Linear Programming Bounds and Partially Ordered Sets

William J. Martin

Department of Mathematical Sciences
and
Department of Computer Science
Worcester Polytechnic Institute

BIRS Workshop April 26, 2011

Outline

The Two Standard Bases

We have a vector space of $v \times v$ symmetric matrices with rows and columns indexed by X

Basis of 01-matrices:

$$\{A_0, A_1, \ldots, A_d\} = \{A_i\}_{i \in \mathcal{I}}$$

Basis of mutually orthogonal idempotents:

$$\{E_0, E_1, \ldots, E_d\} = \{E_j\}_{j \in \mathcal{J}}$$

Change-of-Basis Matrices

First Eigenmatrix: P ($i^{\rm th}$ column gives eigenvalues of A_i)
Second Eigenmatrix: Q ($j^{\rm th}$ column gives the "dual eigenvalues")

$$A_i = \sum_{j \in \mathcal{J}} P_{ji} E_j \qquad E_j = \frac{1}{v} \sum_{i \in \mathcal{I}} Q_{ij} A_i$$

$$PQ = vI \qquad \frac{1}{v_i} P_{ji} = \frac{1}{m_j} Q_{ij}$$

($v_i =$ rowsum of A_i, $\quad m_j = \mathrm{rank}\, E_j$)

Basic Idea

If $C \subseteq X$ with characteristic vector x_C

- $x_C^\top E_j x_C \geq 0$

Basic Idea

If $C \subseteq X$ with characteristic vector x_C

- $x_C^\top E_j x_C \geq 0$
- $x_C^\top A_i x_C$ is combinatorially meaningful

Basic Idea

If $C \subseteq X$ with characteristic vector x_C

- $x_C^\top E_j x_C \geq 0$
- $x_C^\top A_i x_C$ is combinatorially meaningful
- these quantities are related by linear equations

Inner Distribution and Dual Distribution

We have an interesting subset $C \subseteq X$. We define

$$a_i = \frac{1}{|C|} x_C^\top A_i x_C \qquad (i \in \mathcal{I}) \qquad a_i = \frac{|C \times C \cap R_i|}{|C|}$$

and

$$b_j = \frac{v}{|C|} x_C^\top E_j x_C \qquad (j \in \mathcal{J})$$

Vector $\mathbf{a} = [a_0, a_1, \ldots, a_d]$ is called the "inner distribution" of C.

Inner Distribution

$$a_i = \frac{1}{|C|} x_C^\top A_i x_C \qquad (i \in \mathcal{I})$$

Observe

- $a_i \geq 0$ for all i

Inner Distribution

$$a_i = \frac{1}{|C|} x_C^\top A_i x_C \qquad (i \in \mathcal{I})$$

Observe

- $a_i \geq 0$ for all i
- $a_0 = 1$

Inner Distribution

$$a_i = \frac{1}{|C|} x_C^\top A_i x_C \qquad (i \in \mathcal{I})$$

Observe

- $a_i \geq 0$ for all i
- $a_0 = 1$
- $\sum_i a_i = |C|$

Inner Distribution

$$a_i = \frac{1}{|C|} x_C^\top A_i x_C \qquad (i \in \mathcal{I})$$

Observe

- $a_i \geq 0$ for all i
- $a_0 = 1$
- $\sum_i a_i = |C|$
- $a_i = 0$ iff no edge of graph (X, R_i) has both ends in C

Dual Distribution

$$b_j = \frac{v}{|C|} x_C^\top E_j x_C \qquad (\, j \in \mathcal{J})$$

Observe

- $b_j \geq 0$ for all j

Dual Distribution

$$b_j = \frac{v}{|C|} x_C^\top E_j x_C \qquad (j \in \mathcal{J})$$

Observe

- $b_j \geq 0$ for all j
- $b_0 = |C|$

Dual Distribution

$$b_j = \frac{v}{|C|} x_C^\top E_j x_C \qquad (j \in \mathcal{J})$$

Observe

- $b_j \geq 0$ for all j
- $b_0 = |C|$
- $\sum_j b_j = |X| \; (= v)$

Dual Distribution

$$b_j = \frac{v}{|C|} x_C^\top E_j x_C \qquad (j \in \mathcal{J})$$

Observe

- $b_j \geq 0$ for all j
- $b_0 = |C|$
- $\sum_j b_j = |X|$ $(= v)$
- $b_j = 0$ iff $x_C \perp V_j$ (the j^{th} eigenspace, col E_j)

LP Formulation for $\mathcal{A}$-Codes

For $\mathcal{A} \subseteq \mathcal{I}$,
$C \subseteq X$ is an "$\mathcal{A}$-code" provided $(C \times C) \cap R_i = \emptyset$ for $i \in \mathcal{A}$.

LP Formulation for $\mathcal{A}$-Codes

For $\mathcal{A} \subseteq \mathcal{I}$,
$C \subseteq X$ is an "$\mathcal{A}$-code" provided $(C \times C) \cap R_i = \emptyset$ for $i \in \mathcal{A}$.

The size of C is bounded above by the optimal objective value to:

$$\begin{array}{rcll} \max \sum_{i \in \mathcal{I}} a_i & & & \\ \textit{subject to} & & & \\ \sum_{i \in \mathcal{I}} a_i Q_{ij} & \geq & 0 & (j \in \mathcal{J}) \\ a_0 = 1, \ a_i & = & 0 & (i \in \mathcal{A}) \\ a_i & \geq & 0 & (i \in \mathcal{I}) \end{array}$$

Why do we Prefer the Dual?

Note that every $\mathcal{A}$-code gives us a feasible solution
but
Only the optimal solution gives us a true upper bound

What if we don't want to (i.e., can't) solve to optimality?

LP Duality

For simplicity, I'm going to transform this LP into standard form and take its dual.

The dual of the LP

$$\max \quad c^\top x, \qquad s.t. \qquad Ax \leq b, \qquad x \geq 0$$

is

$$\min y^\top b, \qquad s.t. \qquad y^\top A \geq c^\top, \qquad y \geq 0$$

LP Formulation for $\mathcal{A}$-Codes

Same LP, in standard form ($a_0 = 1$, $Q_{0j} = m_j$):

$$
\begin{array}{rcll}
 & & 1 + \max \sum_{i \notin \mathcal{A}} a_i & \\
\textit{subject to} & & & \\
\sum_{i \notin \mathcal{A}} (-Q_{ij}) a_i & \leq & m_j & (\, j \in \mathcal{J}, j \neq 0) \\
a_i & \geq & 0 & (i \notin \mathcal{A})
\end{array}
$$

Dual LP for $\mathcal{A}$-Codes

$$\begin{array}{rcll} 1 + \max \sum_{i \notin \mathcal{A}} a_i & & & \textit{subject to} \\ \sum_{i \notin \mathcal{A}} (-Q_{ij}) a_i & \leq & m_j & (j \in \mathcal{J}^*) \\ a_i & \geq & 0 & (i \notin \mathcal{A}) \end{array}$$

with dual:

$$\begin{array}{rcll} 1 + \min \sum_{j \in \mathcal{J}^*} m_j y_j & & & \textit{subject to} \\ \sum_{j \in \mathcal{J}^*} (-Q_{ij}) y_j & \geq & 1 & (i \notin \mathcal{A}) \\ y_j & \geq & 0 & (j \in \mathcal{J}^*) \end{array}$$

($\mathcal{J}^*$ means omit 0).

Rewriting the Dual

Dual:

$$
\begin{array}{rcll}
1 + \min \sum_{j \in \mathcal{J}^*} m_j y_j & & & \\
\textit{subject to} & & & \\
\sum_{j \in \mathcal{J}^*} (-Q_{ij}) y_j & \geq & 1 & (i \notin \mathcal{A}) \\
y_j & \geq & 0 & (j \in \mathcal{J}^*)
\end{array}
$$

Routine trickery:

$$
b_j := m_j y_j, \qquad b_0 := 1, \qquad \frac{P_{ji}}{v_i} = \frac{Q_{ij}}{m_j}
$$

($\mathcal{J}^*$ means omit 0).

Rewriting the Dual

New Dual:

$$\min \sum_{j \in \mathcal{J}} b_j$$

subject to

$$\sum_{j \in \mathcal{J}} P_{ji} b_j \leq 0 \qquad (i \notin \mathcal{A})$$

$$b_0 = 1, \qquad b_j \geq 0 \qquad (j \in \mathcal{J}^*)$$

Easy Special Case

Suppose $\mathcal{A} = \{2, \ldots, d\}$. Write $P_{j1} = \lambda_j$.

$$\begin{array}{rcl} \min b_0 + b_1 + \cdots + b_d & & \\ \textit{subject to} & & \\ b_0\lambda_0 + b_1\lambda_1 + \cdots + b_d\lambda_d & \leq & 0 \qquad\qquad (\textit{one constraint}) \\ b_0 = 1, \qquad b_1,\ b_2,\ \ldots, b_d & \geq & 0 \end{array}$$

Easy Special Case

Suppose $\mathcal{A} = \{2, \ldots, d\}$. Write $P_{j1} = \lambda_j$.

$$\begin{array}{rcll} & \min b_0 + b_1 + \cdots + b_d & & \\ \textit{subject to} & & & \\ b_0\lambda_0 + b_1\lambda_1 + \cdots + b_d\lambda_d & \leq & 0 & \quad (\textit{one constraint}) \\ b_0 = 1, \qquad b_1,\ b_2,\ \ldots, b_d & \geq & 0 & \end{array}$$

Easy Special Case

Suppose $\mathcal{A} = \{2, \ldots, d\}$. Write $P_{j1} = \lambda_j$.

$$\begin{array}{rcl} & \min b_0 + b_1 + \cdots + b_d & \\ \textit{subject to} & & \\ b_0\lambda_0 + b_1\lambda_1 + \cdots + b_d\lambda_d & \leq & 0 \qquad (\textit{one constraint}) \\ b_0 = 1, \qquad b_1,\ b_2,\ \ldots, b_d & \geq & 0 \end{array}$$

- $b_0 = 1, \qquad \lambda_0 = k \quad \text{(say)}$

Easy Special Case

Suppose $\mathcal{A} = \{2, \ldots, d\}$. Write $P_{j1} = \lambda_j$.

$$\begin{array}{rcll} & \min b_0 + b_1 + \cdots + b_d & & \\ \textit{subject to} & & & \\ b_0\lambda_0 + b_1\lambda_1 + \cdots + b_d\lambda_d & \leq & 0 & (\textit{one constraint}) \\ b_0 = 1, \quad b_1, \ b_2, \ \ldots, b_d & \geq & 0 & \end{array}$$

- $b_0 = 1, \qquad \lambda_0 = k$ (say)
- $b_d = -k/\lambda_d$ (assuming λ_d is smallest)

Easy Special Case

Suppose $\mathcal{A} = \{2, \ldots, d\}$. Write $P_{j1} = \lambda_j$.

$$\begin{array}{rcll} & \min b_0 + b_1 + \cdots + b_d & & \\ \textit{subject to} & & & \\ b_0\lambda_0 + b_1\lambda_1 + \cdots + b_d\lambda_d & \leq & 0 & (\textit{one constraint}) \\ b_0 = 1, \qquad b_1,\ b_2,\ \ldots, b_d & \geq & 0 & \end{array}$$

- $b_0 = 1, \qquad \lambda_0 = k \quad \text{(say)}$
- $b_d = -k/\lambda_d \quad$ (assuming λ_d is smallest)
- $|C| \leq 1 - \frac{k}{\lambda_d}$

LP Formulation for $\mathcal{T}$-Designs

For $\mathcal{T} \subseteq \mathcal{J}$,
$D \subseteq X$ is a "$\mathcal{T}$-design" provided $x_D \perp V_j$ for $j \in \mathcal{T}$.

The size of D is bounded **below** by the optimal objective value to:

$$
\begin{array}{rcll}
 & \min \sum_{i \in \mathcal{I}} a_i & & \\
\textit{subject to} & & & \\
\sum_{i \in \mathcal{I}} a_i Q_{ij} & \geq & 0 & (j \in \mathcal{J}^*) \\
\sum_{i \in \mathcal{I}} a_i Q_{ij} & = & 0 & (j \in \mathcal{T}) \\
a_0 = 1, \quad a_i & \geq & 0 & (i \in \mathcal{I}^*)
\end{array}
$$

Dual LP for $\mathcal{T}$-Designs

$$\begin{array}{rcll} \min \sum_{i\in\mathcal{I}} a_i & & & \\ \textit{subject to} & & & \\ \sum_{i\in\mathcal{I}} a_i Q_{ij} & \geq & 0 & (j \in \mathcal{J}^*) \\ \sum_{i\in\mathcal{I}} a_i Q_{ij} & = & 0 & (j \in \mathcal{T}) \\ a_0 = 1, \quad a_i & \geq & 0 & (i \in \mathcal{I}^*) \end{array}$$

with dual (now using $b_j = -m_j y_j$):

$$\max \sum_{j\in\mathcal{J}} b_j$$

subject to

$$\begin{array}{rcll} \sum_{j\in\mathcal{J}} P_{ji} b_j & \geq & 0 & (i \in \mathcal{I}^*) \\ b_0 = 1, \quad b_j & \leq & 0 & (j \in \mathcal{J}^* - \mathcal{T}) \end{array}$$

Dual width one

Assume we are working in a Q-polynomial scheme.
What subsets D satisfy $x_D \in V_0 \oplus V_1$?
What is the smallest cardinality of D?

Dual width one

Assume we are working in a Q-polynomial scheme.
What subsets D satisfy $x_D \in V_0 \oplus V_1$?
What is the smallest cardinality of D?

$$\begin{aligned} & \max \ b_0 + b_1 \\ & \textit{subject to} \\ & P_{0i} b_0 + P_{1i} b_1 \geq 0 \qquad (i \neq 0) \\ & b_0 = 1 \qquad (b_1 \ \textit{unrestr.}) \end{aligned}$$

We get

$$|D| \geq \frac{v}{1 - \frac{m_1}{Q_{d1}}}$$

where we assume

$$m_1 = Q_{01} > Q_{11} > \cdots > Q_{d1}$$

Subsets of Dual Width One

Assume we are working in a Q-polynomial scheme.
What subsets D satisfy $x_D \in V_0 \oplus V_1$?
Ratio Bound:

$$|D| \geq \frac{v}{1 - \frac{m_1}{Q_{d1}}}$$

where we assume

$$m_1 = Q_{01} > Q_{11} > \cdots > Q_{d1}$$

Hamming scheme: For $H(n, q)$, we get $|D| \geq q^{n-1}$

Johnson scheme: For $J(n, k)$, we get $|D| \geq \binom{v-1}{k-1}$

Projecting onto the Bose-Mesner algebra

With respect to the inner product $\langle M, N \rangle = \mathrm{tr}\,(MN^\top)$,

$$\left\{ \sqrt{\frac{1}{vv_i}} A_i \mid i \in \mathcal{I} \right\} \quad \text{and} \quad \left\{ \sqrt{\frac{1}{m_j}} E_j \mid j \in \mathcal{J} \right\}$$

form orthonormal bases for $\mathbb{A}$.

Projecting onto the Bose-Mesner algebra

With respect to the inner product $\langle M, N \rangle = \mathrm{tr}\,(MN^\top)$,

$$\left\{ \sqrt{\frac{1}{vv_i}} A_i \mid i \in \mathcal{I} \right\} \quad \text{and} \quad \left\{ \sqrt{\frac{1}{m_j}} E_j \mid j \in \mathcal{J} \right\}$$

form orthonormal bases for $\mathbb{A}$.

So, for any matrix M of size $v \times v$

$$\sum_i \frac{\langle M, A_i \rangle}{vv_i} A_i = \sum_j \frac{\langle M, E_j \rangle}{m_j} E_j$$

Projecting onto the Bose-Mesner algebra

So take $M = xx^\top$ to find, for any vector x of length v,

$$\sum_i \frac{x^\top A_i x}{v v_i} A_i = \sum_j \frac{x^\top E_j x}{m_j} E_j$$

Anticode Condition (Godsil-Meagher proof)

Suppose code C has characteristic vector x and anticode A has characteristic vector y.

Anticode Condition (Godsil-Meagher proof)

Suppose code C has characteristic vector x and anticode A has characteristic vector y. If,
for all $i \neq 0$, $x^\top A_i x = 0$ or $y^\top A_i y = 0$,
then

$$\sum_i \frac{x^\top A_i x}{v v_i} y^\top A_i y = \sum_j \frac{x^\top E_j x}{m_j} y^\top E_j y$$

Anticode Condition (Godsil-Meagher proof)

Suppose code C has characteristic vector x and anticode A has characteristic vector y. If,
for all $i \neq 0$, $x^\top A_i x = 0$ or $y^\top A_i y = 0$,
then

$$\sum_i \frac{x^\top A_i x}{v v_i} y^\top A_i y = \sum_j \frac{x^\top E_j x}{m_j} y^\top E_j y$$

$$\frac{|C| \cdot |A|}{v} = \sum_j \frac{1}{m_j} \left(x^\top E_j x \right) \left(y^\top E_j y \right) \geq (x^\top E_0 x)(y^\top E_0 y) = \frac{|C|^2 \cdot |A|^2}{v^2}$$

Anticode Condition (Godsil-Meagher proof)

Suppose code C has characteristic vector x and anticode A has characteristic vector y. If,
for all $i \neq 0$, $x^\top A_i x = 0$ or $y^\top A_i y = 0$,
then

$$\sum_i \frac{x^\top A_i x}{v v_i} y^\top A_i y = \sum_j \frac{x^\top E_j x}{m_j} y^\top E_j y$$

$$\frac{|C| \cdot |A|}{v} = \sum_j \frac{1}{m_j} \left(x^\top E_j x \right) \left(y^\top E_j y \right) \geq (x^\top E_0 x)(y^\top E_0 y) = \frac{|C|^2 \cdot |A|^2}{v^2}$$

…giving $|C| \cdot |A| \leq v$.

Working Without Polynomials

In this next part, we extend two well-known bounds of Delsarte to the setting of association schemes with many vanishing Krein parameters.

Originally, these results were proved by Delsarte for cometric association schemes.

Here, we replace the cometric property with certain vanishing conditions for Krein parameters with reference to a partial order $\trianglelefteq$ on the set $\mathcal{J}$ of eigenspaces of the association scheme.

Big and Small Eigenspaces

For $\mathcal{E}$ and $\mathcal{F}$, subsets of $\mathcal{J}$, define

$$\mathcal{E} \star \mathcal{F} = \{k \in \mathcal{J} : \sum_{i \in \mathcal{E}} \sum_{j \in \mathcal{F}} q_{ij}^k > 0\}.$$

Krein conditions imply

$$k \in \mathcal{E} \star \mathcal{F}$$

whenever

$$q_{ij}^k \neq 0 \text{ for some } i \in \mathcal{E} \text{ and some } j \in \mathcal{F}.$$

Example: In a cometric scheme, if we take $\mathcal{E} = \{0, \ldots, e\}$ and $\mathcal{F} = \{0, \ldots, f\}$, then

$$\mathcal{E} \star \mathcal{F} \subseteq \{0, \ldots, e + f\}.$$

"Fisher-Type" Inequality

Theorem

Let $\mathcal{T} \subseteq \mathcal{J}$. Assume $\mathcal{E} \subseteq \mathcal{J}$ satisfies $\mathcal{E} \star \mathcal{E} \subseteq \mathcal{T}$. Then, for any Delsarte $\mathcal{T}$-design $D \subseteq X$, we have

$$|D| \geq \sum_{j \in \mathcal{E}} m_j .$$

Moreover, if equality holds, then, for $\ell \neq 0$ in $\mathcal{J}$,

$$\sum_{j \in \mathcal{E}} Q_{\ell j} = 0$$

whenever D contains a pair of ℓ-related elements.

LP inside Bose-Mesner Algebra

Proof:
Any matrix $M \in \mathbb{A}$ can be expanded in the form

$$M = v \sum_{j \in \mathcal{J}} \beta_j E_j$$

and also as

$$M = \sum_{i \in \mathcal{I}} \alpha_i A_i$$

where $\alpha_i = \sum_j Q_{ij} \beta_j$ for each $i \in \mathcal{I}$.

LP inside Bose-Mesner Algebra

Restrict to non-negative matrices $M \in \mathbb{A}$ which satisfy the following two conditions:

(a) $\beta_j \leq 0$ for all $j \notin \mathcal{T}$; and

(b) $\beta_0 = 1$.

WOLOG, assume $0 \in \mathcal{T}$.
Let $D \subseteq X$ be a $\mathcal{T}$-design. Abbreviate x_D to x.

LP inside Bose-Mesner Algebra

Expand $x^\top Mx$ in two ways:

$$\begin{aligned} |D|\alpha_0 &= \alpha_0 x^\top A_0 x \\ &\leq \sum_{\mathcal{I}} \alpha_i x^\top A_i x = v \sum_{\mathcal{J}} \beta_j x^\top E_j x \\ &= v x^\top E_0 x + v \sum_{\mathcal{T}-\{0\}} \beta_j x^\top E_j x + v \sum_{j \notin \mathcal{T}} \beta_j x^\top E_j x \\ &\leq v x^\top E_0 x = |D|^2. \end{aligned}$$

This gives us the bound $|D| \geq \alpha_0$.

An Easy Feasible Solution

Let

$$F = \sum_{j \in \mathcal{E}} E_j .$$

Then $F \circ F$ is a non-negative matrix with spectral decomposition

$$F \circ F = \sum_{k \in \mathcal{J}} \left(\frac{1}{v} \sum_{i \in \mathcal{E}} \sum_{j \in \mathcal{E}} q_{ij}^k \right) E_k . \qquad (1)$$

Now, by choice of $\mathcal{E}$, we have $q_{ij}^k = 0$ whenever $i, j \in \mathcal{E}$ and $k \notin \mathcal{T}$.

An Easy Feasible Solution

So condition **(a)** is satisfied by any non-negative multiple of $F \circ F$.
We scale by

$$\gamma = \frac{v^2}{\sum_{j \in \mathcal{E}} m_j}$$

to obtain a non-negative matrix $M = \gamma(F \circ F)$ which satisfies conditions **(a)** and **(b)**,
It is straightforward to check that the diagonal entries of M are all equal to

$$\alpha_0 = \sum_{j \in \mathcal{E}} m_j.$$

What does that Prove?

Theorem

Let $\mathcal{T} \subseteq \mathcal{J}$. Assume $\mathcal{E} \subseteq \mathcal{J}$ satisfies $\mathcal{E} \star \mathcal{E} \subseteq \mathcal{T}$. Then, for any Delsarte $\mathcal{T}$-design $D \subseteq X$, we have

$$|D| \geq \sum_{j \in \mathcal{E}} m_j.$$

What if Equality Holds?

Now if $|D| = \alpha_0$, we return to the above string of equations and inequalities to discover that, for each $\ell \neq 0$,

$$\alpha_\ell \left(x^\top A_\ell x \right) = 0$$

must hold.

Complementary Slackness Conditions

Thus, if D contains a pair of ℓ-related elements, we are forced to have

$$\alpha_\ell = \sum_{k \in \mathcal{J}} \beta_k Q_{\ell k} = 0.$$

Now we find

$$\beta_k = \frac{\gamma}{v} \sum_{i \in \mathcal{E}} \sum_{j \in \mathcal{E}} q_{ij}^k$$

so that

$$\alpha_\ell = \frac{\gamma}{v} \sum_{i \in \mathcal{E}} \sum_{j \in \mathcal{E}} \sum_{k \in \mathcal{J}} q_{ij}^k Q_{\ell k}$$

which gives us

$$\alpha_\ell = \gamma \left(\sum_{j \in \mathcal{E}} Q_{\ell j} \right)^2 = 0 \qquad \text{as desired.}$$

Tight Designs Give Subschemes

Theorem

Let $\mathcal{T} \subseteq \mathcal{J}$ and assume $\mathcal{E} \subseteq \mathcal{J}$ satisfies $\mathcal{E} \star \mathcal{E} \subseteq \mathcal{T}$.

(a) *if D is any Delsarte $\mathcal{T}$-design in our scheme with degree s, then $s+1 \geq |\mathcal{E}|$;*

(b) *if $|\mathcal{E}| = s+1$, then D is a tight design and D is a subscheme;*

(c) *if $|\mathcal{E}| = s$, then either D is a tight design or D is a subscheme.*

Recap Magic Matrix Approach

Designs: Find M in the Bose-Mesner algebra

$$\sum_{i\in\mathcal{I}} \alpha_i A_i = M = v \sum_{j\in\mathcal{J}} \beta_j E_j$$

with $M \geq 0$, $\beta_0 = 1$, $\beta_j \leq 0$ for $j \notin \mathcal{T}$.
Then $|D| \geq \alpha_0$ for any $\mathcal{T}$-design D.

Recap Magic Matrix Approach

Designs: Find M in the Bose-Mesner algebra

$$\sum_{i\in\mathcal{I}} \alpha_i A_i = M = v \sum_{j\in\mathcal{J}} \beta_j E_j$$

with $M \geq 0$, $\beta_0 = 1$, $\beta_j \leq 0$ for $j \notin \mathcal{T}$.
Then $|D| \geq \alpha_0$ for any $\mathcal{T}$-design D.

Codes: Find M in the Bose-Mesner algebra

$$\sum_{i\in\mathcal{I}} \alpha_i A_i = M = v \sum_{j\in\mathcal{J}} \beta_j E_j$$

with $M \succeq 0$, $\beta_0 = 1$, $\alpha_i \leq 0$ for $i \notin \mathcal{A}$.
Then $|C| \leq \alpha_0$ for any $\mathcal{A}$-code C.

A bound on minimum distance

We used a partial order on the eigenspaces. This poset is often the image of a larger poset of anticodes.

These anticodes gives 01-bases for sums of eigenspaces. These have been used to characterize tight designs. But we can sometimes do much more with the anticodes/antidesigns.

A bound on minimum distance

We used a partial order on the eigenspaces. This poset is often the image of a larger poset of anticodes.

These anticodes gives 01-bases for sums of eigenspaces. These have been used to characterize tight designs. But we can sometimes do much more with the anticodes/antidesigns.

Theorem (WJM, 2000)

For any code C in the Hamming graph with $|C| > 1$, $\delta \le t + s^*$ unless C is isomorphic to a binary repetition code.

$\delta =$ min. distance, $t =$ strength, $s^* =$ dual degree $= |S^*(C)|$.

The CGS Lemma

Cameron, Goethals and Seidel: If $u \in V_i$ and $v \in V_j$ and $q_{ij}^k = 0$, then $u \circ v \perp V_k$.

Dual degree set $S^*(C) = \{j \neq 0 \mid E_j x_C \neq 0\}$.

$S^*(C \cap D) \subseteq S^*(C) \star S^*(D)$.

Applications: Roos, WJM, Vanhove.

Posets for Schemes

- regular semilattices
- quantum matroids
- design systems
- width and dual width
- Tanaka's classification of descendants in the classical families

Tanaka's Theorem

In a P- and Q-polynomial scheme, we can define both the *width* and *dual width* of a subset

$$w = \max\{i : x^\top A_i x \neq 0\} \qquad w^* = \max\{j : x^\top E_j x \neq 0\}$$

Brouwer, Godsil, Koolen, WJM (2003): $w + w^* \geq d$. Equality (usually) gives a Q-polynomial subscheme which is also a completely regular code.

Examples: subcubes in n-cubes, all k-sets on a t-set in Johnson graphs.

Tanaka (arXiv, Nov. 2010) recently completed the classification of all sets with $w + w^* = d$ in the 15 classical families.

Tanaka's Theorem

Brief sketch of a big result:

- $H(m, q)$ inside $H(n, q)$
- $J(n - t, k - t)$ inside $J(n, k)$
- the q-analogues of these (Grassmann and bilinear forms)
- classical polar spaces: $[C_d(q)]$, $[B_d(q)]$, $[D_d(q)]$, $[^2D_{d+1}(q)]$, $[^2A_{2d}(\sqrt{q})]$, $[^2A_{2d-1}(\sqrt{q})]$
- twisted Grassmann graphs: what you'd expect
- for Hermitian forms graphs, unitary dual polar graphs (2nd Q-pol. ordering), NONE
- for Alternating forms, Quadratic forms, Half dual polar spaces, halved cubes, Ustimenko: $w = 1$ or $w = d - 1$

The End

Thank you.