & 0 < x < 1 \end{cases}$$

with the periodic boundary condition and $a(u) \geq 0$.

The equation can be rewritten as the system

$$\begin{cases} u_t + f(u)_x = (b(u)q)_x \\ q - B(u)_x = 0 \end{cases}$$

where

$$b(u) = \sqrt{a(u)}, \quad B(u) = \int^u b(u)du.$$

Find $u_h,\ q_h \in V_h$ such that $\forall\ v, w \in V_h$

$$\int_{I_j} (u_h)_t v dx - \int_{I_j} (f(u_h) - b(u_h)q_h) v_x dx$$
$$+ (\hat{f} - \hat{b}\hat{q})_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - (\hat{f} - \hat{b}\hat{q})_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0,$$
$$\int_{I_j} q_h p dx + \int_{I_j} B(u_h) p_x dx - \hat{B}_{j+\frac{1}{2}} p^-_{j+\frac{1}{2}} + \hat{B}_{j-\frac{1}{2}} p^+_{j-\frac{1}{2}} = 0.$$

- $\hat{f}$ is a monotone flux.
- $(\hat{b}, \hat{q}, \hat{B})$ is either of the following

$$\hat{b} = \frac{B(u_h^+) - B(u_h^-)}{u_h^+ - u_h^-}, \quad \hat{q} = q_h^+, \quad \hat{B} = B(u_h^-),$$
$$\hat{b} = \frac{B(u_h^+) - B(u_h^-)}{u_h^+ - u_h^-}, \quad \hat{q} = q_h^-, \quad \hat{B} = B(u_h^+).$$

* 'Cell entropy inequality' and L^2-stability

Proposition: *(1) The solution u_h, q_h to the semi-discrete LDG scheme satisfies the following 'cell entropy inequality'*

$$\frac{1}{2}\frac{d}{dt}\int_{I_j}(u_h)^2dx+\int_{I_j}q_h^2dx+\hat{F}_{j+\frac{1}{2}}-\hat{F}_{j-\frac{1}{2}}\le 0$$

for some consistent 'entropy flux' $\hat{F}=\hat{F}(u_h^-,q_h^-;u_h^+,q_h^+)$ *satisfying* $\hat{F}(u,q;u,q)=F(u)-ub(u)q$ *where, as before,* $F(u)=uf(u)-\int^u f(u)du$.

(2) Furthermore, u_h, q_h satisfies the following L^2-stability

$$\frac{d}{dt}\int_0^1(u_h)^2dx+2\int_0^1 q_h^2dx\le 0,$$

or with $||\cdot||_0$ as the L^2 norm,

$$||u_h(\cdot,t)||_0^2+2\int_0^t||q_h(\cdot,s)||_0^2ds\le||u_h(\cdot,0)||_0^2\le||u_0||_0^2.$$

Proof: The semi-discrete LDG method: find $u_h, q_h \in V_h$, s.t. $\forall v, p \in V_h$, $\forall j$

$$B_j(u_h, q_h; v, p) = 0,$$

$$\begin{aligned} B_j(u, q; v, p) = & \int_{I_j} u_t v dx - \int_{I_j} (f(u) - b(u)q) v_x dx \\ & + (\hat{f} - \hat{b}\hat{q})_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - (\hat{f} - \hat{b}\hat{q})_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} \\ & + \int_{I_j} qp dx + \int_{I_j} B(u) p_x dx - \hat{B}_{j+\frac{1}{2}} p^-_{j+\frac{1}{2}} + \hat{B}_{j-\frac{1}{2}} p^+_{j-\frac{1}{2}} \end{aligned}$$

Step 1:

$$B_j(u_h, q_h; u_h, q_h) = 0, \qquad \forall j$$

Step 2: To show $\forall v, p \in V_h$, with $H(v) = \int^v f(v) dv$

$$B_j(v, p; v, p) = \frac{1}{2} \frac{d}{dt} \int_{I_j} v^2 dx + \int_{I_j} p^2 dx + \hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}} + \Theta_{j-\frac{1}{2}}$$

with $\Theta_{j-\frac{1}{2}} \geq 0$, and $\hat{F}$ is consistent to the ‘entropy flux’ $F(v, p) = vf(v) - H(v) - b(v)pv$.

With $H(v) = \int^v f(v)\,dv$, $B(u) = \int^u b(u)du$,

$$B_j(v,p;v,p) = \int_{I_j} v_t v dx - \int_{I_j} (f(v) - b(v)p)v_x dx$$

$$+ (\hat{f} - \hat{b}\hat{p})_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - (\hat{f} - \hat{b}\hat{p})_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}$$

$$+ \int_{I_j} p^2 dx + \int_{I_j} B(v)p_x dx - \hat{B}_{j+\frac{1}{2}} p^-_{j+\frac{1}{2}} + \hat{B}_{j-\frac{1}{2}} p^+_{j-\frac{1}{2}}$$

$$= \frac{1}{2}\frac{d}{dt}\int_{I_j} v^2 dx + \int_{I_j} p^2 dx - \underbrace{\int_{I_j} \underbrace{(f(v) - b(v)p)v_x - B(v)p_x}_{(H(v)-B(v)p)_x} dx}_{(H(v)-B(v)p)^-_{j+\frac{1}{2}} - (H(v)-B(v)p)^+_{j-\frac{1}{2}}}$$

$$+ (\hat{f} - \hat{b}\hat{p})_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - (\hat{f} - \hat{b}\hat{p})_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} - \hat{B}_{j+\frac{1}{2}} p^-_{j+\frac{1}{2}} + \hat{B}_{j-\frac{1}{2}} p^+_{j-\frac{1}{2}}$$

$$= \frac{1}{2}\frac{d}{dt}\int_{I_j} v^2 dx + \int_{I_j} p^2 dx + \hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}} + \Theta_{j-\frac{1}{2}}$$

With $H(v) = \int^v f(v)\,dv$, $B(u) = \int^u b(u)du$,

$$B_j(v,p;v,p) = \int_{I_j} v_t v dx - \int_{I_j} (f(v) - b(v)p)v_x dx$$

$$+ (\hat{f} - \hat{b}\hat{p})_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - (\hat{f} - \hat{b}\hat{p})_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}$$

$$+ \int_{I_j} p^2 dx + \int_{I_j} B(v)p_x dx - \hat{B}_{j+\frac{1}{2}} p^-_{j+\frac{1}{2}} + \hat{B}_{j-\frac{1}{2}} p^+_{j-\frac{1}{2}}$$

$$= \frac{1}{2}\frac{d}{dt}\int_{I_j} v^2 dx + \int_{I_j} p^2 dx - \underbrace{\int_{I_j} \underbrace{(f(v) - b(v)p)v_x - B(v)p_x}_{(H(v)-B(v)p)_x} dx}_{(H(v)-B(v)p)^-_{j+\frac{1}{2}} - (H(v)-B(v)p)^+_{j-\frac{1}{2}}}$$

$$+ (\hat{f} - \hat{b}\hat{p})_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - (\hat{f} - \hat{b}\hat{p})_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} - \hat{B}_{j+\frac{1}{2}} p^-_{j+\frac{1}{2}} + \hat{B}_{j-\frac{1}{2}} p^+_{j-\frac{1}{2}}$$

$$= \frac{1}{2}\frac{d}{dt}\int_{I_j} v^2 dx + \int_{I_j} p^2 dx + \hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}} + \Theta_{j-\frac{1}{2}}$$

With $H(v) = \int^v f(v)\,dv$, $B(u) = \int^u b(u)du$,

$$B_j(v,p;v,p) = \int_{I_j} v_t v dx - \int_{I_j} (f(v) - b(v)p)v_x dx$$

$$+ (\hat{f} - \hat{b}\hat{p})_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - (\hat{f} - \hat{b}\hat{p})_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}$$

$$+ \int_{I_j} p^2 dx + \int_{I_j} B(v)p_x dx - \hat{B}_{j+\frac{1}{2}} p^-_{j+\frac{1}{2}} + \hat{B}_{j-\frac{1}{2}} p^+_{j-\frac{1}{2}}$$

$$= \frac{1}{2}\frac{d}{dt}\int_{I_j} v^2 dx + \int_{I_j} p^2 dx - \underbrace{\int_{I_j} \underbrace{(f(v) - b(v)p)v_x - B(v)p_x}_{(H(v)-B(v)p)_x} dx}_{(H(v)-B(v)p)^-_{j+\frac{1}{2}} - (H(v)-B(v)p)^+_{j-\frac{1}{2}}}$$

$$+ (\hat{f} - \hat{b}\hat{p})_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - (\hat{f} - \hat{b}\hat{p})_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} - \hat{B}_{j+\frac{1}{2}} p^-_{j+\frac{1}{2}} + \hat{B}_{j-\frac{1}{2}} p^+_{j-\frac{1}{2}}$$

$$= \frac{1}{2}\frac{d}{dt}\int_{I_j} v^2 dx + \int_{I_j} p^2 dx + \hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}} + \Theta_{j-\frac{1}{2}}$$

What remains is to show $\Theta \geq 0$.

$$\begin{aligned}\Theta &= [H(v) - B(v)p] - (\hat{f} - \hat{b}\hat{p})[v] + \hat{B}[p] \\ &= [H(v)] - \hat{f}[v] - \Big([B(v)p] - \hat{b}\hat{p}[v] - \hat{B}[p]\Big) \\ &= [H(v)] - \hat{f}[v]\end{aligned}$$

The last equality is due to the choice of $(\hat{b}, \hat{p}, \hat{B})$, and an equality

$$[ab] = [b]a^- + [a]b^+.$$

Now Θ is the same as that for the conservation law case, therefore $\Theta \geq 0$.

* Error estimate

Proposition: *Let u be the smooth exact solution to the heat equation $u_t = u_{xx}$, and let u_h, q_h be the numerical solution to the semi-discrete LDG method with alternating fluxes, then*

$$||u(\cdot,t) - u_h(\cdot,t)||_0^2 + \int_0^1 ||u_x(\cdot,s) - q_h(\cdot,s)||_0^2 ds \le Ch^{2(k+1)}.$$

here the constant C depends on the exact solution and it is independent of h, and $||\cdot||_0$ is the L^2 norm.

Note:

- similar results can be established for smooth solutions of general nonlinear convection - diffusion equation (Xu and Shu 2007).

Ingredients in the proof:

(1) Projections. $\forall w \in H^{k+1}(0,1)$, define $Pw, Qw \in V_h^k$, such that

$$\int_{I_i} (Pw(x) - w(x))v(x)dx = 0, \qquad Pw(x_{i+\frac{1}{2}}^-) = w(x_{i+\frac{1}{2}})$$

and

$$\int_{I_i} (Qw(x) - w(x))v(x)dx = 0, \qquad Qw(x_{i-\frac{1}{2}}^+) = w(x_{i-\frac{1}{2}})$$

for any $v \in P^{k-1}(I_i)$. They both satisfy

$$||w - Pw||_0 \le C||w||_{H^{k+1}} h^{k+1}, \qquad ||w - Qw||_0 \le C||w||_{H^{k+1}} h^{k+1}.$$

(2) Error equation / Galerkin orthogonality.

$$B_j(u - u_h, q - q_h; v, p) = 0, \qquad \forall v,\ p \in V_h$$

here $q = u_x$ and

$$\begin{aligned} B_j(u, q; v, p) = & \int_{I_j} u_t v dx + \int_{I_j} q v_x dx - \hat{q}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + \hat{q}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} \\ & + \int_{I_j} q p dx + \int_{I_j} v p_x dx - \hat{v}_{j+\frac{1}{2}} p^-_{j+\frac{1}{2}} + \hat{v}_{j-\frac{1}{2}} p^+_{j-\frac{1}{2}} \end{aligned}$$

(3) L^2-stability. $\forall v, q \in V_h$

$$\frac{1}{2}\frac{d}{dt}\int_0^1 v^2 dx + \int_0^1 p^2 dx \le \sum_j B_j(v, p; v, p)$$

* Time discretization

- Strong stability preserving Runge-Kutta methods: 3rd order

$$
\begin{aligned}
w^{(1)} &= \mathbf{U}^n + \Delta t L(\mathbf{U}^n) \\
w^{(2)} &= \frac{3}{4}\mathbf{U}^n + \frac{1}{4}\left(w^{(1)} + \Delta t L(w^{(1)})\right) \\
\mathbf{U}^{n+1} &= \frac{1}{3}\mathbf{U}^n + \frac{2}{3}\left(w^{(2)} + \Delta t L(w^{(2)})\right)
\end{aligned}
$$

Note: *explicit or local time discretization will need* $\Delta t \leq C\Delta x^p$ *(p is the order of the PDE)*

- Other methods (Xia, Xu and Shu 2007): spectral deferred correction, additive Runge-Kutta method, exponential time discretization...

One numerical example: L^2 errors and orders of accuracy for the LDG method to the heat equation on $(0, 2\pi)$ with $u_0(x) = \sin(x)$. $\Delta x = 2\pi/N$.

	P^1		P^2	
N	L^2 error	order	L^2 error	order
u				
20	1.92e-03	-	4.87e-05	-
40	4.81e-04	2.00	6.08e-06	3.00
80	1.20e-04	2.00	7.60e-07	3.00
q				
20	1.93e-03	-	4.87e-05	-
40	4.81e-04	2.00	6.08e-06	3.00
80	1.20e-04	2.00	7.60e-07	3.00

Further discussion

(1) LDG methods can be formulated for multi-dimensional scalar or system of equations:

$$\begin{cases} u_t + \sum_{i=1}^d \left(f_i(u) - \sum_{j=1}^d a_{ij}(u) u_{x_j} \right)_{x_i} = 0, & \mathbf{x} \in (0,1)^d, t > 0 \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \mathbf{x} \in (0,1)^d. \end{cases}$$

here the matrix with the entry $a_{ij}(u)$ is symmetric, semi-positive definite, and $\mathbf{x} = (x_1, \cdots, x_d)$. Examples for multi-dimensional systems include the compressible Navier-Stokes equations, and the equations of the hydrodynamic model for semiconductor device simulation.

(2) LDG methods with provable stability are available for many higher order linear or nonlinear PDEs (Yan, Shu, Xu...). Some examples include

- the KdV equation

$$u_t + f(u)_x + u_{xxx} = 0$$

- the nonlinear Schrödinger equation

$$iu_t + \triangle u + f(|u|^2)u = 0$$

- the Camassa-Holm equation

$$u_t - u_{xxt} + 2\kappa u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

(3) Revisit the 'inconsistent' DG method for the heat equation:

$$\int_{I_j} u_{h,t} v dx + \int_{I_j} u_{h,x} v_x dx - (\widehat{u_{h,x}})_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + (\widehat{u_{h,x}})_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0,$$

and $\widehat{u_{h,x}} = \{u_{h,x}\}$.

Some remedy:

- Additional term on the left (Baumann and Oden 1999):

$$-\frac{1}{2}\left((v_x^- [u_h])_{j+\frac{1}{2}} + (v_x^+ [u_h])_{j-\frac{1}{2}}\right)$$

- One more integration by parts (Cheng and Shu 2007): $k \geq 1$

$$\int_{I_j} (u_{h,t} v - u_h v_{xx})\, dx - (\widehat{u_{h,x}})_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + (\widehat{u_{h,x}})_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}$$
$$+ \tilde{u}_{h,j+\frac{1}{2}} (v_x)^-_{j+\frac{1}{2}} - \tilde{u}_{h,j-\frac{1}{2}} (v_x)^+_{j-\frac{1}{2}} = 0,$$

- Use other discretizations for 2nd order elliptic equations (Arnold, Brezzi, Cockburn and Marini 2002)

DG Methods for Hamilton-Jacobi Equations

Consider

$$\phi_t + H(\nabla_{\mathbf{x}}\phi, \mathbf{x}) = 0$$

with suitable initial and boundary conditions, $H(\cdot, \cdot)$ is the Hamiltonian.

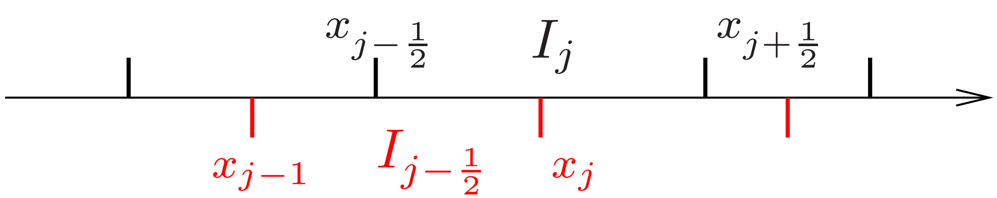
Challenges to design DG methods: Hamilton-Jacobi (HJ) equations in general are not in the divergence form.

Outline

- A central DG method
  - Formulation for conservation laws
  - How to solve Hamilton-Jacobi equations?
- Other DG methods

A central DG method

Central DG methods, based on the central scheme and the DG method, were introduced for hyperbolic conservation laws in (Liu, Shu, Tadmor and Zhang, 2007 2008)



Two discrete spaces:

$$V_h = V_h^k = \{v : v|_{I_j} \in P^k(I_j), \forall j\}$$

$$W_h = W_h^k = \{v : v|_{I_{j-\frac{1}{2}}} \in P^k(I_{j-\frac{1}{2}}), \forall j\}$$

Some notations:

$$I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], \qquad I_{j-\frac{1}{2}} = [x_{j-1}, x_j]$$

*** Formulation for conservation laws**

Suppose $\phi_h^n \in V_h, \psi_h^n \in W_h$ are two approximations to the solution of $\phi_t + f(\phi)_x = 0$ at t^n, to update the solution at $t^{n+1} = t^n + \Delta t^n$:

With the DG discretization in space and the forward Euler in time, $\phi_h^{n+1} \in V_h$ can be updated as follows: $\forall \eta \in V_h$

$$\int_{I_j} \phi_h^{n+1} \eta dx = \int_{I_j} \phi_h^n \eta dx$$

$$+ \Delta t^n \left(\int_{I_j} f(\phi_h^n) \eta_x dx - \hat{f}_{j+\frac{1}{2}} \eta_{j+\frac{1}{2}}^- + \hat{f}_{j-\frac{1}{2}} \eta_{j-\frac{1}{2}}^+ \right)$$

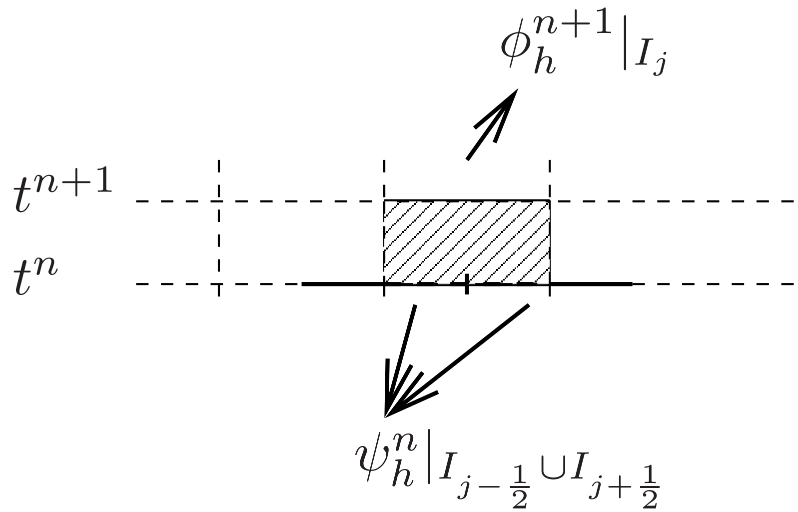
here $\hat{f}$ is a consistent numerical flux

$$\hat{f}_{j+\frac{1}{2}} = \hat{f}_{j+\frac{1}{2}}(\phi_{h,j+\frac{1}{2}}^-, \phi_{h,j+\frac{1}{2}}^+)$$

With the spirit of the central scheme on overlapping meshes, $\phi_h^{n+1} \in V_h$ can be updated based on: $\forall \eta \in V_h$

$$\int_{I_j} \phi_h^{n+1} \eta dx = \int_{I_j} \psi_h^n \eta dx$$

$$+ \Delta t^n \left(\int_{I_j} f(\psi_h^n) \eta_x dx - f(\psi_{h,j+\frac{1}{2}}^n) \eta_{j+\frac{1}{2}}^- + f(\psi_{h,j-\frac{1}{2}}^n) \eta_{j-\frac{1}{2}}^+ \right)$$



Moreover

$$\int_{I_j} \phi_h^{n+1} \eta dx = \theta_n \int_{I_j} \psi_h^n \eta dx + (1-\theta_n) \int_{I_j} \phi_h^n \eta dx$$

$$+ \Delta t^n \left(\int_{I_j} f(\psi_h^n) \eta_x dx - f(\psi^n_{h,j+\frac{1}{2}}) \eta^-_{j+\frac{1}{2}} + f(\psi^n_{h,j-\frac{1}{2}}) \eta^+_{j-\frac{1}{2}} \right)$$

where $\theta_n = \Delta t^n / \Delta \tau^n \in [0,1]$, $\Delta \tau^n$ is the upper bound of the time step due to the stability condition. With $\Delta t^n \to 0$, one gets the semi-discrete central DG method

$$\int_{I_j} \phi_{h,t} \eta dx = \frac{1}{\tau_{\max}} \int_{I_j} (\psi_h - \phi_h) \eta dx + \int_{I_j} f(\psi_h) \eta_x dx$$

$$- f(\psi_{h,j+\frac{1}{2}}) \eta^-_{j+\frac{1}{2}} + f(\psi_{h,j-\frac{1}{2}}) \eta^+_{j-\frac{1}{2}}$$

- Properties:
  - Using overlapping meshes and two sets of numerical solutions
  - Being free of the numerical flux (Riemann solver)
  - Allowing larger CFL number than DG methods for $k \geq 1$
  - Sharing some features of standard DG methods
- The L^2 stability and the error estimate: $f(\phi) = a\phi$,

$$\frac{1}{2}\frac{d}{dt}\int (\phi_h)^2 + (\psi_h)^2 dx = -\frac{1}{\tau_{\max}}\int (\phi_h - \psi_h)^2 dx \leq 0$$

$$||\phi - \phi_h||_{L^2} + ||\phi - \psi_h||_{L^2} = O(h^k)$$

Numerical order of accuracy: $O(h^{k+1})$

- Central DG methods for MHD equations with the *exactly* divergence-free magnetic field (L., Xu and Yakovlev)

*** How to solve Hamilton-Jacobi equations?**

The main observation is a reformulation of the central DG methods for conservation laws: To look for $\phi_h(\cdot,t) \in V_h$ and $\psi_h(\cdot,t) \in W_h$, s.t.

$$\int_{I_j} (\phi_{h,t} + f(\psi_h)_x)\,\eta dx = \frac{1}{\tau_{\max}} \int_{I_j} (\psi_h - \phi_h)\eta dx - [f(\psi_h)]_j \eta_j$$

$$\int_{I_{j+\frac{1}{2}}} (\psi_{h,t} + f(\phi_h)_x)\,\xi dx = \frac{1}{\tau_{\max}} \int_{I_{j+\frac{1}{2}}} (\phi_h - \psi_h)\xi dx - [f(\phi_h)]_{j+\frac{1}{2}} \xi_{j+\frac{1}{2}}$$

for any $\eta \in V_h$, $\xi \in W_h$ and j.

Note. DG method with the upwinding flux for $u_t + au_x = 0$ ($a > 0$):

$$\underbrace{\int_{I_j} (u_{h,t} + au_{h,x})\, v_h dx}_{\text{residual}} + \underbrace{a[u_h]_{j-\frac{1}{2}} v^+_{h,j-\frac{1}{2}}}_{\text{jump at interface}} = 0$$

A central DG method for $\phi_t + H(\phi_x, x) = 0$ (L. and Yakovlev 2010): look for $\phi_h(\cdot, t) \in V_h$ and $\psi_h(\cdot, t) \in W_h$, s.t. for any $\eta \in V_h$, $\xi \in W_h$

$$\int_{I_j} (\phi_{h,t} + H(\psi_{h,x}, x))\, \eta dx = \frac{1}{\tau_{\max}} \int_{I_j} (\psi_h - \phi_h)\eta dx - H_1(\phi_{h,x}, x)|_{x_j} [\psi_h]_j \eta_j$$

$$\int_{I_{j+\frac{1}{2}}} (\psi_{h,t} + H(\phi_{h,x}, x))\, \xi dx = \frac{1}{\tau_{\max}} \int_{I_{j+\frac{1}{2}}} (\phi_h - \psi_h)\xi dx - H_1(\psi_{h,x}, x)|_{x_{j+\frac{1}{2}}} [\phi_h]_{j+\frac{1}{2}} \xi_{j+\frac{1}{2}}$$

Note: - $H_1(\phi_x, x) = \frac{\partial H(\phi_x, x)}{\partial \phi_x}$.

- The method is the same as the central DG method for conservation laws when $H(\nabla\phi) = a\nabla\phi$. The interface treatment is also used in (Cheng and Shu 2007).

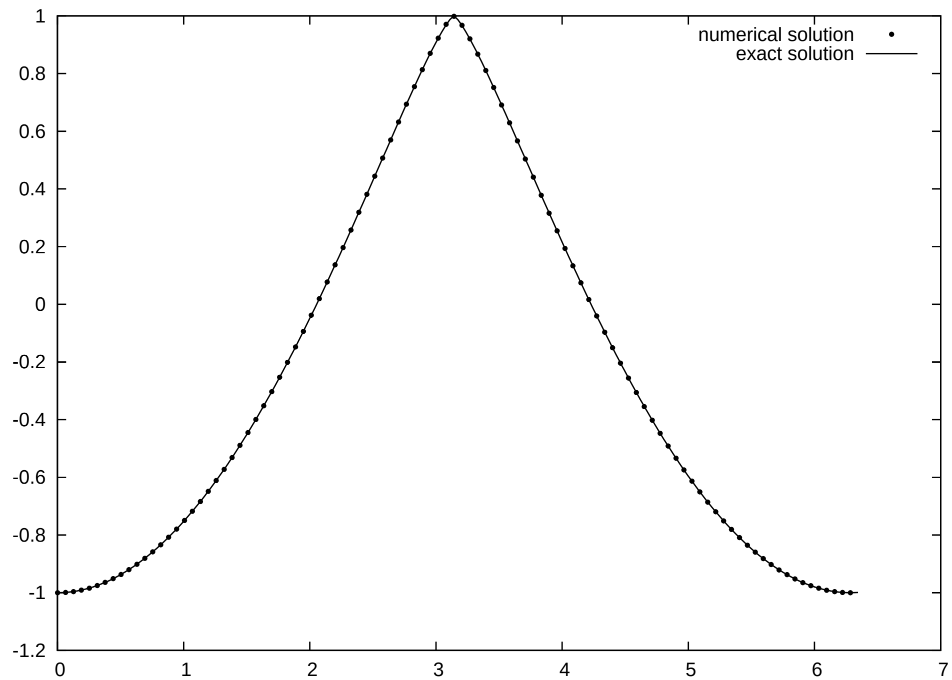
- The L^2 stability and the error estimate are obtained for linear Hamiltonians.

Example 1: $\begin{cases} \phi_t + \frac{\phi_x^2}{2} = 0, & 0 < x < 2\pi \\ \phi(x,0) = -\cos(x), & \phi(0,t) = \phi(2\pi,t) \end{cases}$

	P^1		P^2	
N	L^2 error	order	L^2 error	order
20	1.19e-02	-	5.97e-04	-
40	2.99e-03	2.00	7.60e-05	2.97
80	7.46e-04	2.00	9.54e-06	2.99
160	1.87e-04	2.00	1.20e-06	2.99
320	4.67e-05	2.00	1.50e-07	3.00
640	1.17e-05	2.00	1.87e-08	3.00

Convex Hamiltonian with smooth solution at $t = 0.5$

(Cont'd)



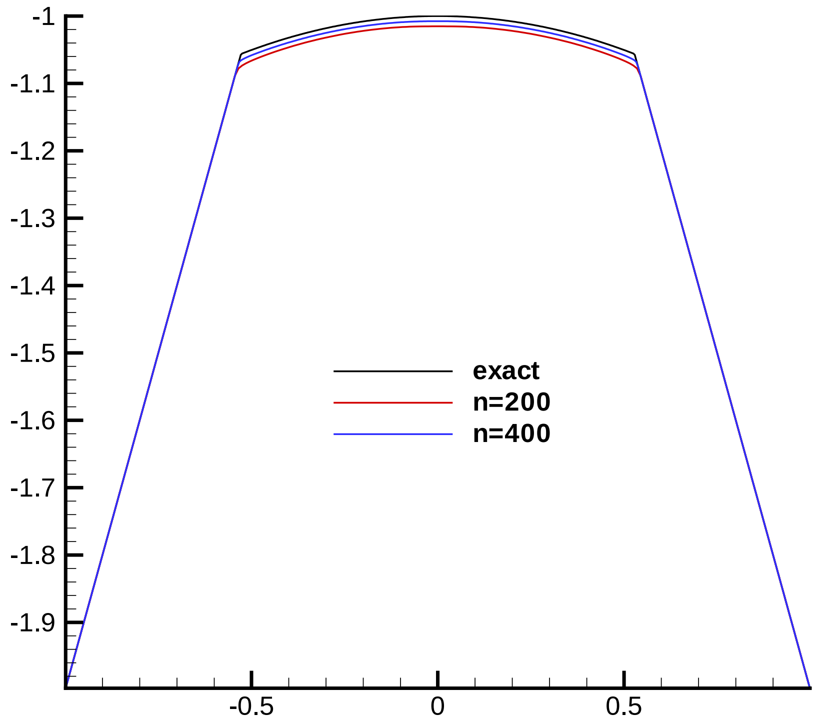
At $t = 1$ after the shock forms in ϕ_x

(Cont'd)

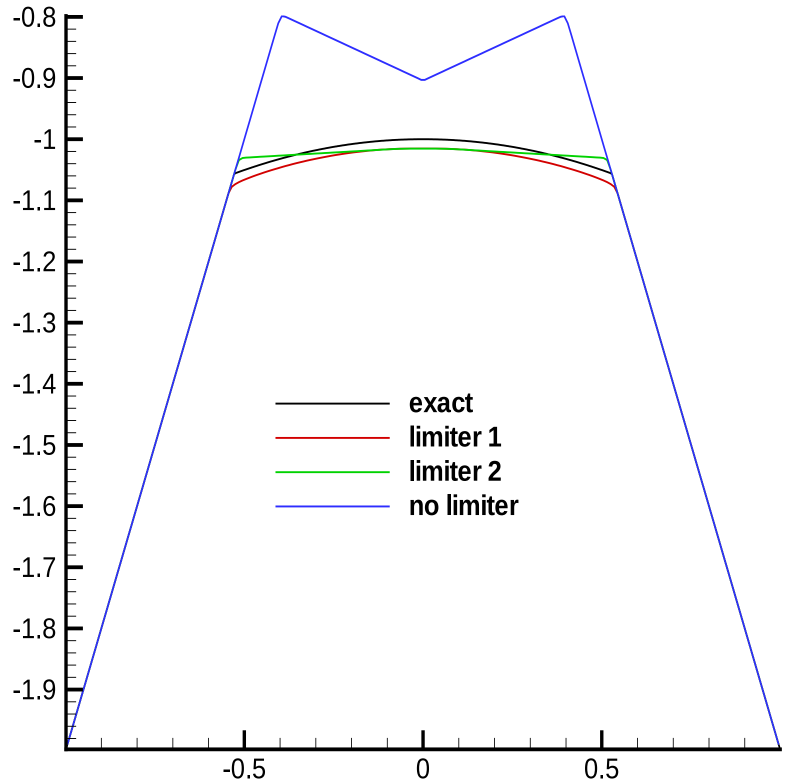
	P^1		P^2	
N	L^2 error	order	L^2 error	order
20	1.05e-02	-	1.69e-03	-
40	3.16e-03	1.73	4.80e-04	1.82
80	5.61e-04	2.49	2.80e-05	4.10
160	1.37e-04	2.04	1.55e-06	4.18
320	3.43e-05	2.00	1.82e-07	3.08
640	8.55e-06	2.00	2.28e-08	3.00

Errors and orders in the smooth region: $[0, 2\pi] \backslash [3.0, 3.28]$

Example 2: $\begin{cases} \phi_t + \frac{1}{4}(\phi_x^2 - 1)(\phi_x^2 - 4) = 0, & -1 < x < 1 \\ \phi(x,0) = -2|x|, & \textbf{outflow bc} \end{cases}$



Non-convex Hamiltonian at $t = 1$



The role of the nonlinear limiter to capture the viscosity solution, P^2 with $N = 100$. Limiter 1: TVB minmod limiter based on two copies of numerical solutions, Limiter 2: TVB minmod limiter

Other DG methods

*** Method of (Hu and Shu 1999, L. and Shu 2005)**: it is based on the relation of HJ equations and hyperbolic conservation laws

$$\mathbf{w}_t + \nabla_{\mathbf{x}} \cdot \mathbf{F}(\mathbf{w}, \mathbf{x}) = 0, \quad \text{with } \mathbf{w} = \nabla_{\mathbf{x}} \phi.$$

New ingredients:

(1) $\mathbf{w}$ is approximated by locally curl-free polynomial $\mathbf{w}_h$,

$$\mathbf{w}_h \in V_h^{k-1} = \{\mathbf{v} : \mathbf{v} \in [P^{k-1}(T)]^d, \nabla_{\mathbf{x}} \times \mathbf{v}|_T = 0, \ \forall\, T \in \mathcal{T}_h\}$$

Note that $\nabla_{\mathbf{x}} \times \mathbf{w} = \nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \phi = 0$.

(2) The missing constant in ϕ_h is recovered either

- by requiring for each j

$$\int_{I_j} \phi_t + H(\phi_x, \phi_y) dx = 0 \tag{3}$$

- or, by using (3) to update only one or a few elements, then after reaching the final time T of the computation, use

$$\phi(B,t) = \phi(A,t) + \int_A^B (\phi_x dx + \phi_y dy), \qquad t = T.$$

Note: The accuracy is optimal for $\nabla\phi$ and suboptimal for ϕ. Generally, it is necessary to consider the dependence of the components of $\mathbf{w}$ for the accuracy and for resolving steady state solutions.

*** Method of (Cheng and Shu 2006, Bokanowski, Cheng and Shu)**: look for $\phi_h \in V_h$,

$$\int_{I_j} (\phi_{h,t} + H(\phi_{h,x}, x))\, v dx =$$

$$-\underbrace{\frac{1}{2}\left(\max_{x\in I_{j-\frac{1}{2}}} H_1(\widetilde{\phi_{h,x}}, x) + |\max_{x\in I_{j-\frac{1}{2}}} H_1(\widetilde{\phi_{h,x}}, x)|\right)}_{\alpha_{l,j}(\phi_{h,x},x)}[\phi_h]_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}$$

$$-\underbrace{\frac{1}{2}\left(\min_{x\in I_{j+\frac{1}{2}}} H_1(\widetilde{\phi_{h,x}}, x) - |\min_{x\in I_{j+\frac{1}{2}}} H_1(\widetilde{\phi_{h,x}}, x)|\right)}_{\alpha_{r,j}(\phi_{h,x},x)}[\phi_h]_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}}$$

Here $H_1(\phi_x, x) = \frac{\partial H(\phi_x,x)}{\partial \phi_x}$, $\widetilde{\phi_{h,x}}$ is certain reconstruction of $\phi_{h,x}$ on $I_{j\pm\frac{1}{2}}$.

or written as

$$
\begin{aligned}
&\int_{I_j} (\phi_{h,t} + H(\phi_{h,x}, x))\, v dx \\
&= -\alpha_{l,j}(\phi_{h,x}, x)[\phi_h]_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} - \alpha_{r,j}(\phi_{h,x}, x)[\phi_h]_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}}
\end{aligned}
$$

with an entropy correction step.

Note:

- The entropy condition is violated at $x_{j+\frac{1}{2}}$ if

$$
(H_1(\phi_{h,x}))^-_{j+\frac{1}{2}} < 0, \qquad (H_1(\phi_{h,x}))^+_{j+\frac{1}{2}} > 0
$$

- When $H(\phi_x, x) = a\phi_x$, the scheme is the same as the upwind DG method.

Method of (Yan and Osher 2011)

$$\phi_t + H(\phi_x) = 0 \Leftrightarrow \begin{cases} \phi_t + \hat{H}(p,q) = 0 \\ p = \phi_x, \quad q = \phi_x \end{cases}$$

where $\hat{H}$ is a consistent monotone numerical Hamiltonian. The method is: look for $\phi_h, p_h, q_h \in V_h$, such that $\forall \psi, u, v \in V_h$,

$$\int_{I_j} \left(\phi_{h,t} + \hat{H}(p_h, q_h)\right) \psi dx = 0$$

$$\int_{I_j} p_h u dx + \int_{I_j} \phi_h u_x dx - \widehat{(\phi_h)}_{j+\frac{1}{2}} u^-_{j+\frac{1}{2}} + \widehat{(\phi_h)}_{j-\frac{1}{2}} u^+_{j-\frac{1}{2}} = 0,$$

$$\int_{I_j} q_h v dx + \int_{I_j} \phi_h v_x dx - \widetilde{(\phi_h)}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + \widetilde{(\phi_h)}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0$$

with $\widehat{w} = w^-$ and $\widetilde{w} = w^+$.

Note: The method is the standard monotone scheme for piecewise constant case.