An Introduction to Lift-And-Project Systems

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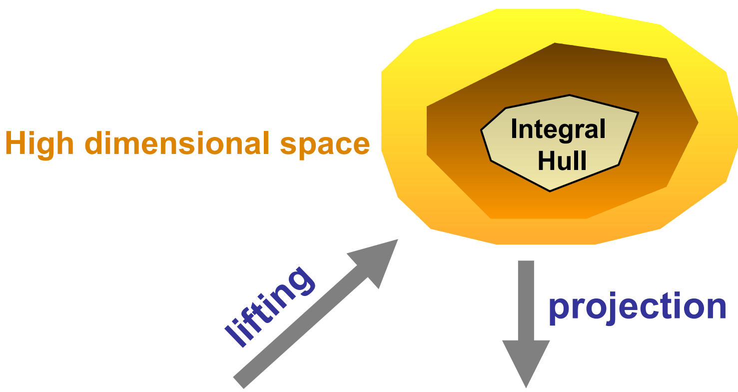
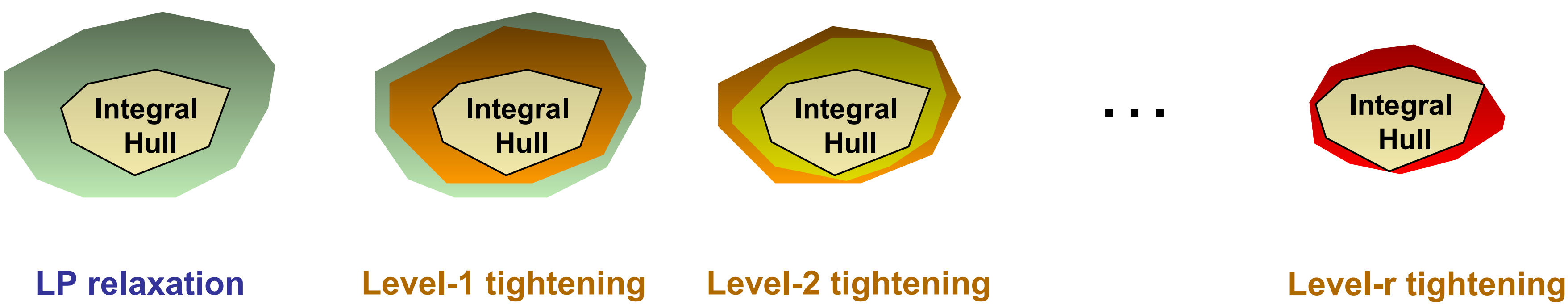
Lift-And-Project in a Nutshell

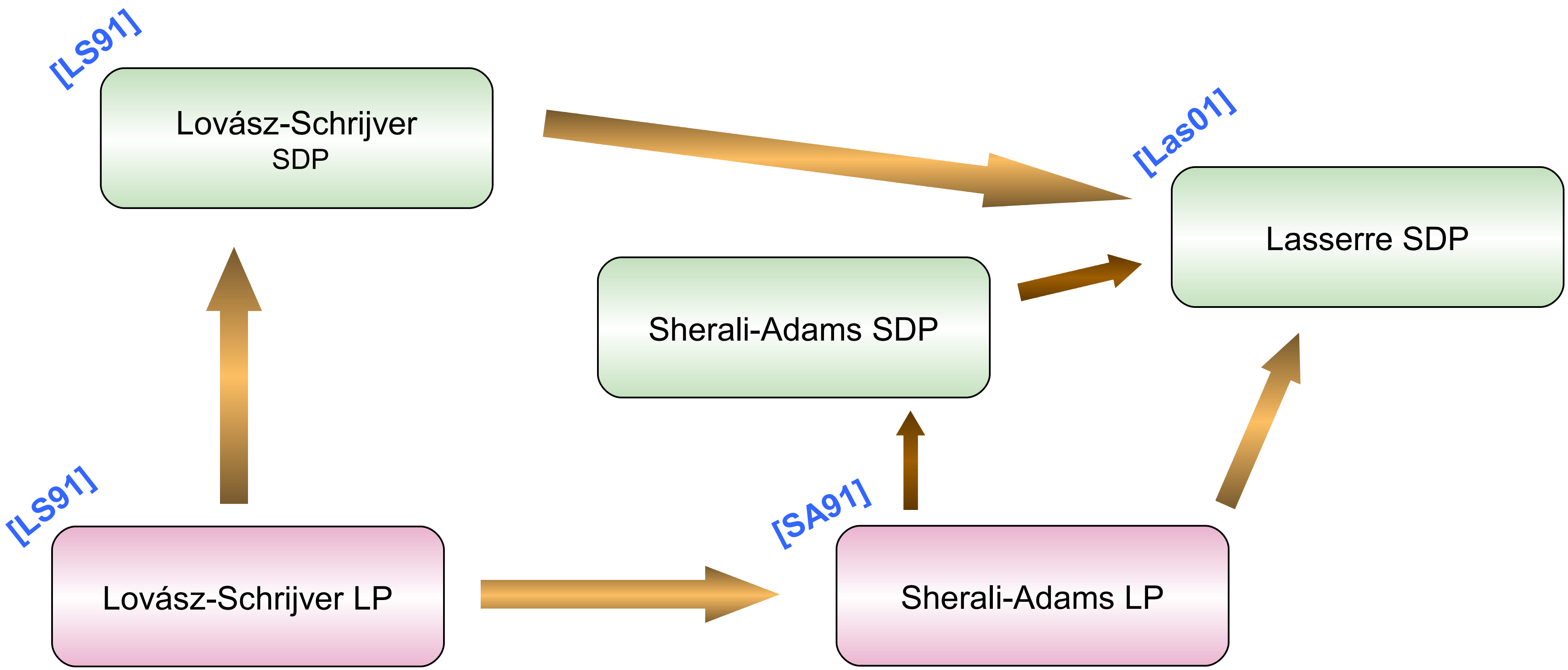


Objective: Attack a combinatorial problem starting with an "elementary" **0-1 LP relaxation**



High level: Derive systematically a **hierarchy** of tighter & tighter LP (SDP) relaxations.

The Realm of Lift-and-Project Systems



Algorithmic aspects

- Level-n tightening gives Integral hull.
- Level-r tightening optimizable in time $n^{O(r)}$.

The Lovász-Schrijver (LS) LP system

<u>Our toy example</u>

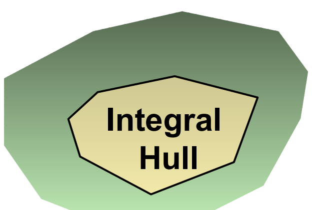
Stable Set Relaxation

on input G=(V,E)

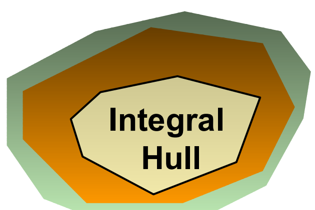
$$\min \sum_{i \in V} x_i$$

$$x_i + x_j \le 1, \forall ij \in E$$

$$x_i \in [0,1]$$



LP relaxation



Level-1 tightening

Level-1 derivation rule:

$$(x_i + x_j)x_k \le x_k$$

$$(x_i + x_j)(1 - x_k) \le 1 - x_k$$

$$x_k \le x_k^2$$

Any conical combination of the above

New linear constraints

Definition: **Level-1 LS tightening**

Original relaxation

+ new linear constraints

The LS System in Action

<u>Our toy example</u>
Stable Set Relaxation
on input G=(V,E)

$$\min \sum_{i \in V} x_i$$

$$x_i + x_j \le 1, \forall ij \in E$$

$$x_i \in [0,1]$$

$x_1 + x_2 \le 1$	$\cdot\, x_1$
$x_2 + x_3 \le 1$	$\cdot\, (1 - x_1)$
$x_3 + x_4 \le 1$	$\cdot\, x_1$
$x_4 + x_5 \le 1$	$\cdot\, (1 - x_1)$
$x_1 + x_5 \le 1$	$\cdot\, x_1$

$\boxed{2}$

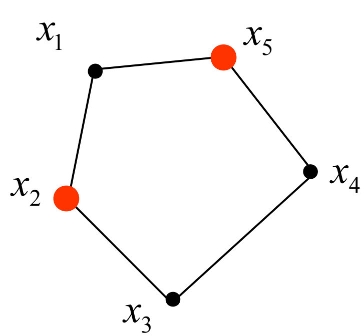
$x_1 \le x_1^2$

\+ derivation rule

$$x_1^2 + \cancel{x_1 x_2} \le \cancel{x_1}$$
$$x_2 + x_3 - \cancel{x_1 x_2} - \cancel{x_1 x_3} \le 1 - \cancel{x_1}$$
$$\cancel{x_1 x_3} + \cancel{x_1 x_4} \le \cancel{x_1}$$
$$x_4 + x_5 - \cancel{x_1 x_4} - \cancel{x_1 x_5} \le 1 - \cancel{x_1}$$
$$x_1^2 + \cancel{x_1 x_5} \le x_1$$

$$x_2 + x_3 + x_4 + x_5 + 2x_1^2 \le 2 + x_1$$

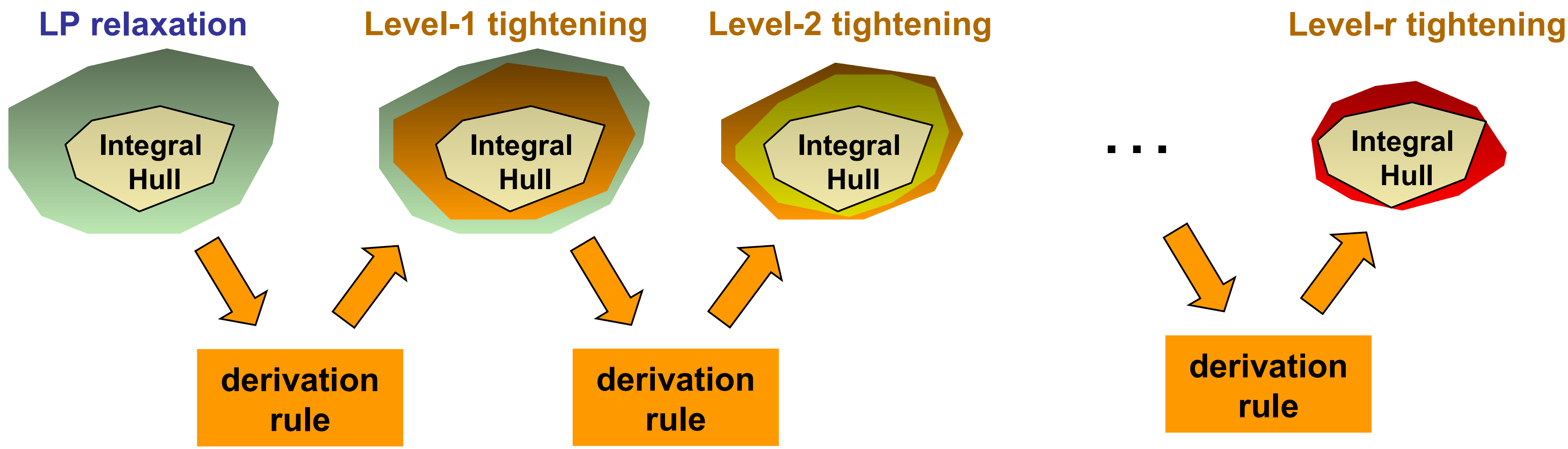
$$2x_1 \le 2x_1^2$$



The odd cycle constraint

$$x_1 + x_2 + x_3 + x_4 + x_5 \le 2$$

The LS System – Subsequent Tightenings



Level-1 derivation rule:

$$(x_i + x_j)x_k \le x_k$$

$$(x_i + x_j)(1 - x_k) \le 1 - x_k$$

$$x_k \le x_k^2$$

Any conical combination of the above

New linear constraints

Definition: The level-r tightening is the relaxation we obtain by applying the derivation rule on the level-(r-1) relaxation.

The LS System (Derivation Rule Made Formal)

Our toy example

Stable Set **Cone K**

on input G=(V,E)

$$x_i + x_j \le x_0, \forall ij \in E$$
$$x_i \in [0, x_0]$$
$$x = (x_0, x_1, \dots, x_n) \in \Re^{\binom{V}{1}}$$

Level-1 derivation rule:

$$(x_i + x_j)x_k \le x_k$$
$$(x_i + x_j)(1 - x_k) \le 1 - x_k$$
$$x_k \le x_k^2$$

Any conical combination of the above

New linear constraints

Constraints on the lifted space

Introduce $y \in \Re^{1+\binom{V}{1}+\binom{V}{2}}$ with $y_{\{i,k\}}$ simulating $x_i x_k$

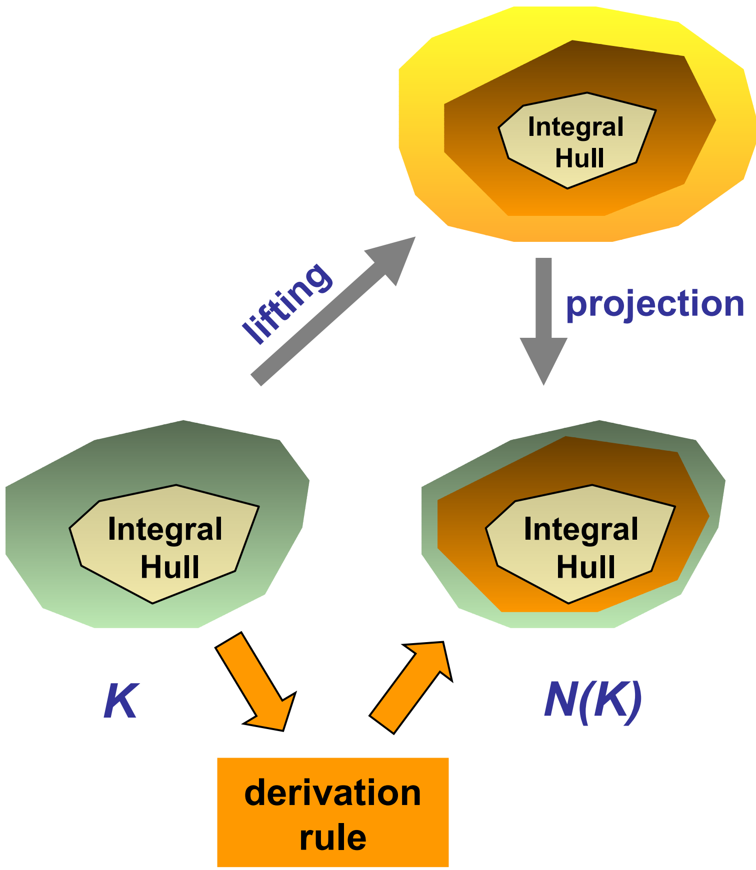
$$M^{(y)} = \begin{array}{cccccc} \varnothing & \{1\} & \{2\} & \cdots & \{n\} & \\ \left(\begin{array}{c} 1 \\ y_{\{1\}} \\ y_{\{2\}} \\ \vdots \\ y_{\{n\}} \end{array}\right. & \begin{array}{c} y_{\{1\}} \\ y_{\{1\}} \\ y_{\{1,2\}} \\ \vdots \\ y_{\{1,n\}} \end{array} & \begin{array}{c} y_{\{2\}} \\ y_{\{1,2\}} \\ y_{\{2\}} \\ \vdots \\ y_{\{2,n\}} \end{array} & \begin{array}{c} \cdots \\ \cdots \\ \cdots \\ \ddots \\ \cdots \end{array} & \left.\begin{array}{c} y_{\{n\}} \\ y_{\{1,n\}} \\ y_{\{2,n\}} \\ \vdots \\ y_{\{n\}} \end{array}\right) & \begin{array}{c} \varnothing \\ \{1\} \\ \{2\} \\ \vdots \\ \{n\} \end{array} \end{array}$$

$$x = Me_0 = \operatorname{diag}(M) \in K$$

$$(x_i + x_j)x_2 \le x_0 x_2 \approx y_{\{i,2\}} + y_{\{j,2\}} \le y_{\{2\}} \Leftrightarrow Me_2 \in K$$

$$(x_i + x_j)(x_0 - x_2) \le x_0(x_0 - x_2) \approx M(e_0 - e_2) \in K$$

A Remarkable Yet Simple Implication



$$M_{ij}^{(y)} := y_{\{i,j\}}$$

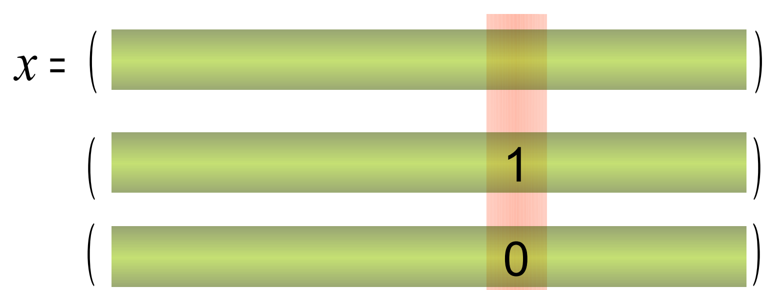
$$Me_0 = \mathrm{diag}(M) \in K$$
$$Me_k \in K$$
$$M(e_0 - e_k) \in K$$

$$x \in N(K) \Leftrightarrow \exists\, y \in \Re^{1+\binom{n}{1}+\binom{n}{2}} : M^{(y)}$$

satisfying the above

Remarkable Implication:

If $x \in N(K)$ then for every index ***t***,
x can be written as convex combination
of vectors in ***K*** that are integral on ***t***



Towards Proof of Convergence

Remarkable Implication:

If $x \in N(K)$ then for every index ***t***, *x* can be written as convex combination of vectors in the cone ***K***, integral on ***t***

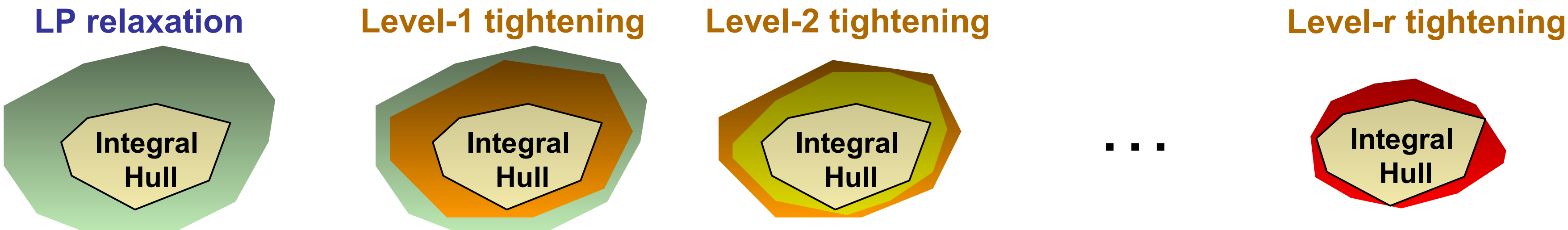
$$M^{(y)} = \begin{array}{c|ccccc|c} & \emptyset & \{1\} & \{2\} & \cdots & \{n\} & \\ \hline & 1 & y_{\{1\}} & y_{\{2\}} & \cdots & y_{\{n\}} & \emptyset \\ & y_{\{1\}} & y_{\{1\}} & y_{\{1,2\}} & \cdots & y_{\{1,n\}} & \{1\} \\ & y_{\{2\}} & y_{\{1,2\}} & y_{\{2\}} & \cdots & y_{\{2,n\}} & \{2\} \\ & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & y_{\{n\}} & y_{\{1,n\}} & y_{\{2,n\}} & \cdots & y_{\{n\}} & \{n\} \end{array}$$

Proof: (t={2})

$$x = \begin{pmatrix} 1 \\ y_{\{1\}} \\ y_{\{2\}} \\ \vdots \\ y_{\{k\}} \\ \vdots \\ y_{\{n\}} \end{pmatrix} = \begin{pmatrix} y_{\{2\}} \\ y_{\{1,2\}} \\ y_{\{2\}} \\ \vdots \\ y_{\{k,2\}} \\ \vdots \\ y_{\{n,2\}} \end{pmatrix} + \begin{pmatrix} 1 - y_{\{2\}} \\ y_{\{1\}} - y_{\{1,2\}} \\ 0 \\ \vdots \\ y_{\{2\}} - y_{\{k,2\}} \\ \vdots \\ y_{\{n\}} - y_{\{n,2\}} \end{pmatrix}$$

$$\begin{aligned} x &= Me_2 + M(e_0 - e_2) \\ &= y_{\{2\}}\left(\tfrac{1}{y_{\{2\}}} Me_2\right) + (1 - y_{\{2\}})\left(\tfrac{1}{1 - y_{\{2\}}} M(e_0 - e_2)\right) \end{aligned}$$

And … the Proof of Convergence



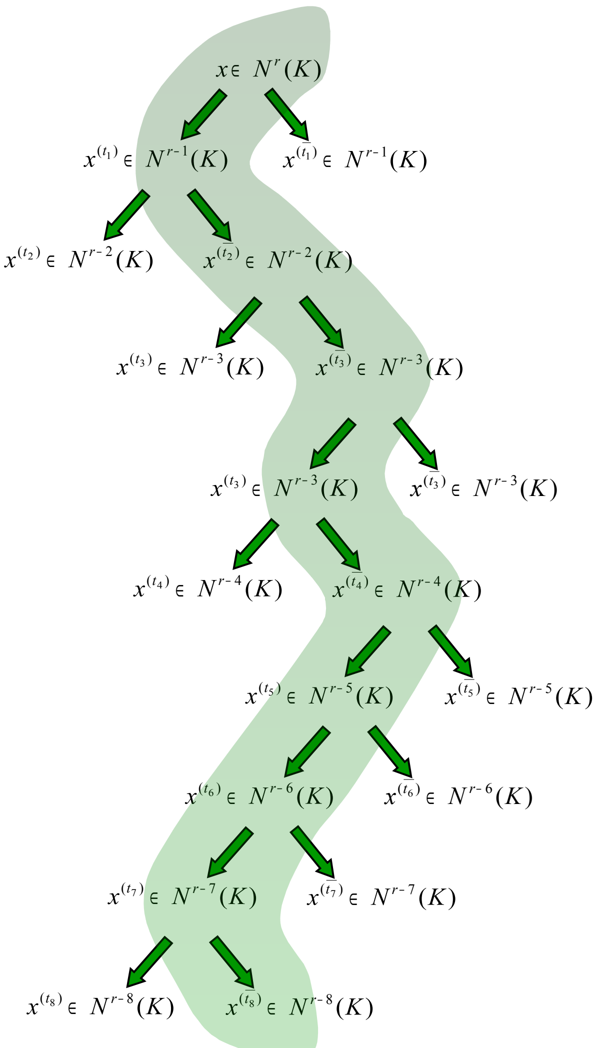
DOUBLE Remarkable Implication: $x \in N^r(K)$

$$x^{(t_1)} \in N^{r-1}(K) \qquad x^{(\bar{t}_1)} \in N^{r-1}(K)$$

$$x^{(t_2)} \in N^{r-2}(K) \qquad x^{(\bar{t}_2)} \in N^{r-2}(K)$$

$$x^{(t_3)} \in N^{r-3}(K) \qquad x^{(\bar{t}_3)} \in N^{r-3}(K)$$

And … the Proof of Convergence (more formally)



Claim: Let $x \in N^r(K)$.

Then, for every **SEQUENCE** $\left(t_1, \overline{t_2}, \dots, t_r\right)$ x can be written as convex combination of vectors in K that are integral in $\left\{t_1, t_2, \dots, t_r\right\}$.

Corollary: The level-n relaxation gives the integral hull

Corollary: The level-r relaxation satisfies all constraints of support at most r.

Utilizing/Fooling the LS System

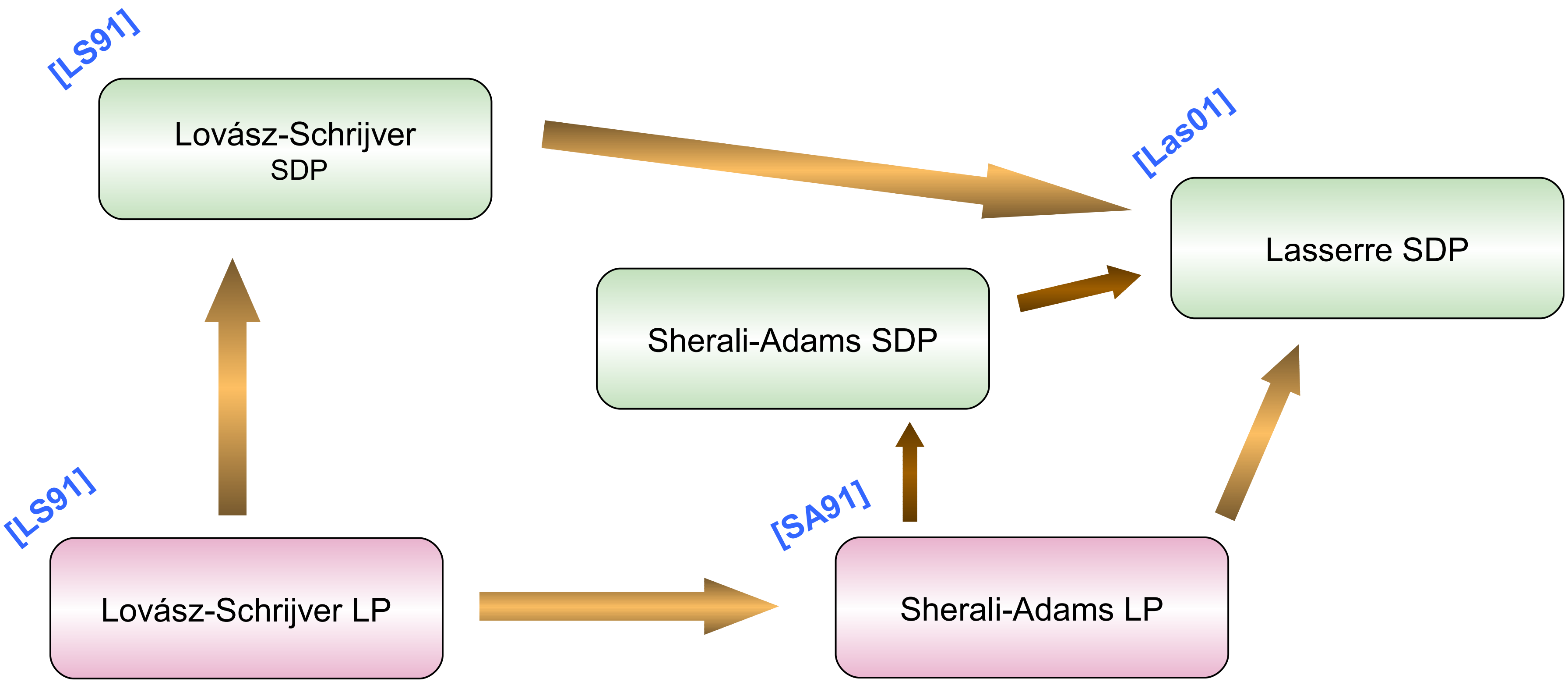
Combinatorial Problem	Approximation	Level
Densest-k-Subgraph [BCCFV10]	$n^{\frac{1}{4}+\varepsilon}$	$\frac{1}{\varepsilon}$

& who knows what else …

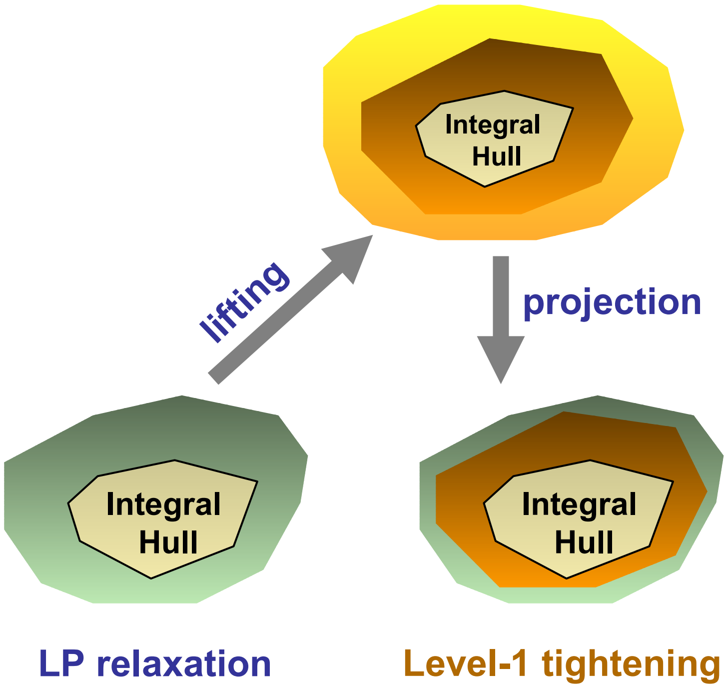
Combinatorial Problem	Integrality gap	Level
Vertex Cover [STT07]	$2-\varepsilon$	$\Theta(n)$
Max-Cut [STT07]	$2-\varepsilon$	$\Theta(n)$

& many rank lower bounds

The Realm of Lift-and-Project Systems



The Lovász-Schrijver (LS+) SDP system



$$M_{ij}^{(y)} = y_{\{i,j\}} = x_i x_j$$

$$M^{(y)} = \begin{pmatrix} y_{\emptyset} & y_{\{1\}} & y_{\{2\}} & \cdots & y_{\{n\}} \\ y_{\{1\}} & y_{\{1\}} & y_{\{1,2\}} & \cdots & y_{\{1,n\}} \\ y_{\{2\}} & y_{\{1,2\}} & y_{\{2\}} & \cdots & y_{\{2,n\}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{\{n\}} & y_{\{1,n\}} & y_{\{2,n\}} & \cdots & y_{\{n\}} \end{pmatrix}$$

is ~~rank 1~~, positive semidefinite (PSD)

<u>Algorithmic aspects of **LS+**</u>

- Convergence
- Level-t utilizable in time $n^{O(t)}$
- Constant level tightenings derive celebrated relaxations, e.g. **[GW95]**, **[ARV04]**

Utilizing/Fooling the LS+ System

Combinatorial Problem	Approximation	Level
Max-Cut [GW95]	1.139	1
Sparsest Cut [ARV04]	$O\left(\sqrt{\log n}\right)$	3

& who knows what else …

Combinatorial Problem	Integrality gap	Level
Vertex Cover [STT07]	$\frac{7}{6}$	$\Theta(n)$
Vertex Cover [GMPT07]	$2-\varepsilon$	$\Omega\left(\sqrt{\log n / \log\log n}\right)$
Max-k-XOR [BOGHMT06]	$2-\varepsilon$	$\Theta(n)$
Hypegraph Vertex Cover [AAT05]	$k-1-\varepsilon$	$\Theta(n)$
Hypegraph Vertex Cover [Tou05]	$k-\varepsilon$	$\Omega(\log\log n)$
Set Cover [AAT05]	$(1-\varepsilon)\ln n$	$\Theta(n)$
Independent Set (rand instances) [FK03]	$\Theta\left(\sqrt{n} / 2^{r/2}\log n\right)$	r

Deriving the GW SDP for Max-Cut

LP relaxation

$$\max \sum_{ij \in E} d_{ij}$$

K

$d_{ij} \geq$ … $, \forall\, i, j \in V$

$d_{ij} \leq x_i +$ … $i, j \in V$

$d_{ij} \leq 2 - (x_i + x_j), \forall\, i, j \in V$

$x_i \in [0,1], \forall\, i \in V$

IG = 2

SDP relaxation

m… $\frac{1}{\,}\left\|v_i - v_j\right\|^2$

$\left\|v_i\right\|^2 =$ …

IG ≤ 1.139

Claim: New constraint of ***N+(K)***

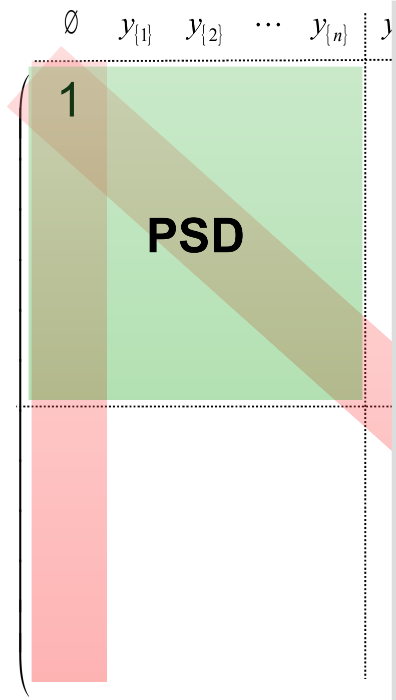
$$d_{ij} = y_{\{i\}} + y_{\{j\}} - 2y_{\{i,j\}}$$
$$= u_i^2 + u_j^2 - 2u_i \cdot u_j$$
$$= \left\|u_i - u_j\right\|^2$$

Level-1 derivation rule:

$$d_{ij}x_i \geq (x_i - x_j)x_i$$
$$d_{ij}(1 - x_i) \geq (x_i - x_j)(1 - x_i)$$
$$+ \quad x_i^2 \geq x_i$$

$$d_{ij} \geq x_i^2 + x_j^2 - 2x_i x_j$$

$M^{(y)} =$

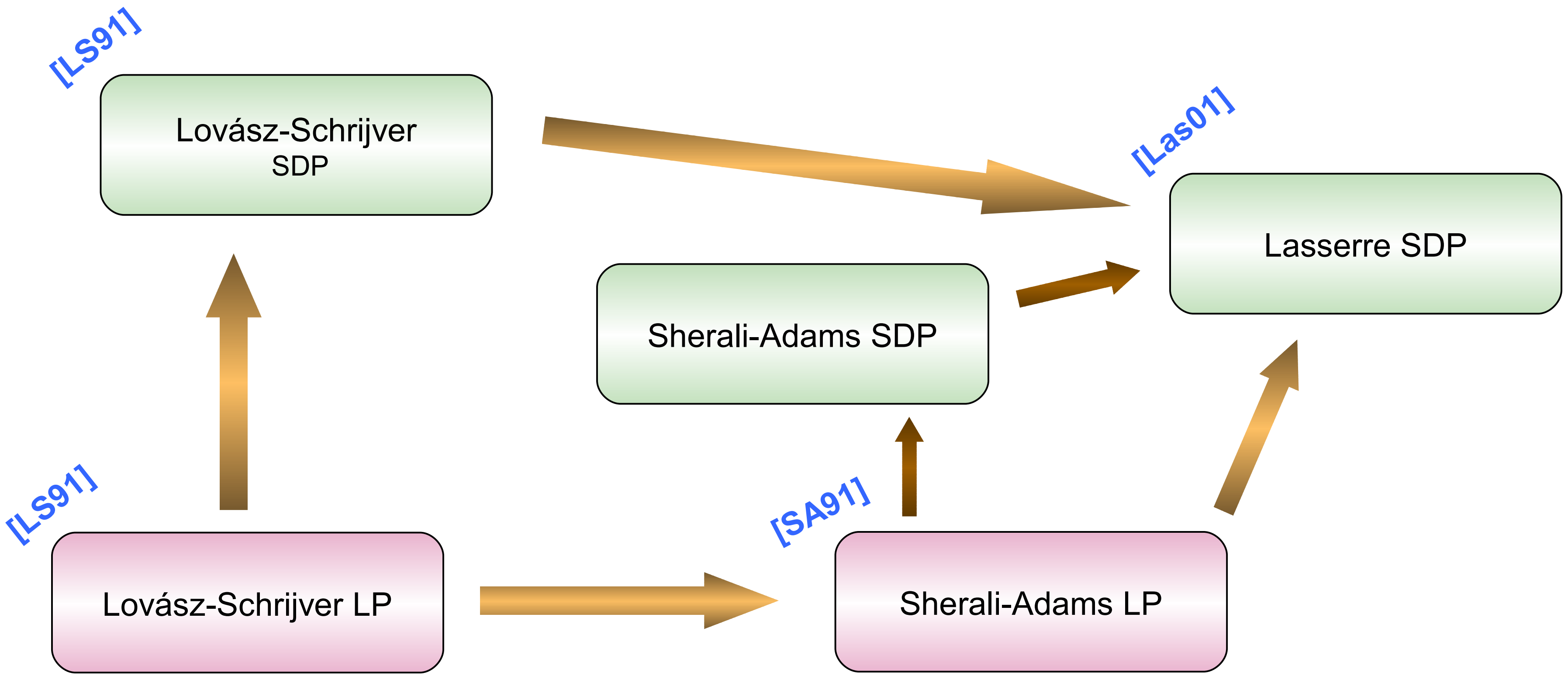


Finish up the proof:

$$v_i := u_0 - 2u_i$$
$$\Downarrow$$
$$d_{ij} = \frac{1}{4}\left\|v_i - v_j\right\|^2$$

$$v_i^2 = (u_0 - 2u_i)^2$$
$$= u_0^2 + 4u_i^2 - 4u_0 \cdot u_i$$
$$= 1 + 4M_{ii} - 4M_{0i}$$
$$= 1$$

The Realm of Lift-and-Project Systems



The Sherali-Adams (SA) LP system

<u>Our toy example</u>

Stable Set Relaxation

on input G=(V,E)

$$\min \sum_{i \in V} x_i$$

$$x_i + x_j \le 1, \forall ij \in E$$

$$x_i \in [0,1]$$

Level-r derivation rule: $\forall A \subseteq [n], |A| \le r$

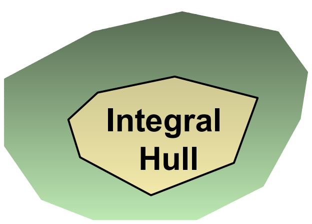
$$(x_i + x_j) \prod_{k \in Y} x_k \prod_{k \in N} (1 - x_k) \le \prod_{k \in Y} x_k \prod_{k \in N} (1 - x_k)$$

A Y/N

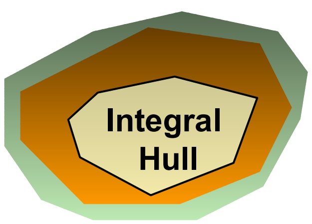
$$x_k \le x_k^2 \le \cdots \le x_k^r$$

Any conical combination of the above

New linear constraints



LP relaxation



Level-r tightening

Definition: **Level-r SA tightening**

Original relaxation

+ new linear constraints

The SA System (Derivation Rule Made Formal)

Our toy example

Stable Set **Cone K**

on input G=(V,E)

$$x_i + x_j \le x_0, \forall ij \in E$$

$$x_i \in [0, x_0]$$

$$x = (x_0, x_1, \ldots, x_n) \in \Re^{\binom{V}{1}}$$

Level-r derivation rule:

$$\forall I \subseteq [n], |I| \le r$$

$$(x_i + x_j)\prod_{k \in A} x_k \prod_{k \in I \setminus A} (1 - x_k) \le \prod_{k \in I} x_k \prod_{k \in I \setminus A} (1 - x_k)$$

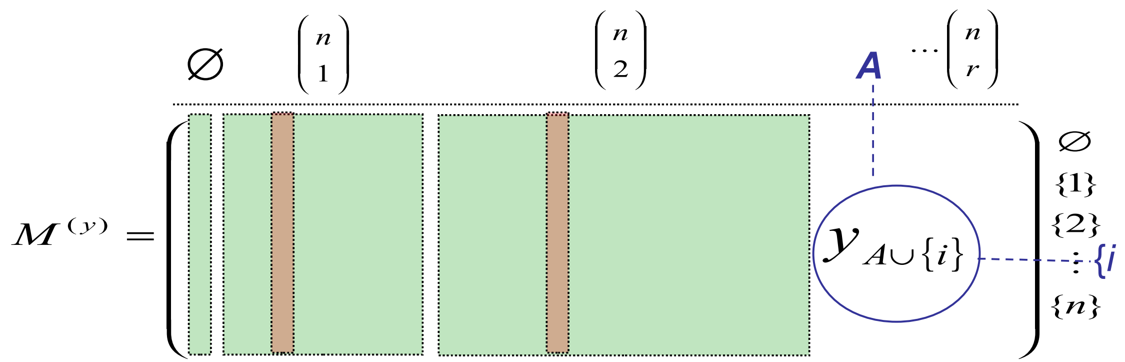
$$x_k^{\,t} \le x_k^r, \forall t \le r$$

Any conical combination of the above

New linear constraints

Constraints on the lifted space

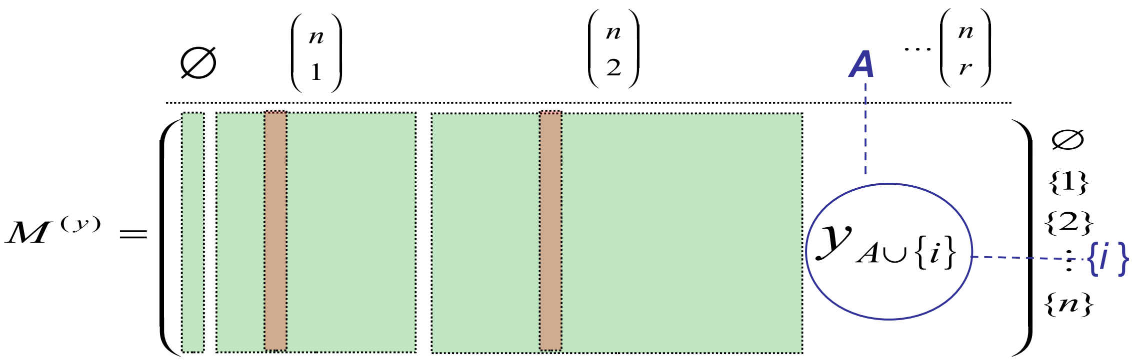
Introduce $y \in \Re^{\sum_{t \le r+1}\binom{n}{t}}$ with y_A simulating $\prod_{i \in A} x_i$



Example: $A = \{2,3\}$, $x_2(1 - x_3)$

$$M^{(y)}\left(e_{\{2\}} - e_{\{2,3\}}\right) \in K$$

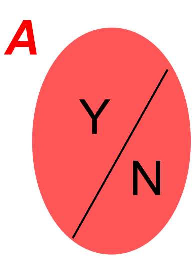
SA & Distributions of 0-1 Assignments



Claim: Level-**r** SA relaxation associates every subset ***A***, of size at most **r**, with a distribution of 0-1 assignments ***D(A)*** such that

- **Consistency:** ***D(A)*** and ***D(B)*** agree on the marginals.
- **Feasibility:** Assignments in ***D(A)*** are locally feasible.

Proof: What is ***D(A)*** ?



$$a \in \{0,1\}^A$$

$$N := \{i : a(i) = 0\}$$

$$Y := \{i : a(i) = 1\}$$

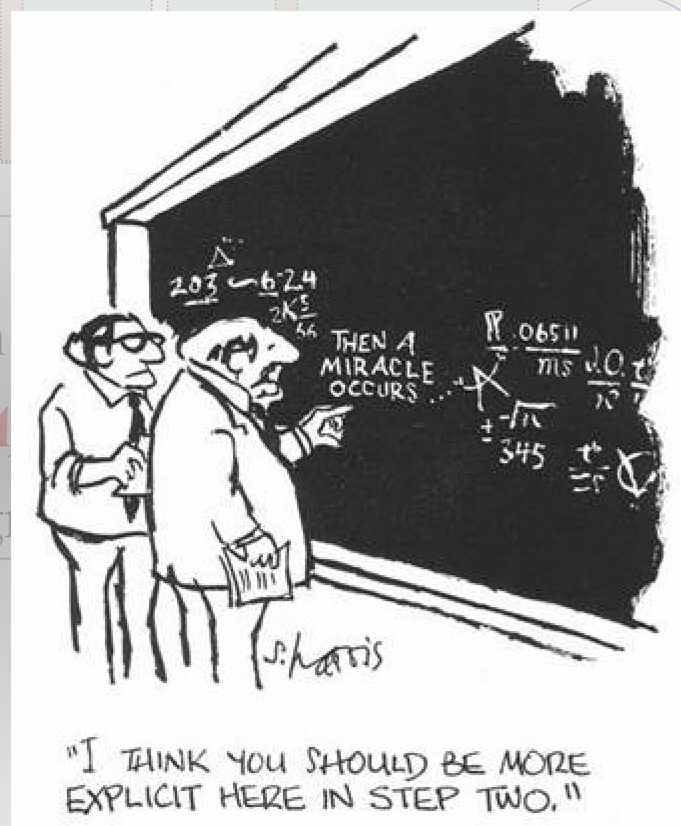
$$\mathbf{Pr}_{D(A)}[a] := \sum_{R \subseteq N} (-1)^{|R|} y_{R \cup Y}$$

SA & Distributions of 0-1 Assignments

$M^{(y)} =$

Claim: Level-**r** SA … ubset ***A***, of size at most **r**, with … ents ***D(A)*** such that

- **Consistency:** ***D(A*** … inals.
- **Feasibility:** Assig … asible.



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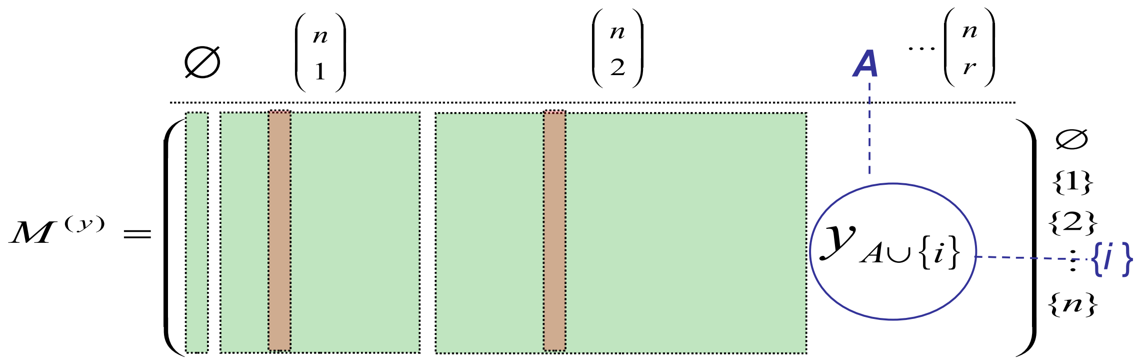
$\ldots -1)^{|R|} y_{R \cup Y}$

SA & Distributions of 0-1 Assignments



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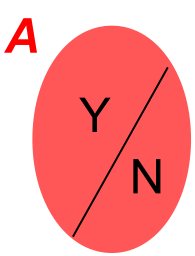


$$a \in \{0,1\}^A$$

$$N := \{i : a(i) = 0\}$$

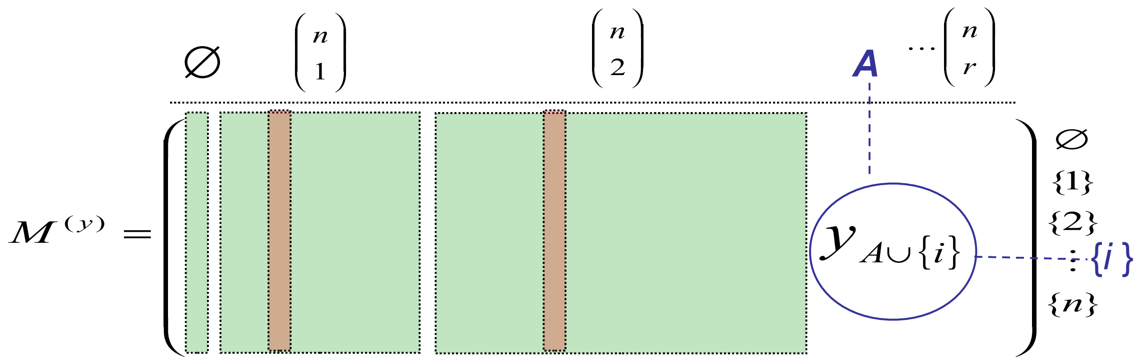
$$Y := \{i : a(i) = 1\}$$

$$\Pr_{D(A)}[a] := \sum_{R\subseteq N} (-1)^{|R|}\, y_{R\cup Y} \qquad z_{Y,N}$$

$$y_A = \prod_{i\in A} x_i$$

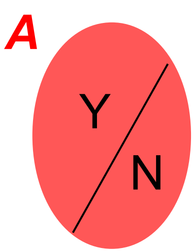
$$P(Y,N) := \prod_{i\in Y} x_i \prod_{i\in N} (1 - x_i)$$

SA & Distributions of 0-1 Assignments



Proof: What is ***D(A)*** ?

$$\Pr_{D(A)}[a] = \sum_{R\subseteq N} (-1)^{|R|} y_{R\cup Y} := z_{Y,N}$$



$$a \in \{0,1\}^A$$

$$N := \{i : a(i) = 0\}$$

$$Y := \{i : a(i) = 1\}$$

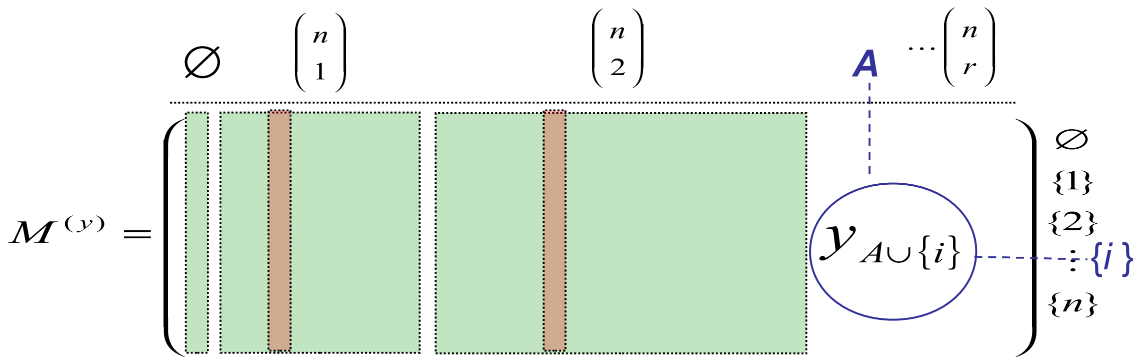
$$P(Y,N) := \prod_{i\in Y} x_i \prod_{i\in N} (1 - x_i)$$

Is really ***D(A)*** a distribution? $z_{Y,N} \geq 0$ **?**

$x \geq 0$ valid for ***K***

$$P(Y,N) \mapsto \sum_{R\subseteq N} (-1)^{|R|} M^{(y)} e_{R\cup Y} \in K$$

SA & Distributions of 0-1 Assignments



Proof: What is ***D(A)*** ?

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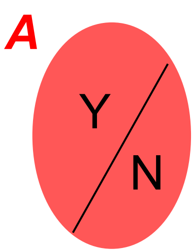

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$$Y := \{i : a(i) = 1\}$$

$$P(Y,N) := \prod_{i\in Y} x_i \prod_{i\in N} (1 - x_i)$$

Is really ***D(A)*** a distribution? $\displaystyle\sum_{Y,N\subseteq A} z_{Y,N} = 1$ **?**

$P(Y,N) \mapsto z_{Y,N}$ is a homomorphism because

$$P(Y,N) = P(Y,N)\cdot x_i + P(Y,N)\cdot(1 - x_i) \qquad P(\varnothing,\varnothing) = 1$$

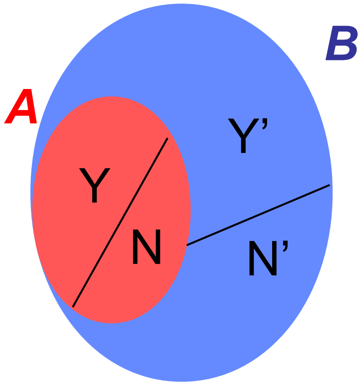
$$z_{Y,N} = z_{Y\cup\{i\},N} + z_{Y,N\cup\{i\}} \qquad z_{\varnothing,\varnothing} = 1$$

SA & Distributions of 0-1 Assignments

Claim: Level-**r** SA relaxation associates every subset ***A***, of size at most **r**, with a distribution of 0-1 assignments ***D(A)*** such that

- **Consistency:** ***D(A)*** and ***D(B)*** agree on the marginals.
- **Feasibility:** Assignments in ***D(A)*** are locally feasible.

Local Consistency & Feasibility



$$a \in \{0,1\}^A$$

$$N := \{i : a(i) = 0\}$$

$$Y := \{i : a(i) = 1\}$$

$$\Pr_{D(A)}[a] = \sum_{R \subseteq N} (-1)^{|R|} y_{R \cup Y} := z_{Y,N}$$

Local Consistency: $\forall B \supseteq A, |B| \leq r \quad \Pr_{D(B)}[a] = \Pr_{D(A)}[a]$ **?**

$$\Pr_{D(B)}[a] = \sum_{Y',N' \subseteq B \setminus A} z_{Y \cup Y', N \cup N'} = z_{Y,N}$$

Due to the previous homomorphism

Feasibility:

> **Claim:** Let $x \in S^r(K)$
> Then, for every <u>**SET**</u> $A = \{t_1, t_2, \ldots, t_r\}$
> *x* can be written as convex combination
> of vectors in *K* that are integral in $\{t_1, t_2, \ldots, t_r\}$

D(A) defines the convex combination

Solving Problems of Bounded Treewidth

<u>Our toy example</u>

Stable Set Relaxation

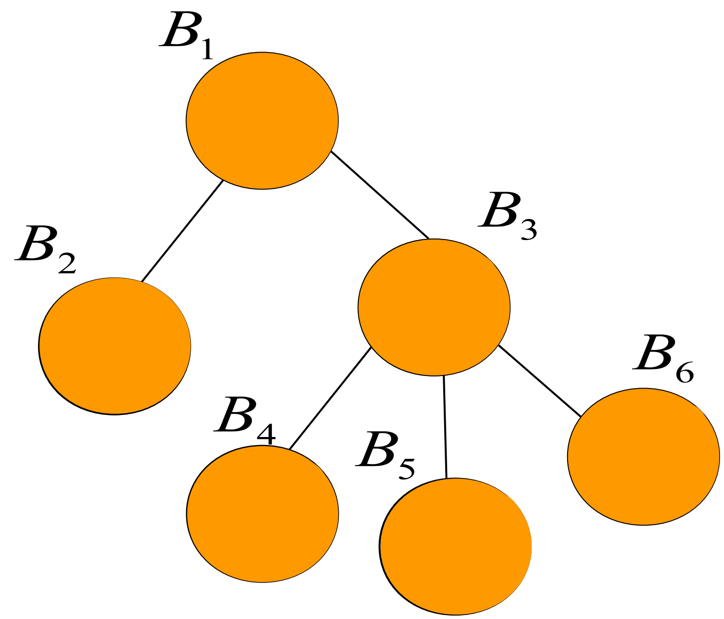
on input G=(V,E)

$$\min \sum_{i \in V} x_i$$

$$x_i + x_j \le 1, \forall\, ij \in E$$

$$x_i \in [0,1]$$

Level-r SA tihgtening



G=(V,E) of **treewidth r**

- Union of bags (of **size ≤r**) is V
- vertices of every edge, in at least one bag
- Bags containing any vertex form one connected component.

Claim: **Level-r** SA tightening solves problem exactly

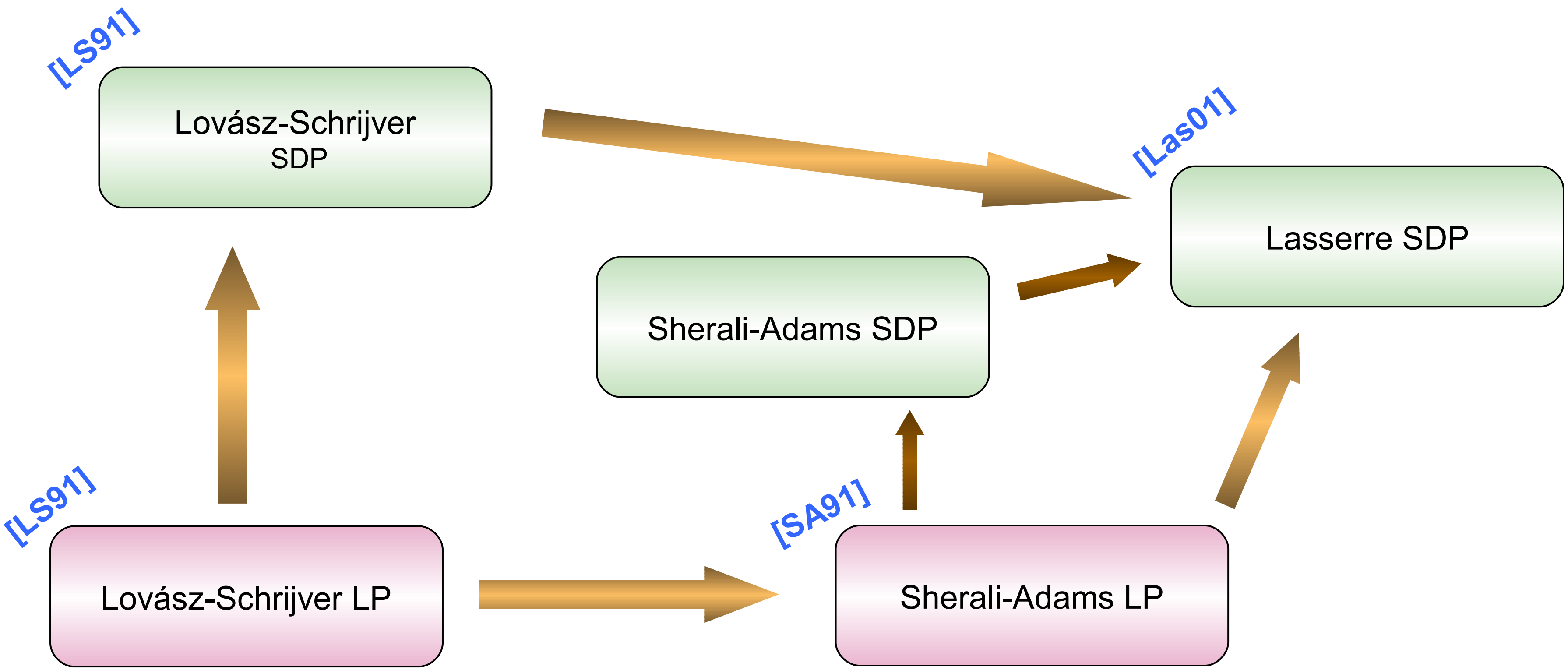
Theorem ([WJ04]): **Level-r SA** LP solves exactly any polytope of **treewidth r**.

Utilizing/Fooling the SA LP System

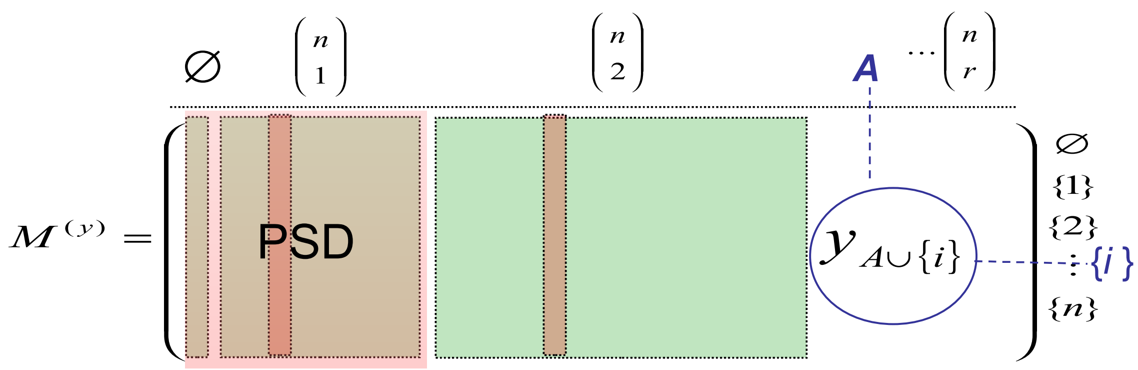
Combinatorial Problem	Approximation	Level
Max-Cut (dense graphs) [FdlVM07]	$1+\varepsilon$	$O\left(\log\frac{1}{\varepsilon}\Big/\varepsilon^{4}\right)$
Vertex Cover (planar graphs) [MM09]	$1+\varepsilon$	$O\left(\frac{1}{\varepsilon}\right)$
Independent Set (planar graphs) [MM09]	$1+\varepsilon$	$O\left(\frac{1}{\varepsilon}\right)$
Max Min Allocation [BCG09]	n^{ε}	$O\left(\frac{1}{\varepsilon}\right)$
Sparsest Cut (treewidth r) [CKR10]	$2^{2^{r}}$	$O(r)$

Combinatorial Problem	Integrality gap	Level
Vertex Cover [CMM09]	$2-\varepsilon$	$\Omega\left(n^{\delta}\right)$
Max Cut [CMM09]	$2-\varepsilon$	$\Omega\left(n^{\delta}\right)$
Unique Games [CMM09]	$1-\varepsilon$ vs γ	$\Omega\left(n^{\delta}\right)$
Max Acyclic Subgraph [CMM09]	$2-\varepsilon$	$\Omega\left(n^{\delta}\right)$
Sparsest Cut [CMM09]	$\Omega\left(\sqrt{\log n \big/ (\log r + \log\log n)}\right)$	r
Knapsack [KMN11]	$2-\varepsilon$	$\Theta(n)$

The Realm of lift-and-project Systems



The Notorious Sherali-Adams SDP (SA+) System



$$\sum_{R \subseteq N} (-1)^{|R|} M^{(y)} e_{R \cup Y} \in K$$

$M_{ij}^{(y)} = y_{\{i,j\}} = x_i x_j$ is ~~rank 1~~, positive semidefinite (PSD)

The level-r **SA+** SDP relaxation:

- Start with your favorite 0-1 relaxation.
- Impose level-r SA linear constraints.
- Require low level PSDness.

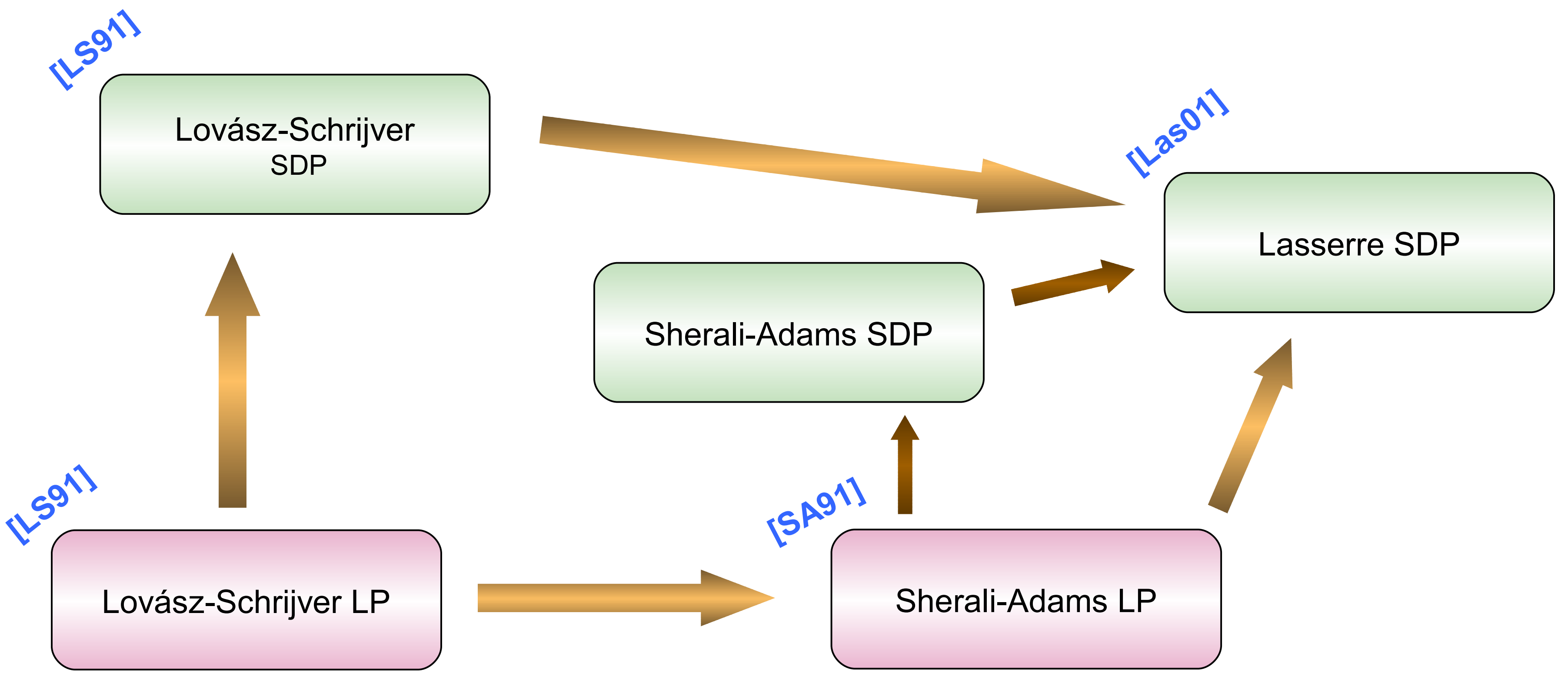
Utilizing/Fooling the SA+ System

Combinatorial Problem	Approximation	Level
Max CSPs [Rag08]	OPT	$O(1)$
Max Cut (random dense graphs) [AU03]	$1+\Theta\left(\frac{1}{r}\right)$	r

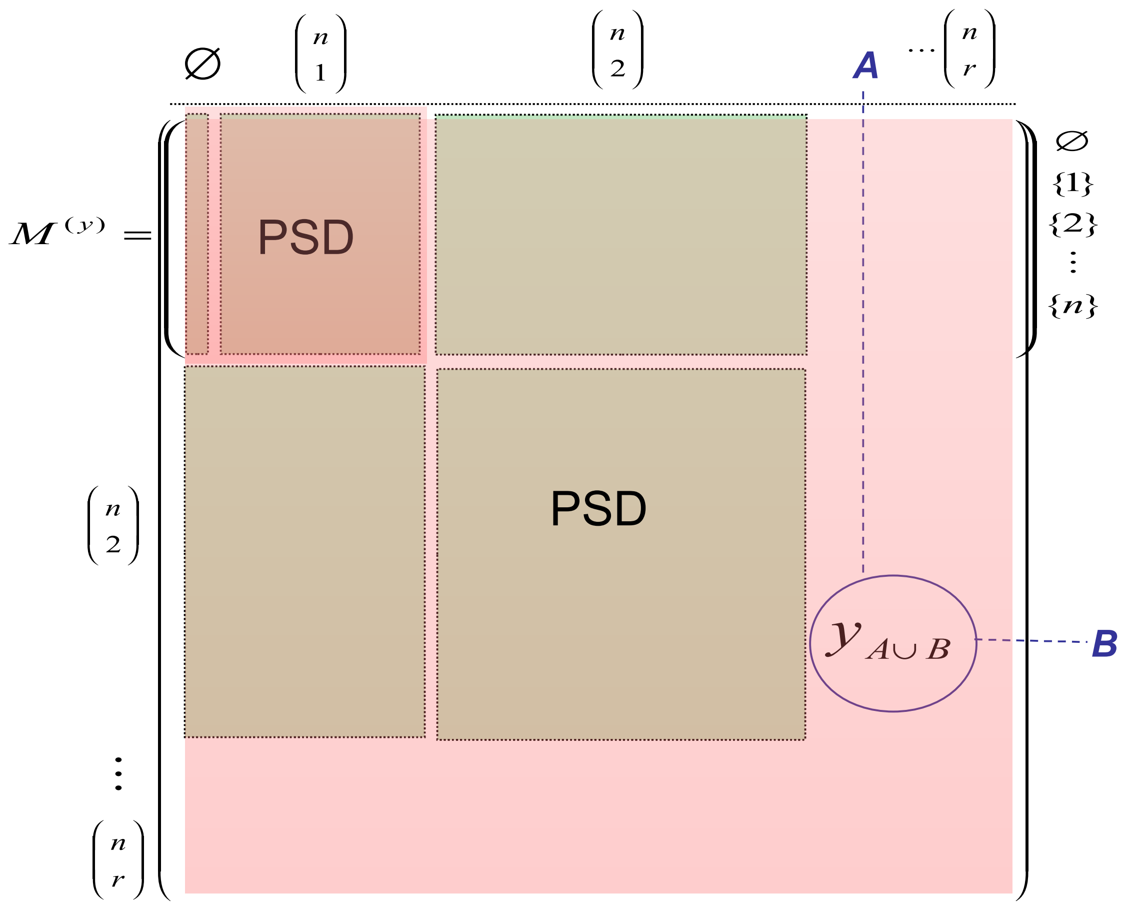
… and I am sure, more are coming.

Combinatorial Problem	Integrality gap	Level
Max Cut [KS09]	1.139	$\Omega\left(\log^{\frac{1}{6}}\log\log n\right)$
Some Max CSPs [BGMT11]	tight	$\Theta\left(n\right)$
Unique Games [RS09]	$1-\varepsilon$ vs γ	$\Omega\left(\log^{\frac{1}{4}}\log n\right)$
Quadratic Programming [BM10]	$\Omega\left(\log n\right)$	$\Omega\left(n^{\delta}\right)$
MaxCut Gain [BM10]	$\Omega\left(1\right)$	$\Omega\left(1\right)$
Vertex Cover [BCGM11]	$2-\varepsilon$	6

The Realm of lift-and-project Systems



The Renowned Lasserre System



$$\sum_{R \subseteq N} (-1)^{|R|} M^{(y)} e_{R \cup Y} \in K$$

$$M^{(y)}_{ij} = y_{\{i,j\}}$$

is positive semidefinite

$$M^{(y)}_{A,B} = y_{A \cup B} = \prod_{i \in A \cup B} x_i$$

is positive semidefinite

An Application of the Lasserre System

<u>Our toy example</u>

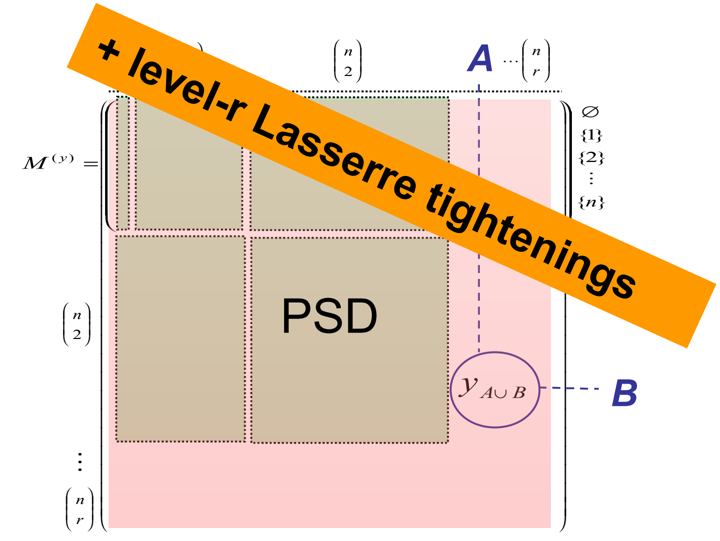
Stable Set Relaxation

on input G=(V,E)

$$\min \sum_{i \in V} x_i$$

$$x_i + x_j \leq 1, \forall\, ij \in E$$

$$x_i \in [0,1]$$



$$\min \sum_{i \in V} v_0 \cdot v_i$$

$$v_I^{\,2} = 0, \forall I \subseteq V, |I| \leq 2r, I \text{ not stable}$$

$$v_I \cdot v_J = v_{I \cup J}^2, \forall I, J \subseteq V, |I|, |J| \leq r$$

$$v_{\varnothing}^2 = 1$$

Set r=1 to get the Lovász theta function!

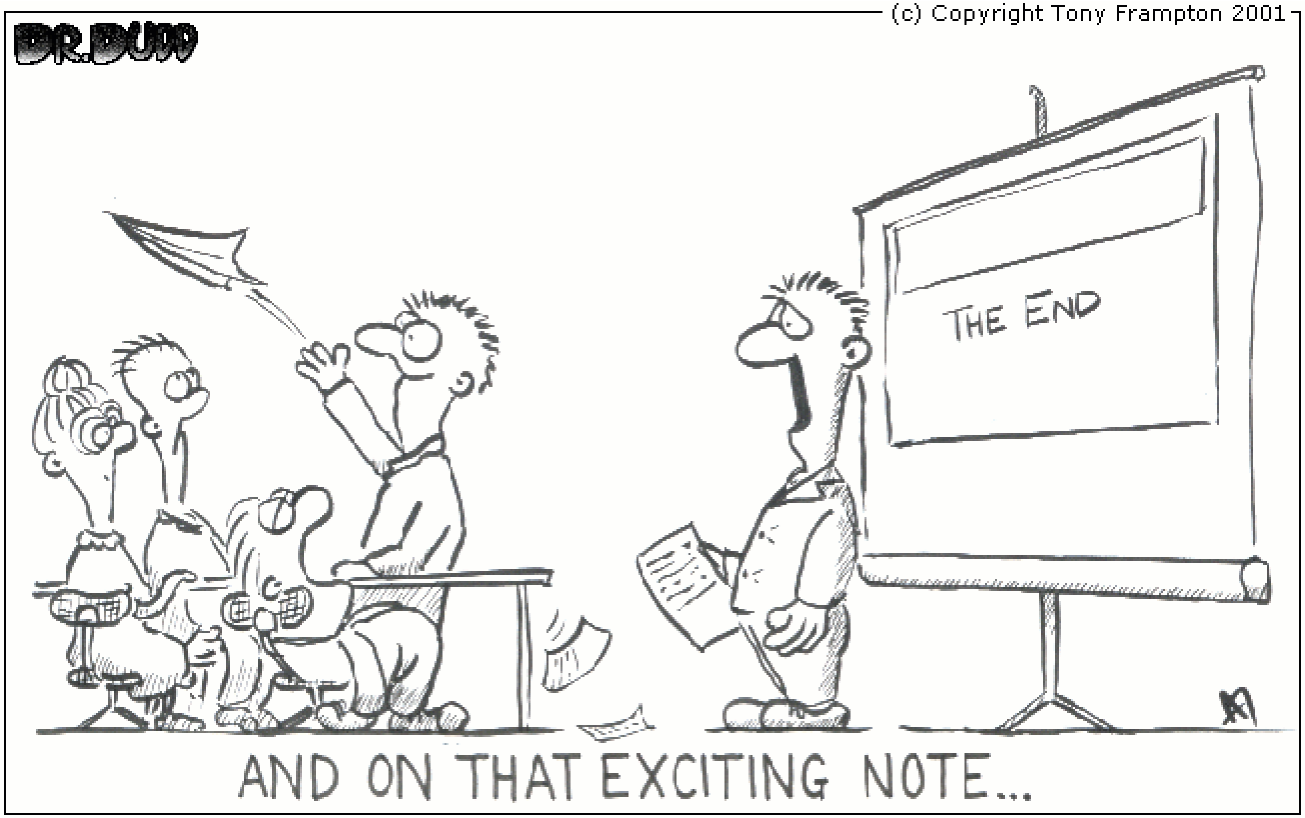
Utilizing/Fooling the Lasserre System

Combinatorial Problem	Approximation	Level
Knapsack [KMN11]	$1+\frac{1}{r}$	$O(r^2)$
Coloring (3 colorable graphs) [Chl07]	$O(n^{0.2072})$	$O(1)$
Independent Set (3-uniform hypergraph with solution rn) [CS08]	abs $O(n^{r^2})$	$O\left(\frac{1}{r^2}\right)$
2-CSPs [BRS11]	$1+\varepsilon$	$\frac{1}{\text{poly}(\varepsilon)}$
Directed Steiner Tree [Rot11]	$O(\log^3 \lvert R \rvert)$	$\log \lvert R \rvert$

… new exciting stuff
to be presented shortly!

Combinatorial Problem	Integrality gap	Level
Vertex Cover [Sch08]	$\frac{7}{6}$	$\Theta(n)$
Vertex Cover [Tul09]	1.36	$\Omega(n^{\delta})$
Max-k-XOR [Sch08]	$2-\varepsilon$	$\Theta(n)$
Max-k-CSPs [Toul09]	$\frac{2^k}{2k}-\varepsilon$	$\Theta(n)$

… and try if you dare to show a level-2 tight integrality gap for any problem with hard constraints.


DR.DUDD
THE END
AND ON THAT EXCITING NOTE...