Complex spherical designs and nonsymmetric association schemes

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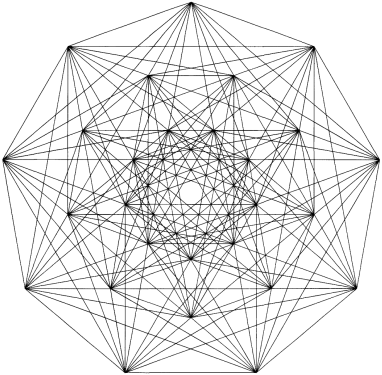


Association schemes from real spherical designs

For $X \subseteq S(\mathbb{R}^d)$, define

$$A(X) = \{x^T y : x, y \in X, x \neq y\}.$$

If $A(X) = \{\alpha_1, \ldots, \alpha_s\}$, define relations

$$R_i = \{(x, y) \in X^2 : x^T y = \alpha_i\}$$

Theorem (Delsarte, Goethals, Seidel '75)

Let X be a t-design with $|A(X)| = s$. Then:

(i) $t \leq 2s$.

(ii) *If $t \geq 2(s-1)$, then $\{I, R_1, \ldots, R_s\}$ is an association scheme.*

Association schemes from complex projective designs

For $X \subseteq S(\mathbb{C}^d)$, define

$$A(X) = \{|x^* y| : x, y \in X, x \neq y\}.$$

Theorem (Delsarte, Goethals, Seidel '75)

Let X be a complex projective t-design with $|A(X)| = s$. Then:

(i) $t \leq 2s$.

(ii) *If $t \geq 2(s-1)$, then $\{I, R_1, \ldots, R_s\}$ is an association scheme.*

Example [rotations of MUBs]

What about

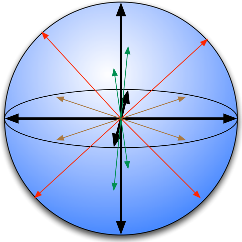
$$A(X) = \{x^* y : x, y \in X, x \neq y\}$$

for $X \subseteq S(\mathbb{C}^d)$?

eg) $\epsilon = \frac{1+i}{2}$,

$$L = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \cup \left\{ \epsilon \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \epsilon \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \cup \left\{ \epsilon \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}, \epsilon \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} \right\},$$

$$X = L \cup \mathrm{i}L \cup -L \cup -\mathrm{i}L.$$



$$A(X) = \{-1, \pm\mathrm{i}, \frac{\pm 1 \pm \mathrm{i}}{2}, 0\},$$

8-class nonsymmetric association scheme.

Designs from association schemes

- $\mathcal{A}$: symmetric association scheme, n vertices, s classes
- E_1: first idempotent, rank m
- $\frac{n}{m}E_1$: Gram matrix of unit vectors $\{x_1, \ldots, x_n\}$ in $\mathbb{R}^m$

Theorem (Cameron, Goethals, Seidel '78)

Let $\mathcal{A}$ be a symmetric association scheme and identify the points of $\mathcal{A}$ with unit vectors X whose Gram matrix is a scalar multiple of E_1. Then:

(i) *X is a real spherical 2-design.*

(ii) *X is a 3-design if and only if $q_{1,1}^1 = 0$.*

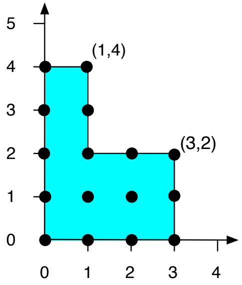
Complex spherical designs

$\mathrm{Hom}(k,l)$: homogeneous polynomials on $z=(z_1,\ldots,z_d)\in S(\mathbb{C}^d)$ of degree k in $z_1,\ldots,z_d$, degree l in $\overline{z_1},\ldots,\overline{z_d}$.

eg) $z_1^2\overline{z_1}+z_1z_2\overline{z_3}\in \mathrm{Hom}(2,1)$

Lower set: $\mathcal{T}\subseteq\mathbb{N}^2$ such that if $(k,l)\in\mathcal{T}$, so is (m,n) for all $0\le m\le k, 0\le n\le l$.

eg) $\mathcal{T}=\overline{\{(1,4),(3,2)\}}$



Complex spherical designs

Complex spherical $\mathcal{T}$-design: $X \subseteq S(\mathbb{C}^d)$ such that for every polynomial $f \in \mathrm{Hom}(k, l)$, $(k, l) \in \mathcal{T}$,

$$\frac{1}{|X|} \sum_{z \in X} f(z) = \int_{S(\mathbb{C}^d)} f(z)\, \mathrm{d}z. \tag{1}$$

eg) $\mathcal{T} = \{(1, 0)\}$:

$$\frac{1}{|X|} \sum_{z \in X} z = 0.$$

- $\overline{\{(t, t)\}}$-design in $S(\mathbb{C}^d) \Rightarrow$ projective t-design.
- $\{(k, l) : k + l \leq t\}$-design in $S(\mathbb{C}^d) \Leftrightarrow t$-design in $S(\mathbb{R}^{2d})$.

Complex spherical codes

For $X \subseteq S(\mathbb{C}^d)$,

$$A(X) := \{x^* y : x, y \in X, x \neq y\}.$$

Complex spherical code of degree s: $|A(X)| = s$.
Annihilator polynomial F of X: $F(\alpha) = 0$ for all $\alpha \in A(X)$.
$\mathcal{S}$-code: annihilator polynomial in $\mathrm{span}\{x^k \overline{x}^l : (k,l) \in \mathcal{S}\}$.

eg) If $|\alpha| = c$ for all $\alpha \in A(X)$: $F(x) = x\overline{x} - c^2$, $S = \overline{\{(1,1)\}}$.

- degree $s \Rightarrow \overline{\{(s,0)\}}$-code.
- degree s in $S(\mathbb{C}^d) \Rightarrow$ degree $\leq s$ in $S(\mathbb{R}^{2d})$.
- degree s in $S(\mathbb{C}^d) \Rightarrow$ degree $\leq s$ in $\mathbb{C}P^{d-1}$.

Harmonic polynomials

$\mathrm{Harm}(k,l)$: irreducible $U(d)$-module such that

$$\mathrm{Hom}(k,l) = \mathrm{Harm}(k,l) \bigoplus \mathrm{Hom}(k-1,l-1).$$

$$\dim(\mathrm{Hom}(k,l)) = \tbinom{d+k-1}{d-1}\tbinom{d+l-1}{d-1}$$
$$\dim(\mathrm{Harm}(k,l)) = \tbinom{d+k-1}{d-1}\tbinom{d+l-1}{d-1} - \tbinom{d+k-2}{d-1}\tbinom{d+l-2}{d-1}.$$

Absolute bounds

$$\mathcal{E} * \mathcal{E} := \{(k_1 + l_2, k_2 + l_1) : (k_1, l_1), (k_2, l_2) \in \mathcal{E}\}.$$

Theorem

If X is a $\mathcal{T}$-design with $\mathcal{E} * \mathcal{E} \subseteq \mathcal{T}$, then

$$|X| \geq \sum_{(k,l)\in\mathcal{E}} \dim(\mathrm{Harm}(k,l)).$$

If X is an $\mathcal{S}$-code, then

$$|X| \leq \sum_{(k,l)\in\mathcal{S}} \dim(\mathrm{Harm}(k,l))$$

where $\dim(\mathrm{Harm}(k,l)) = \binom{d+k-1}{d-1}\binom{d+l-1}{d-1} - \binom{d+k-2}{d-1}\binom{d+l-2}{d-1}$.

Tightness equivalence

Tight $\mathcal{S}$-code:

$$|X| = \sum_{(k,l)\in\mathcal{S}} \dim(\mathrm{Harm}(k,l)).$$

Tight design with respect to $\mathcal{E}$: $\mathcal{T}$-design with $\mathcal{E} * \mathcal{E} \subseteq \mathcal{T}$,

$$|X| = \sum_{(k,l)\in\mathcal{E}} \dim(\mathrm{Harm}(k,l))$$

.

Theorem

The following are equivalent:

(i) X *is an* $\mathcal{S}$*-code and a* $\mathcal{T}$*-design with* $\mathcal{S} * \mathcal{S} \subseteq \mathcal{T}$.

(ii) X *is a tight* $\mathcal{S}$*-code.*

(iii) X *is a tight design with respect to* $\mathcal{S}$.

Association schemes

Let $X \subseteq S(\mathbb{C}^d)$ have inner product set $A(X) = \{\alpha_1, \ldots, \alpha_s\}$.
For $x, y \in X$, define

$$(A_i)_{x,y} = \begin{cases} 1, & x^* y = \alpha_i; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem

Let X be a $\mathcal{T}$-design with $\mathcal{E} * \mathcal{E} \subseteq \mathcal{T}$ and degree s. Then:

(i) $|\mathcal{E}| \leq s + 1$.

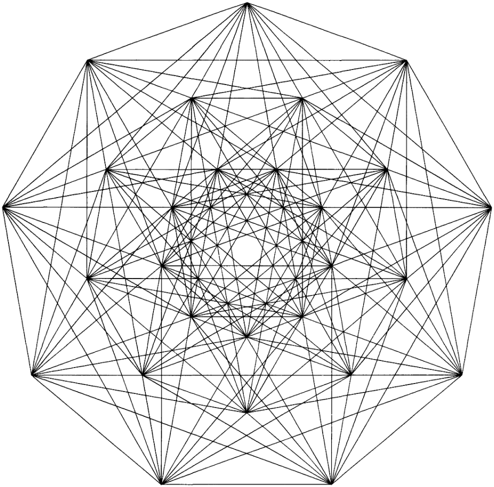
(ii) *If $|\mathcal{E}| \geq s$, then X carries an association scheme.*

(iii) *If $|\mathcal{E}| = s + 1$, then X is a tight design with respect to $\mathcal{E}$.*

Example [Coxeter]

Let $w^3 = 1$, $X =$

$$\frac{1}{\sqrt{2}}\left\{(0, w^i, -w^j), (-w^i, 0, w^j), (w^i, -w^j, 0) : i, j \in \{0, 1, 2\}\right\}.$$



Example [Coxeter]

Let $w^3 = 1$, $X =$

$$\frac{1}{\sqrt{2}} \left\{ (0, w^i, -w^j), (-w^i, 0, w^j), (w^i, -w^j, 0) : i, j \in \{0, 1, 2\} \right\}.$$

Then $|X| = 27$, $|A(X)| = 5$ and X is $\mathcal{T}$-design where

$$\mathcal{T} = \overline{\{(5, 0), (3, 2), (2, 3), (0, 5)\}}.$$

$\Rightarrow$ tight design with respect to

$$\mathcal{E} = \overline{\{(2, 0), (1, 1), (0, 2)\}}$$

.

- Absolute bound $\Rightarrow$ 5-class nonsymmetric scheme.

Symmetrization

Define $R_i^T = \{(y,x) : (x,y) \in R_i\}$.

- If $\mathcal{A} = \{R_0, \ldots, R_s\}$ is a commutative association scheme, then $\{R_i \cup R_i^T : R_i \in \mathcal{A}\}$ is a symmetric association scheme.

Real designs from complex designs

Define

$$\phi(x_1, \ldots, x_d) = (\mathrm{Re}(x_1), \mathrm{Im}(x_1), \ldots, \mathrm{Re}(x_d), \mathrm{Im}(x_d)). \qquad (2)$$

Then

$$\phi(x)^T \phi(y) = \mathrm{Re}(x^* y).$$

Theorem

If X is a tight design with respect to $\mathcal{E} = \{(k, l) : k + l \leq t\}$ in $\mathbb{C}^d$, then:

- *$\phi(X)$ is a tight t-design in S^{2d-1}.*
- *The inner product scheme of $\phi(X)$ is a fusion scheme of the inner product scheme of X.*

Projective designs from complex designs

X is n-antipodal if $X = L \mathbin{\dot\cup} \omega L \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} \omega^{n-1} L$, where $\omega^n = 1$.

Define

$$P(X) = \{xx^* : x \in X\}.$$

Theorem

Let X be an n-antipodal $\mathcal{T}$-design with degree s. If some $\mathcal{E}$ satisfies $\mathcal{E} * \mathcal{E} \subseteq \mathcal{T}$ and $|\mathcal{E}| \geq s$, then:

- *$P(L)$ is a projective t-design, where t is the largest integer with $(t,t) \in \mathcal{T}$, $t \leq n$.*
- *The inner product scheme of $P(L)$ is a quotient scheme of the inner product scheme of X.*

Designs from nonsymmetric association schemes

Let $E_{\widehat{i}} = E_i^T$.

Theorem

Let $\mathcal{A}$ be a commutative association scheme and identify the points of $\mathcal{A}$ with unit vectors X whose Gram matrix is a scalar multiple of E_1. Then:

(i) X *is a* $\overline{\{(1,1)\}}$*-design.*

(ii) X *is a* $\overline{\{(2,0)\}}$*-design if and only if* $\widehat{1} \neq 1$.

(iii) X *is a* $\overline{\{(2,1)\}}$*-design if and only if* $q_{1,1}^{1} = 0$.

(iv) X *is a* $\overline{\{(3,0)\}}$*-design if and only if* $q_{1,1}^{\widehat{1}} = 0$.

Reference

"Complex spherical designs and codes"
arxiv.org:1104.4692