Complex Hadamard Matrices and Strongly Regular Graphs

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Theorem (Goethals and Seidel 1970)

Let X be a strongly regular graph.
Then $I - A(X) + A(\overline{X})$ is a Hadamard matrix if and only if X or $\overline{X}$ has one of the following parameters:

i. $(4\theta^2, 2\theta^2 - \theta, \theta^2 - \theta, \theta^2 - \theta)$ *(Latin square type)*
ii. $(4\theta^2, 2\theta^2 + \theta, \theta^2 + \theta, \theta^2 + \theta)$ *(negative Latin square type)*

A Generalization

Theorem

Let X be a strongly regular graph with at least 5 vertices. Then $I + xA(X) + yA(\overline{X})$ is a complex Hadamard matrix if and only if X or $\overline{X}$ has one of the following parameters:

i. $(4\theta^2, 2\theta^2 - \theta, \theta^2 - \theta, \theta^2 - \theta)$

ii. $(4\theta^2, 2\theta^2 + \theta, \theta^2 + \theta, \theta^2 + \theta)$

iii. $(4\theta^2 - 1, 2\theta^2, \theta^2, \theta^2)$

iv. $(4\theta^2 + 4\theta + 1, 2\theta^2 + 2\theta, \theta^2 + \theta - 1, \theta^2 + \theta)$

v. $(4\theta^2 + 4\theta + 2, 2\theta^2 + \theta, \theta^2 - 1, \theta^2)$

Definition

A complex Hadamard matrix is a $v \times v$ matrix H such that

- $|H_{ij}| = 1$, for all i, j
- $HH^* = vI$.

We say H and H' are equivalent if

$$H' = D_1 P_1 H P_2 D_2$$

for some unitary diagonal matrices D_1 and D_2, and permutation matrices P_1 and P_2.

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A type II matrix is a $v \times v$ matrix W satisfying

$$WW^{(-)T} = vI,$$

where $W^{(-)}_{ij} = \frac{1}{W_{ij}}$.

Example

complex Hadamard matrices
spin models, four-weight spin models

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For $a, b = 1, \ldots, v$, define

$$Y_{ab} = \begin{pmatrix} \frac{W_{1a}}{W_{1b}} \\ \\ \frac{W_{2a}}{W_{2b}} \\ \\ \vdots \\ \\ \frac{W_{va}}{W_{vb}} \end{pmatrix}$$

$$(Y_{aa} = [1, 1, \ldots, 1]^T)$$

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The Nomura algebra of W is

$$\mathcal{N}_W = \{M : Y_{ab} \text{ is an eigenvector of } M \quad \forall a, b\}$$

- $I \in \mathcal{N}_W$
- $\sum_x \frac{W_{xa}}{W_{xb}} = v\delta_{ab} \Longrightarrow J \in \mathcal{N}_W$
- $\mathcal{N}_W$ is closed under matrix multiplication.
- $\mathcal{N}_W$ is commutative and is closed under entrywise product and transpose.

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Theorem (Jaeger et al. 1998)

$\mathcal{N}_W$ and $\mathcal{N}_{W^T}$ give a formally dual pair of association schemes.

Spin model W

- Type I: constant diagonal, constant row and column sum
- Type II
- Type III: $W \in \mathcal{N}_W$.

Example

Potts model $W = -u^3 I + u^{-1} J$ where $(u^2 + u^{-2})^2 = n$
Jones polynomial

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Let X be a strongly regular graph with v vertices and eigenvalues k, θ and τ ($\theta \geq 0 > \tau$).

Let E_0, E_1 and E_2 be the orthogonal projection onto the eigenspace for k, θ, and τ, respectively.

Consider

$$\begin{aligned} W =& I + xA(X) + yA(\overline{X}) \\ =& [1 + kx + (v - k - 1)y]E_0 + \\ & [1 + \theta x + (-1 - \theta)y]E_1 + \\ & [1 + \tau x + (-1 - \tau)y]E_2 \end{aligned}$$

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Then

$$WW^{(-)T} = vI$$

if and only if

$$\begin{cases} (1+kx+(v-k-1)y)\left(1+k\frac{1}{x}+(v-k-1)\frac{1}{y}\right) & = v \\ (1+\theta x+(-1-\theta)y)\left(1+\theta\frac{1}{x}+(-1-\theta)\frac{1}{y}\right) & = v \\ (1+\tau x+(-1-\tau)y)\left(1+\tau\frac{1}{x}+(-1-\tau)\frac{1}{y}\right) & = v \end{cases}$$

Theorem (Godsil and C, 2010)

W is a type-II matrix if and only if one of the following holds

(a) $y = x = \frac{1}{2}(2 - v \pm \sqrt{v^2 - 4v})$ *and W is the Potts model.*

(b) $x = \frac{1}{2}(2 - v \pm \sqrt{v^2 - 4v})$ *and* $y = (1 + xk)/(x + k)$, *and X is isomorphic to mK_{k+1} for some $m > 1$.*

(c) $x = 1$ *and* $y = \frac{v+2(1+\theta)(1+\tau)\pm\sqrt{v^2+4v(1+\theta)(1+\tau)}}{2(1+\theta)(1+\tau)}$, *and $A(\overline{X})$ is the incidence matrix of a symmetric design.*

(d) $x = -1$ *and* $y = \frac{-v+2\theta^2+2\pm\sqrt{(v-4)(v-4\theta^2)}}{2(1+\theta)(1+\tau)}$, *and $A(X)$ is the incidence matrix of a symmetric design.*

(e) $x + x^{-1}$ *is a zero of a quadratic equation and* $y = \frac{\theta\tau x^3-[v(\theta+\tau+1)-2\theta-2\tau-1]x^2-(v+2\theta+2\tau+\tau\theta)x-1}{(x^2-1)(1+\theta)(1+\tau)}$.

When $|x| = |y| = 1$,

$$v = (1 + \theta x + (-1-\theta)y)\left(1 + \theta\frac{1}{x} + (-1-\theta)\frac{1}{y}\right)$$

$$= 1 + \theta^2 + (-1-\theta)^2 + \theta(x + \frac{1}{x}) + (-1-\theta)(y + \frac{1}{y}) + \theta(-1-\theta)(\frac{x}{y} + \frac{y}{x})$$

$$\leq 4(\theta+1)^2.$$

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Theorem

$W = I + xA(X) + yA(\overline{X})$ is a complex Hadamard matrix if and only if X or $\overline{X}$ has one of the following parameters:

- i $(4\theta^2, 2\theta^2 - \theta, \theta^2 - \theta, \theta^2 - \theta)$
- ii $(4\theta^2, 2\theta^2 + \theta, \theta^2 + \theta, \theta^2 + \theta)$
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- iv $(4\theta^2 + 4\theta + 1, 2\theta^2 + 2\theta, \theta^2 + \theta - 1, \theta^2 + \theta)$.
- v $(4\theta^2 + 4\theta + 2, 2\theta^2 + \theta, \theta^2 - 1, \theta^2)$

Cases (i) and (ii):

All solutions of x and y give Hadamard matrices.

Case (iii): $(4\theta^2 - 1, 2\theta^2, \theta^2, \theta^2)$

- $\begin{pmatrix} 0 & \mathbf{1}^T \\ \mathbf{1} & S(X) \end{pmatrix} + I$ is a Hadamard matrix.
  Szöllősi's construction (2010)
- $J - A(X)$ is the incidence matrix of a Hadamard design.
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Case (iv): $(4(\theta^2+\theta)+1, 2(\theta^2+\theta), (\theta^2+\theta)-1, \theta^2+\theta)$

$\begin{pmatrix} 0 & \mathbf{1}^T \\ \mathbf{1} & S(X) \end{pmatrix}$ is a symmetric conference matrix.

Szöllősi's construction (2010)

Case (v): $(4\theta^2+4\theta+2, 2\theta^2+\theta, \theta^2-1, \theta^2)$

- $S(X)$ is a regular symmetric conference matrix, $x=-y=\pm i$.
  Turyn's construction (1970)
-
  $$x=\frac{-1\pm\sqrt{4\theta^2(\theta+1)^2-1}\;i}{2\theta(\theta+1)} \quad \text{and} \quad y=\bar{x}$$

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Diţă's construction

Let $M = [m_{ij}]$ be a $k \times k$ complex Hadamard matrix.
Let $N_1, \ldots, N_k$ be $v \times v$ complex Hadamard matrix.
Then

$$W = \begin{pmatrix} m_{11}N_1 & m_{12}N_2 & \ldots & m_{1k}N_k \\ m_{21}N_1 & m_{22}N_2 & \ldots & m_{2k}N_k \\ \vdots & \vdots & \ddots & \vdots \\ m_{k1}N_1 & m_{k2}N_2 & \ldots & m_{kk}N_k \end{pmatrix}$$

is a $vk \times vk$ complex Hadamard matrix.

Generalized tensor product (Hosoya and Suzuki 2003)

Let M_1, M_2, ..., M_v be $k \times k$ type II matrices.
Let $N_1, \dots, N_k$ be $v \times v$ type II matries.

Then the matrix $(M_1, M_2, \dots M_v) \otimes (N_1, \dots, N_k)$ with (i, j)-block being

$$\begin{pmatrix} (M_1)_{i,j} & 0 & \dots & 0 \\ 0 & (M_2)_{i,j} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & (M_v)_{i,j} \end{pmatrix} N_j$$

is a $vk \times vk$ type II matrix.

Diţă's construction is $(M, \ldots, M) \otimes (N_1, \ldots, N_k)$.

Theorem (Hosoya and Suzuki, 2003)

A type II matrix is equivalent to $(M_1, \ldots, M_n) \otimes (N_1, \ldots, N_k)$ if and only if $J_k \otimes I_n \in P\mathcal{N}_W P^T$ for some permutation matrix P.

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Theorem (Hosoya and Suzuki, 2003)

A type II matrix is equivalent to $(M_1, \ldots, M_n) \otimes (N_1, \ldots, N_k)$ *if and only if* $J_k \otimes I_n \in P\mathcal{N}_W P^T$ *for some permutation matrix* P.

For Case (iii) to (v), $\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}^{\otimes 2}$ is the only complex Hadamard matrix that is of Diţă-type.

Distance regular cover of K_n

Theorem

If a distance regular cover of K_n gives a complex Hadamard matrix then $n \leq 16$.

C_6
the cube
the line graph of the Petersen graph

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