The circular law for random matrices with independent log-concave rows

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Definition

For an $n \times n$ matrix A let μ_A denote its spectral measure, i.e.

$$\mu_A = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(A)},$$

where $\lambda_i(A)$ are the eigenvalues of A.

Theorem (Tao, Vu (2008))

Let $(X_{ij})_{i,j<\infty}$ be an infinite array of i.i.d. mean zero, variance one complex random variables. Let $A_n = (X_{ij})_{i,j\le n}$. Then the spectral measure of $n^{-1/2}A_n$ converges almost surely as $n \to \infty$ to the uniform measure on the unit disc.

Previous contributions:

Ginibre, Mehta, Girko, Edelman, Bai, Götze-Tikhomirov, Pan-Zhou

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The general approach

- Reduction to the Hermitian matrix $M_n = (n^{-1/2} A_n - z\mathrm{Id})(n^{-1/2} A_n - z\mathrm{Id})^*$
- Needed: bounds on the smallest singular value of $n^{-1/2} A_n - z\mathrm{Id}$.

Question:

- Can the independence assumption on the entries of A_n be relaxed?
- The first idea: independent entries –> independent rows?
- What could step in for independence? Looking for some geometric conditions.

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Natural candidates for the rows:

- Random vectors distributed on ℓ_p^n balls (properly normalized)

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Natural candidates for the rows:

- Random vectors distributed on ℓ_p^n balls (properly normalized)
- More generally isotropic log-concave vectors

Isotropicity, log-concavity

- A random vector X in $\mathbb{R}^n$ is **isotropic** if
$$\mathbb{E}X = 0$$
and
$$\mathbb{E}X \otimes X = \mathrm{Id}$$
or equivalently for all $y \in \mathbb{R}^n$,
$$\mathbb{E}\langle X, y\rangle^2 = |y|^2.$$
- A random vector X in $\mathbb{R}^n$ is log-concave if its law μ satisfies a Brunn-Minkowski type inequality
$$\mu(\theta A + (1-\theta)B) \geq \mu(A)^\theta \mu(B)^{1-\theta}.$$

Theorem (Borell)

A random vector not supported on any $(n-1)$ dimensional hyperplane is log-concave iff it has density of the form $\exp(-V(x))$, where $V\colon \mathbb{R}^n \to (-\infty, \infty]$ is convex.

Theorem (R.A. (2010–2011))

Let A_n be a sequence of $n \times n$ random matrices with independent rows $X_1^{(n)}, \ldots, X_n^{(n)}$ (defined on the same probability space). Assume that for each n and $i \leq n$, $X_i^{(n)}$ has a log-concave isotropic distribution. Then, with probability one, the spectral measure $\mu_{\frac{1}{\sqrt{n}} A_n}$ converges weakly to the uniform distribution on the unit disc.

Strategy of proof (Girko)

Definition

Let μ be a probability measure on $\mathbb{C}$ integrating $\log(|\cdot|)$ at infinity. The logarithmic potential of μ is defined as

$$U_\mu(z) = \int_{\mathbb{C}} \log(|x - z|) d\mu(x).$$

Fact

$$\mu = -\frac{1}{2\pi} \Delta U_\mu.$$

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Let μ be a probability measure on C integrating $\log(|\cdot|)$ at infinity. The logarithmic potential of μ is defined as

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For the empirical spectral measure of $n^{-1/2}A_n$,

$$U_{\mu_n}(z) = \frac{1}{n} \log|\det(n^{-1/2}A_n - z)| = \frac{1}{2n} \log|\det(A_n - z)|^2$$
$$= \frac{1}{2} \int \log x d\nu_z(x),$$

where ν_z is the empirical spectral measure of the (Hermitian) matrix $(n^{-1/2}A_n - z)(n^{-1/2}A_n - z)^*$.

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Strategy

- Prove that $(\mu_n)_n$ is tight and ν_n converge weakly. Use the log-potential to identify the limit.

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Strategy

- Prove that $(\mu_n)_n$ is tight and ν_n converge weakly. Use the log-potential to identify the limit.
- Problem: singularities of the logarithm

Theorem (Replacement principle - Tao, Vu (2008))

Suppose for each n that A_n, B_n are ensembles of $n \times n$ random matrices defined on a common probability space. Assume that

(i) *The expression*

$$\frac{1}{n^2}\|A_n\|_{HS}^2 + \frac{1}{n^2}\|B_n\|_{HS}^2$$

is bounded almost surely,

(ii) *For almost all complex numbers z,*

$$\frac{1}{n}\log|\det(\frac{1}{\sqrt{n}}A_n - z\mathrm{Id})| - \frac{1}{n}\log|\det(\frac{1}{\sqrt{n}}B_n - z\mathrm{Id})|$$

converges almost surely to zero.

Then $\mu_{\frac{1}{\sqrt{n}}A_n} - \mu_{\frac{1}{\sqrt{n}}B_n}$ converges almost surely to 0.

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To prove the circular law one checks the hypothesis e.g. with $B_n = (g_{ij})$, where g_{ij} – i.i.d. $\mathcal{N}(0,1)$.

Log-concave toolkit

Theorem (Prekopa-Leindler (1970's))

Marginals of log-concave isotropic random vectors are themselves isotropic and log-concave.

Theorem (Hensley (1980))

The density of a one-dimensional variance one log-concave variable is bounded by a universal constant.

Theorem (Klartag's thin shell concentration (2007))

Let X be an isotropic log-concave random vector in $\mathbb{R}^n$. There exist numerical positive constants C and c such that for all $\varepsilon \in (0, 1)$,

$$\mathbb{P}\left(\left|\frac{|X|^2}{n} - 1\right| \geq \varepsilon\right) \leq C \exp(-c\varepsilon^C n^c).$$

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- It remains to show that for each z, with probability one,

$$\frac{1}{n}\log|\det(\frac{1}{\sqrt{n}}A_n - z\mathrm{Id})| - \frac{1}{n}\log|\det(\frac{1}{\sqrt{n}}B_n - z\mathrm{Id})| \to 0.$$

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- Let X_i be the rows of A_n, Z_i the rows of $n^{-1/2}A_n - z\mathrm{Id}$, Y_i – rows of $n^{-1/2}B_n - z\mathrm{Id}$

$$\frac{1}{n}\log|\det(\frac{1}{\sqrt{n}}A_n - z\mathrm{Id})| = \frac{1}{n}\sum_{i=1}^{n}\log \mathrm{dist}(Z_i, span\{Z_j\}_{j<i})$$

$$\frac{1}{n}\log|\det(\frac{1}{\sqrt{n}}B_n - z\mathrm{Id})| = \frac{1}{n}\sum_{i=1}^{n}\log \mathrm{dist}(Y_i, span\{Y_j\}_{j<i})$$

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  - One takes care of $i > n - n^{\varepsilon}$ by employing the bound on $\max\{|Z_i|, |Y_i|\}$ and the smallest singular value –> Hensley

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  - One can discard $i \in [(1-\delta)n, n - n^{\varepsilon}]$ using an estimate for a distance of a row to a subspace of codimension at least n^{ε} –> Klartag
  - For the sum over $i \in (1-\delta)n$ one can use convergence of the empirical spectral measures of $(n^{-1/2}A_n - z\mathrm{Id})(n^{-1/2}A_n - z\mathrm{Id})^*$ and $(n^{-1/2}B_n - z\mathrm{Id})(n^{-1/2}B_n - z\mathrm{Id})^*$ –> Klartag

Bounds on the smallest singular value

Proposition

Let A_n be an $n \times n$ matrix with independent log-concave isotropic rows and let M_n be any deterministic matrix. Let σ_n be the smallest singular value of $A_n + M_n$. Then with probability at least $1 - n^{-2}$,

$$\sigma_n \geq cn^{-4}.$$

Proof (now standard).

Let X_i be the rows of $A_n + M_n$. We have

$$\sigma_n \geq \frac{1}{\sqrt{n}} \min_{i \leq n} (\mathrm{dist}(X_i, span\{X_j\}_{j \neq i}).$$

This can be easily bounded by conditioning and using the fact that the densities of one dimensional marginals are bounded. □

Digression: $M_n = 0$

Theorem (Guédon, Litvak, Pajor, Tomczak-Jaegermann, R.A. (2010))

Let A_n be an $n \times n$ matrix with independent log-concave isotropic rows and let σ_n be the smallest singular value of A_n. Then for every $\varepsilon \in (0,1)$,

$$\mathbb{P}\left(\sigma_n \le c\varepsilon n^{-1/2}\right) \le C\varepsilon \log^2\left(\frac{2}{\varepsilon}\right).$$

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Problems:

- get rid of the log,
- extend to nonzero M_n (for Gaussian matrix - Sankar, Spielman, Teng (2003)).

Distance from a subspace

We need a good lower estimate on $\mathrm{dist}(\mathrm{X}, \mathrm{E})$, where E is a deterministic subspace of $\mathbb{C}^n$ of codimension $k \geq n^\alpha$.

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We need a good lower estimate on $\mathrm{dist}(\mathrm{X}, \mathrm{E})$, where E is a deterministic subspace of $\mathbb{C}^n$ of codimension $k \geq n^\alpha$. For $\mathbb{R}^n$ it follows directly from Klartag's result, since $P_{E^c}X$ is an isotropic log-concave random vector on E^c (by Prekopa-Leindler) and thus

$$\mathbb{P}\Big(|P_{E^c}X|^2 - k| \geq \varepsilon k\Big) \leq C\exp(-c\varepsilon^C k^c).$$

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For the general case we can use

Lemma (after Pajor&Pastur)

For any $p \geq 1$ and any complex matrix A with $\|A\| = \|A\|_{\ell_2 \to \ell_2} \leq 1$,

$$\mathbb{E}|\langle AX, X\rangle - \mathrm{tr}A|^p \leq C_p n^{(1-\beta)p},$$

where C_p depends only on p and $\beta > 0$ is a universal constant.

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- For large p it gives $\mathrm{dist}(\mathrm{X}, \mathrm{E}) \geq \mathrm{c}\sqrt{\mathrm{k}}$ with probab. high enough for the union bound and Borel-Cantelli lemma (for $\alpha \in (1-\beta, 1)$).

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- For large p it gives $\mathrm{dist}(X, E) \geq c\sqrt{k}$ with probab. high enough for the union bound and Borel-Cantelli lemma (for $\alpha \in (1-\beta, 1)$).
- The proof relies on reduction to real diagonal matrices (log-concavity and isotropicity is rotationally invariant) and then by convexity to diagonal ± 1 matrices. It uses Borell's lemma for truncation and Klartag's inequality for bounding the essential part.

Distance from a subspace – alternate approach

Instead of Klartag's result one may also use the following

Theorem (Paouris (2009))

Let X be an isotropic log-concave random vector in $\mathbb{R}^n$ and let A be an $n \times n$ real nonzero matrix. Then for $y \in \mathbb{R}^n$ and $\varepsilon \in (0, c_1)$,

$$\mathbb{P}(|AX - y| \leq \varepsilon \|A\|_{HS}) \leq \varepsilon^{c_1(\|A\|_{HS}/\|A\|)},$$

where $c_1 > 0$ is a universal constant.

In our case (after passing to real matrices) $\|A\|_{HS} = \sqrt{k}$, $\|A\| \leq 1$.

Limiting spectral distribution for hermitian matrices

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- By general properties of random matrices with independent rows(exponential concentration for the Stieltjes transform), it is enough to prove the convergence of expected spectral measure.

Lemma (folklore(?)– Corollary to Azuma's inequality)

Let A be any $n \times N$ random matrix with independent rows and let $S\colon \mathbb{C}^+ \to \mathbb{C}$ be the Stieltjes transform of the spectral measure of $H = AA^*$. Then for any $\alpha = x + iy \in \mathbb{C}_+$ and any $\varepsilon > 0$,

$$\mathbb{P}(|S_n(\alpha) - \mathbb{E}S_n(\alpha)| \geq \varepsilon) \leq C\exp(-cn\varepsilon^2 y^2).$$

Theorem (R.A. (2011), following Dozier-Silverstein)

Let $N = N_n$ and assume that $n/N \to c > 0$. Let R_n be a deterministic $n \times N$ matrix such that the spectral measure of $\frac{1}{N} R_n R_n^*$ converges to some probability measure H. Let A_n be an $n \times N$ random matrix with independent rows $X_i = X_i^{(n)}$ such that

$$\max_{i \le n} \sup_{\|C\| \le 1} \frac{1}{N} \mathbb{E} |\langle C X_i, X_i \rangle - \operatorname{tr} C| = o(1).$$

Then the spectral measure of the matrix $M_n = \frac{1}{N}(R_n + A_n)(R_n + A_n)^*$ converges a.s. to a deterministic probability measure μ, whose Stieltjes transform $S(z) = \int_0^\infty \frac{1}{x - z} \mu(dx)$ is characterized by

$$S(z) = \int_0^\infty \frac{1}{\frac{t}{1 + cS(z)} - (1 + S(z))z + 1 - c} H(dt).$$

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- The Marchenko-Pastur theorem for general matrices with independent rows such that

$$\max_{\|C\|\leq 1} \frac{1}{N^2}\mathbb{E}|\langle CX, X\rangle - \mathrm{tr}\, C|^2 = o(1)$$

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Remarks

- The Marchenko-Pastur theorem for random matrices with independent rows distributed uniformly on ℓ_p^n balls was proved by Aubrun (2006) by reduction to the independent case
- The Marchenko-Pastur theorem for general matrices with independent rows such that

$$\max_{\|C\|\le 1} \frac{1}{N^2}\mathbb{E}|\langle CX, X\rangle - \mathrm{tr}\, C|^2 = o(1)$$

  was obtained by Pajor and Pastur (2007)
- If one is interested in the expected spectral distribution (and not a.e. convergence), one can obtain results similar to those by Götze-Tikhomirov for matrices with a martingale structure.

Further extensions (work in progress)

Definition

A random vector $X = (X_1, \ldots, X_n)$ is called **unconditional** *if its distribution is equal to the distribution of $(\varepsilon_1 X_1, \ldots, \varepsilon_n X_n)$ for any choice of $\varepsilon_i \in \{-1, +1\}$.*

- If in addition to independence of rows one assumes unconditionality then one can obtain the circular law under the assumption that projections of rows on coordinate subspaces are sufficiently (polynomially) concentrated.
- the general case would require bounds on the smallest singular value for square matrices with independent isotropic rows with some concentration property (open).
- if one assumes unconditionality of the matrix law (in the standard basis) and some concentration properties, then one should be able to get rid of the independence of rows (partial results).

Theorem (R.A. (2010))

Let $A_n = [X_{ij}^{(n)}]_{1 \le i \le n, 1 \le j \le n}$*. Let us assume that the following assumptions are satisfied*

(A1) *for every* $k \in N$*,* $\sup_n \max_{i \le n, j \le n} \mathbb{E}|X_{ij}^{(n)}|^k < \infty$*,*

(A2) *for every* n, i, j*,* $\mathbb{E}(X_{ij}^{(n)}|\mathcal{F}_{ij}) = 0$*, where* $\mathcal{F}_{ij}$ *is the* σ*-field generated by* $\{X_{kl}^{(n)} : (k,l) \neq (i,j)\}$*,*

(A3) $|R_n|/\sqrt{n}, |C_n|/\sqrt{n} \to 1$ *in probability, where* R_n *and* C_n *are resp. random row and column of* A_n*.*

Then the expected spectral measure of

$$(n^{-1/2}A_n - z\mathrm{Id})(n^{-1/2}A_n - z\mathrm{Id})^*$$

converges to a measure which does not depend on the distribution of A_n*.*

Thank you