Tail estimates for sums of independent log-concave random vectors

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Log-concave vectors

A random vector in $\mathbb{R}^n$ is called *logarithmically concave* (*log-concave* in short) if for any compact nonempty sets $A, B \subset \mathbb{R}^n$ and $\lambda \in [0,1]$,

$$\mathbb{P}(X \in \lambda A + (1-\lambda)B) \geq \mathbb{P}(X \in A)^{\lambda}\mathbb{P}(X \in B)^{1-\lambda}.$$

Theorem (Borell)

A random vector X with a full dimensional support is log-concave iff it has a log-concave density, i.e a density of the form $e^{-h(x)}$ with $h\colon \mathbb{R}^d \to (-\infty, \infty]$ convex.

Examples of log-concave vectors

- Gaussian vectors
- Vectors with independent log-concave coordinates (in particular vectors with the product exponential distribution)
- Vectors uniformly distributed on convex bodies
- Affine images of log-concave vectors
- Sums of independent log-concave vectors

It may be shown that the class of log-concave distributions is the smallest class that contains uniform distributions on convex bodies and is closed under affine transformations and weak limits.

Notation

For $x \in \mathbb{R}^n$,

- $|x| = \|x\|_2 = \left(\sum_{i=1}^n x_i^2 \right)^{1/2}$
- $\|x\|_r = \left(\sum_{i=1}^n |x_i|^r \right)^{1/r}$, $1 \leq r < \infty$, $\|x\|_\infty = \max_i |x_i|$
- $P_I x$ - canonical projection of x onto $\{y \in \mathbb{R}^n \colon \ \mathrm{supp}(y) \subset I\}$, $\emptyset \neq I \subset \{1, \ldots, n\}$.

For a random variable S and $p > 0$, $\|S\|_p := (\mathbb{E}|S|^p)^{1/p}$

We say that a random vector $X = (X_1, \ldots, X_n)$ is *isotropic* if $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i X_j = \delta_{i,j}$.

Paouris Large Deviation for Euclidean Norm

Theorem (Paouris'06)

For any isotropic log-concave vector X in $\mathbb{R}^n$,

$$\mathbb{P}(|X| \geq t) \leq \exp\Big(-\frac{1}{C}t\Big) \quad \textit{for } t \geq C\sqrt{n},$$

equivalently

$$(\mathbb{E}|X|^p)^{1/p} \leq C\Big(\sqrt{n} + p\Big) \quad \textit{for } p \geq 2.$$

Problem. What is the concentration for other norms of X? In particular how it is for $\|X\|_r$?

Concentration of l_r norms, $1 \leq r < 2$

Case $1 \leq r \leq 2$ reduces to the Paouris result for $r = 2$, since by the Hölder's inequality $\|X\|_r \leq n^{1/r-1/2}|X|$. Thus

$$(\mathbb{E}\|X\|_r^p)^{1/p} \leq C(n^{1/r} + n^{1/r-1/2}p)$$

and

$$\mathbb{P}(\|X\|_r \geq t) \leq \exp\Big(-\frac{1}{C}tn^{1/2-1/r}\Big) \quad \text{for } t \geq Cn^{1/r}.$$

It is not hard to construct examples showing that these bounds are optimal.

Concentration of l_r norms, $r > 2$

Example If $X_1, \ldots, X_n$ are independent symmetric exponential r.v.'s with variance one then

$$(\mathbb{E}\|X\|_r^p)^{1/p} \geq \frac{1}{C}(rn^{1/r} + p) \quad \text{for } p \geq 2, r \geq 2, n \geq C^r.$$

Problem Can we reverse this bound?
We have

$$\|X\|_r = \Big(\sum_{i=1}^n |X_i|^r\Big)^{1/r} = \Big(\sum_{i=1}^n |X_i^*|^r\Big)^{1/r} \leq \Big(2\sum_{k=0}^{s-1} 2^k |X_{2^k}^*|^r\Big)^{1/r},$$

where $s = \lfloor \log_2 n \rfloor$ and $X_1^* \geq X_2^* \geq \ldots \geq X_n^*$ we denote the nonincreasing rearrangement of $|X_1|, \ldots, |X_n|$ (order statistics of vectors X).
So to get concentration inequalities for l_r norms we may look at the tail inequalities for X_l^*, $1 \leq l \leq n$.

Union bound

We have for isotropic logconcave vectors X,

$$\mathbb{P}(X_k^* \geq t) = \mathbb{P}\Big(\bigcup_{i_1<\ldots<i_k} \{|X_{i_1}| \geq t, \ldots, |X_{i_k}| \geq t\}\Big)$$
$$\leq \sum_{i_1<\ldots<i_k} \sum_{\eta_1=\pm1,\ldots,\eta_k=\pm1} \mathbb{P}(\eta_1 X_{i_1} \geq t, \ldots, \eta_k X_{i_k} \geq t)$$
$$\leq \sum_{i_1<\ldots<i_k} \sum_{\eta_1=\pm1,\ldots,\eta_k=\pm1} \mathbb{P}\Big(\frac{1}{\sqrt{k}}(\eta_1 X_{i_1} + \ldots + \eta_k X_{i_k}) \geq t\sqrt{k}\Big)$$
$$\leq \binom{n}{k} 2^k \exp\Big(-\frac{1}{C} t\sqrt{k}\Big).$$

Therefore

$$\mathbb{P}(X_k^* \geq t) \leq \exp\Big(-\frac{1}{C} t\sqrt{k}\Big) \quad \text{for } t \geq C\sqrt{k}\log\Big(\frac{en}{k}\Big).$$

The threshold $\sqrt{k}\log(\frac{en}{k})$ is very bad.
If coordinates of X_i are independent symmetric exponential r.v. with variance 1 then $\mathrm{Med}(X_k^*) \sim \log(en/k)$ for $k \leq n/2$.
The right threshold should be $\log(en/k)$, not $\sqrt{k}\log(\frac{en}{k})$.

Order Statistics for isotropic log-concave vectors

Theorem

Let X be n-dimensional log-concave isotropic vector. Then

$$\mathbb{P}(X_k^* \geq t) \leq \exp\Big(-\frac{1}{C}\sqrt{k}t\Big) \quad \text{for } t \geq C\log\Big(\frac{en}{k}\Big).$$

The proof is based on the estimate of moments of the process

$$N_X(t) := \sum_{i=1}^{n} \mathbb{1}_{\{X_i \geq t\}} \quad t \geq 0.$$

Theorem

For any isotropic log-concave vector X and $p \geq 1$ we have

$$\mathbb{E}(t^2 N_X(t))^p \leq (Cp)^{2p} \quad \text{for } t \geq C\log\Big(\frac{nt^2}{p^2}\Big).$$

Estimate for N_X implies tail inequality for X_k^*

Recall:

$$\mathbb{E}(t^2 N_X(t))^p \leq (Cp)^{2p} \quad \text{for } t \geq C\log\left(\frac{nt^2}{p^2}\right),\ p \geq 1.$$

where

$$N_X(t) := \sum_{i=1}^{n} \mathbb{1}_{\{X_i \geq t\}} \quad t \geq 0.$$

To get estimate for order statistics we observe that $X_k^* \geq t$ implies that $N_X(t) \geq k/2$ or $N_{-X}(t) \geq k/2$ and vector $-X$ is also isotropic and log-concave. Estimates for N_X and Chebyshev's inequality gives

$$\mathbb{P}(X_k^* \geq t) \leq \left(\frac{2}{k}\right)^p (\mathbb{E}N_X(t)^p + \mathbb{E}N_{-X}(t)^p) \leq 2\Big(\frac{Cp}{t\sqrt{k}}\Big)^{2p}$$

provided that $t \geq C\log(nt^2/p^2)$. We take $p = \frac{1}{eC}t\sqrt{k}$ and notice that the restriction on t follows by the assumption that $t \geq C\log(en/k)$.

Estimate for N_X implies Paouris concentration

Proposition

Suppose that X is a random vector in $\mathbb{R}^n$ such that

$$\mathbb{E}(t^2 N_{UX}(t))^l \le (A_1 l)^{2l} \quad \textit{for } t \ge A_2,\ l \ge \sqrt{n},\ U \in O(n),$$

where A_1, A_2 are finite constants. Then

$$\mathbb{P}(|X| \ge t\sqrt{n}) \le \exp\Big(-\frac{1}{CA_1} t\sqrt{n}\Big) \quad \textit{for } t \ge \max\{CA_1, A_2\}.$$

Idea of the proof. For any $U_1, \ldots, U_l \in O(n)$,

$$\mathbb{E}\prod_{i=1}^{l} N_{U_i X}(t) \le \Big(\prod_{i=1}^{l} \mathbb{E} N_{U_i X}(t)^l\Big)^{1/l} \le \Big(\frac{A_1 l}{t}\Big)^{2l} \quad \text{for } l \ge \sqrt{n}.$$

If $U_1, \ldots, U_l$ are random rotations then one may show that

$$\mathbb{E}_X \mathbb{E}_U \prod_{i=1}^{l} N_{U_i X}(t) = \mathbb{E}_X(\mathbb{E}_{U_1} N_{U_1 X}(t))^l \ge n^l C^{-l} \mathbb{P}(|X| \ge 2t\sqrt{n})$$

and we take $l = \lceil \sqrt{n} t / (\sqrt{eC_1} A_1) \rceil$.

Concentration of l_r norms, $r > 2$

Theorem

For any $\delta > 0$ there exist constants $C_1(\delta), C_2(\delta) \le C\delta^{-1/2}$ such that for any isotropic logconcave vector X and $r \ge 2 + \delta$,

$$(\mathbb{E}\|X\|_r^p)^{1/p} \le C_2(\delta)\Big(rn^{1/r} + p\Big) \quad \text{for } p \ge 2.$$

Equivalently

$$\mathbb{P}(\|X\|_r \ge t) \le \exp\Big(-\frac{1}{C_1(\delta)}t\Big) \quad \text{for } t \ge C_1(\delta)rn^{1/r}.$$

We suspect that there should be no dependence on δ.

Uniform Paouris-type estimate

Theorem

For any $m \le n$ and any isotropic log-concave vector X in $\mathbb{R}^n$ we have for $t \ge 1$,

$$\mathbb{P}\Big(\sup_{|I|=m} |P_I X| \ge Ct\sqrt{m}\log\Big(\frac{en}{m}\Big)\Big) \le \exp\Big(-\frac{t\sqrt{m}}{\sqrt{\log(em)}}\log\Big(\frac{en}{m}\Big)\Big).$$

Idea of the proof. We have

$$\sup_{\substack{I\subseteq\{1,\ldots,N\}\\ |I|=m}} |P_I X| = \Big(\sum_{k=1}^{m} |X_k^*|^2\Big)^{1/2} \le 2\Big(\sum_{i=0}^{s-1} 2^i |X_{2^i}^*|^2\Big)^{1/2},$$

where $s = \lceil \log_2 m \rceil$.

Weak parameter

For a vector X in $\mathbb{R}^n$ we define

$$\sigma_X(p) := \sup_{t \in S^{n-1}} (\mathbb{E}|\langle t, X\rangle|^p)^{1/p} \quad p \geq 2.$$

Examples

- For isotropic log-concave vectors X, $\sigma_X(p) \leq p/\sqrt{2}$.
- For subgaussian vectors X, $\sigma_X(p) \leq C\sqrt{p}$.
- We say that an isotropic vector X is ψ_α if $\sigma_X(p) \leq Cp^{1/\alpha}$ (uniform distributions on suitable normalized B_r^n balls are ψ_α with $\alpha = \min(r, 2)$)

Paouris theorem with weak parameter

Theorem (Paouris)

For any log-concave random vector X,

$$(\mathbb{E}|X|^p)^{1/p} \le C\Big((\mathbb{E}|X|^2)^{1/2} + \sigma_X(p)\Big) \quad \text{for } p \ge 2,$$

$$\mathbb{P}(|X| \ge t) \le \exp\Big(-\sigma_X^{-1}\Big(\frac{t}{C}\Big)\Big) \quad \text{for } t \ge C(\mathbb{E}|X|^2)^{1/2}.$$

Corollary

For any log-concave vector X in $\mathbb{R}^n$, any Euclidean norm $\| \ \|$ on $\mathbb{R}^n$ and $p \ge 1$ we have

$$(\mathbb{E}\|X\|^p)^{1/p} \le C\Big((\mathbb{E}\|X\|^2)^{1/2} + \sup_{\|t\|_* \le 1} (\mathbb{E}|\langle t, X\rangle|^p)^{1/p}\Big), \qquad (1)$$

where $(\mathbb{R}^n, \|\cdot\|_*)$ is a dual space to $(\mathbb{R}^n, \|\cdot\|)$.

It is an open problem whether (1) holds for arbitrary norms

Bounds with use of weak parameter

Theorem

For any n-dimensional log-concave isotropic vector X,

$$\mathbb{P}(X_l^* \geq t) \leq \exp\Big(-\sigma_X^{-1}\Big(\frac{1}{C}t\sqrt{l}\Big)\Big) \quad \text{for } t \geq C\log\Big(\frac{en}{l}\Big).$$

As before the proof is based on a suitable estimate of N_X:

Theorem

Let X be an isotropic log-concave vector in $\mathbb{R}^n$. Then

$$\mathbb{E}(t^2 N_X(t))^p \leq (C\sigma_X(p))^{2p} \quad \text{for } p \geq 2, t \geq C\log\Big(\frac{nt^2}{\sigma_X^2(p)}\Big).$$

Uniform bound for projections with weak parameter

Theorem

Let X be an isotropic log-concave vector in $\mathbb{R}^n$. Then for any $t \geq 1$,

$$\mathbb{P}\left(\sup_{|I|=m} |P_I X| \geq Ct\sqrt{m}\log\left(\frac{en}{m}\right)\right) \leq \exp\left(-\sigma_X^{-1}\left(\frac{t\sqrt{m}\log\left(\frac{en}{m}\right)}{\sqrt{\log(em/m_0)}}\right)\right),$$

where

$$m_0 = m_0(X,t) = \sup\left\{k \leq m \colon\ k\log\left(\frac{en}{k}\right) \leq \sigma_X^{-1}\left(t\sqrt{m}\log\left(\frac{en}{m}\right)\right)\right\}.$$

Weak parameter for convolution of log-concave measures

Proposition

Let $X^{(1)}, \ldots, X^{(d)}$ be independent isotropic log-concave vectors and $Y = \sum_{i=1}^d x_i X^{(i)}$. Then

$$\sigma_Y(p) \le C(\sqrt{p}|x| + p\|x\|_\infty). \quad \text{for } p \ge 2.$$

Sketch of the proof. Fix $t \in S^{n-1}$. Let E_i be independent symmetric exponential random variables with variance 1. The result of Borell gives $\mathbb{E}|\langle t, X^{(i)}\rangle|^p \le C^p \mathbb{E}|E_i|^p$ for $p \ge 1$. Hence

$$\begin{aligned}(\mathbb{E}|\langle t, Y\rangle|^p)^{1/p} &= \Big(\mathbb{E}\Big|\sum_{i=1}^d x_i \langle t, X^{(i)}\rangle\Big|^p\Big)^{1/p} \le C\Big(\mathbb{E}\Big|\sum_{i=1}^d x_i E_i\Big|^p\Big)^{1/p} \\ &\le C(\sqrt{p}|x| + p\|x\|_\infty),\end{aligned}$$

where the last inequality follows by the Gluskin and Kwapień bound. □

Order statistics of convolutions

Corollary

Let $X^{(1)}, \ldots, X^{(m)}$ *be independent isotropic log-concave vectors and* $Y = \sum_{i=1}^{m} x_i X^{(i)}$. *Then*

$$\mathbb{P}(Y_l^* \geq t) \leq \exp\Big(-\frac{1}{C}\min\Big\{\frac{t^2 l}{|x|^2}, \frac{t\sqrt{l}}{\|x\|_\infty}\Big\}\Big) \quad \text{for } t \geq |x|\log\Big(\frac{en}{l}\Big).$$

Uniform bound for projections of convolutions

Theorem

Let $Y = \sum_{i=1}^d x_i X^{(i)}$, where $X^{(1)}, \dots, X^{(d)}$ are independent isotropic n-dimensional log-concave vectors. Assume that $|x| \le 1$ and $\|x\|_\infty \le b \le 1$.

i) If $b \ge \frac{1}{\sqrt{m}}$, then for any $t \ge 1$,

$$\mathbb{P}\Bigg(\sup_{\substack{I \subseteq \{1,\dots,n\} \\ |I|=m}} |P_I Y| \ge Ct\sqrt{m}\log\left(\frac{en}{m}\right)\Bigg) \le \exp\left(- \frac{t\sqrt{m}\log\left(\frac{en}{m}\right)}{b\sqrt{\log(e^2 b^2 m)}}\right).$$

ii) If $b \le \frac{1}{\sqrt{m}}$ then for any $t \ge 1$,

$$\mathbb{P}\Bigg(\sup_{\substack{I \subseteq \{1,\dots,n\} \\ |I|=m}} |P_I Y| \ge Ct\sqrt{m}\log\left(\frac{en}{m}\right)\Bigg)$$
$$\le \exp\left(- \min\left\{ t^2 m \log^2\left(\frac{en}{m}\right), \frac{t}{b}\sqrt{m}\log\left(\frac{en}{m}\right)\right\}\right).$$

Uniform bound for norms of submatrices

Let A be an $d \times n$ random matrix with independent log-concave isotropic rows $X^{(1)}, \ldots, X^{(d)} \in \mathbb{R}^n$. For $k \leq d, m \leq n$ we define

$$A_{k,m} = \sup\{\|A_{|I \times J}\|_{\ell_2^m \to \ell_2^k} \colon I \subseteq \{1, \ldots, d\}, J \subseteq \{1, \ldots, n\}, \\ |I| = k, |J| = m\}$$

the maximal operator norm of a $k \times m$ submatrix of A.

Theorem

For any integers d, n, $k \leq d$, $m \leq n$ and any $t \geq 1$, we have

$$\mathbb{P}\Big(A_{k,m} \geq Ct\lambda_{mk}\Big) \leq \exp\Big(-\frac{t\lambda_{mk}}{\sqrt{\log(3m)}}\Big),$$

where

$$\lambda_{mk} = \sqrt{\log\log(3m)}\sqrt{m}\log\Big(\frac{e\max(n,d)}{m}\Big) + \sqrt{k}\log\Big(\frac{ed}{k}\Big).$$

The bound is essentially optimal up to $\sqrt{\log\log m}$ factor.

Scheme of the proof

Proof consists of two parts depending on the relation between k and the quantity

$$k' = \inf\{l \geq 1 \colon m \log(en/m) \leq l \log(ed/l)\}.$$

Step 1. Reduction to the case $k \leq k'$. We follow the method of Adamczak-Litvak-Pajor-Tomczak Jaegermann (JAMS'2010), where $A_{k,n}$ was estimated. The new ingredient is uniform Paouris-type estimate for projections of X.

Step 2. Case $k \leq k'$. Chaining argument with the use of uniform estimate for projections of convolutions. At this step we loose $\log \log m$.

RIP property for matrices with independent rows

Theorem

Let A be an $d \times n$ random matrix with independent log-concave isotropic rows $X^{(1)}, \ldots, X^{(d)} \in \mathbb{R}^n$ and $d \le n$. There exists an absolute constant $c > 0$ such that if $m \le n$ satisfies

$$m \log\log(3m) \Big(\log \frac{en}{m} \Big)^2 \le cd$$

then with high probability A satisfies RIP property of order m, i.e. every vector in $x \in \mathbb{R}^n$ with $|\mathrm{supp}(x)| \le m$ may be reconstructed from its compressed image $Ax \in \mathbb{R}^d$ by l_1-minimization method.

In the unconditional case we are able to remove $\log\log$ factor.

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Thank you for your attention