Small deviation probabilities of Gaussian processes and path regularity

Frank Aurzada
TU Berlin

Banff, October 12, 2011
High Dimensional Probability

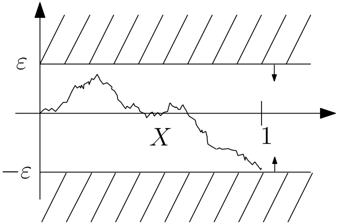
Outline

Outline

Small deviation probabilities

Let $(X_t)_{t\geq 0}$ be a stochastic process with $X_0 = 0$
Goal: find asymptotic rate of

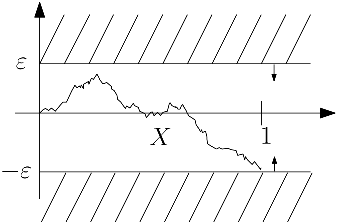
$$\mathbb{P}\left[\sup_{0\leq t\leq 1} |X_t| \leq \varepsilon\right] \approx \ ? \,, \qquad \text{with } \varepsilon \to 0$$



Small deviation probabilities

Let $(X_t)_{t\geq 0}$ be a stochastic process with $X_0 = 0$
Goal: find asymptotic rate of

$$\mathbb{P}\left[\sup_{0\leq t\leq 1} |X_t| \leq \varepsilon\right] \approx \ ? \,, \qquad \text{with } \varepsilon \to 0$$



In many examples,

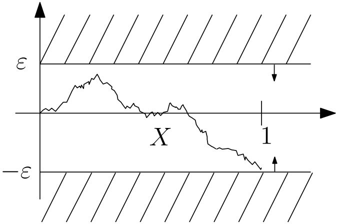
$$\mathbb{P}\left[\sup_{0\leq t\leq 1} |X_t| \leq \varepsilon\right] = e^{-\kappa\varepsilon^{-r}(1+o(1))}, \qquad \text{with } \varepsilon \to 0$$

with $\gamma > 0$ und $\kappa > 0$.

Small deviation probabilities

Let $(X_t)_{t\geq 0}$ be a stochastic process with $X_0 = 0$
Goal: find asymptotic rate of

$$\mathbb{P}\left[\sup_{0\leq t\leq 1} |X_t| \leq \varepsilon\right] \approx \; ? \,, \qquad \text{with } \varepsilon \to 0$$



Therefore, we study

$$\phi_X(\varepsilon) := -\log \mathbb{P}\left[\sup_{0\leq t\leq 1} |X_t| \leq \varepsilon\right] = \kappa\varepsilon^{-r}(1 + o(1)), \qquad \text{with } \varepsilon \to 0$$

the so-called small deviation function of X.

Small deviation probabilities

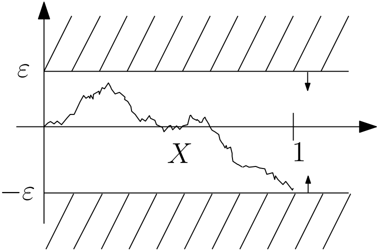
In the setup of Gaussian processes, there are various connections to:

- functional analytic quantities (later in this talk)
- entropy of function classes
- convergence rate of series representations
- coding quantities for the process
- approximation quantitites for the process
- Chung's law of the iterated logarithm
- statistical problems
- ...

Outline

Motivation

$$\mathbb{P}\left[\sup_{0\le t\le 1} |X_t| \le \varepsilon\right] \approx \ ? \, , \qquad \text{with } \varepsilon \to 0$$



Exemplary result

Theorem

Let $(X_t)_{t\in[0,1]}$ be a centred Gaussian process and n an integer.
If (a modif. of) X is n-times differentiable with $X^{(n)} \in L_2[0,1]$ then

$$\phi_X(\varepsilon) = -\log \mathbb{P}\left[\sup_{0\le t\le 1} |X_t| \le \varepsilon\right] \preceq \varepsilon^{-1/n}.$$

Exemplary result

Theorem

Let $(X_t)_{t\in[0,1]}$ be a centred Gaussian process and n an integer.
If (a modif. of) X is n-times differentiable with $X^{(n)} \in L_2[0,1]$ then

$$\phi_X(\varepsilon) = -\log \mathbb{P}\left[\sup_{0\le t\le 1} |X_t| \le \varepsilon\right] \preceq \varepsilon^{-1/n}.$$

Corollary

Let $(X_t)_{t\in[0,1]}$ be a centred Gaussian process.
If X has a $\mathcal{C}^\infty$-modif. then for any $\delta > 0$

$$\lim_{\varepsilon\to 0} \varepsilon^{\delta} \left(-\log \mathbb{P}\left[\sup_{0\le t\le 1} |X_t| \le \varepsilon\right]\right) = 0.$$

Exemplary result

Theorem

Let $(X_t)_{t\in[0,1]}$ be a centred Gaussian process and n an integer.
If (a modif. of) X is n-times differentiable with $X^{(n)} \in L_2[0,1]$ then

$$\phi_X(\varepsilon) = -\log \mathbb{P}\left[\sup_{0\le t\le 1} |X_t| \le \varepsilon\right] \preceq \varepsilon^{-1/n}.$$

Corollary

Let $(X_t)_{t\in[0,1]}$ be a centred Gaussian process.
If X has a $\mathcal{C}^\infty$-modif. then for any $\delta > 0$

$$\lim_{\varepsilon\to 0} \varepsilon^{\delta}\left(-\log \mathbb{P}\left[\sup_{0\le t\le 1} |X_t| \le \varepsilon\right]\right) = 0.$$

Now, what happens when n (above) is non-integer?

Main result

Define fractional differentiation: Let $\gamma > 0$ (recall $X_0 = 0$)

$$X_t^{(\gamma)} = x(t) \qquad \text{if} \qquad X_t = \int_0^t (t-s)^{\gamma-1} x(t) \mathrm{d}t.$$

Main result

Define fractional differentiation: Let $\gamma > 0$ (recall $X_0 = 0$)

$$X_t^{(\gamma)} = x(t) \qquad \text{if} \qquad X_t = \int_0^t (t-s)^{\gamma-1} x(t) \mathrm{d}t.$$

Theorem

Let $(X_t)_{t\in[0,1]}$ be a centred Gaussian process and $\gamma > 1/2$. If $X^{(\gamma)}$ exists and $X^{(\gamma)} \in L_2[0,1]$ then

$$\phi_X(\varepsilon) = -\log \mathbb{P}\left[\sup_{0\le t\le 1} |X_t| \le \varepsilon\right] \preceq \varepsilon^{-1/\gamma}.$$

Main result

Define fractional differentiation: Let $\gamma > 0$ (recall $X_0 = 0$)

$$X_t^{(\gamma)} = x(t) \qquad \text{if} \qquad X_t = \int_0^t (t-s)^{\gamma-1} x(t) \mathrm{d}t.$$

Theorem

Let $(X_t)_{t\in[0,1]}$ be a centred Gaussian process and $\gamma > 1/2$. If $X^{(\gamma)}$ exists and $X^{(\gamma)} \in L_2[0,1]$ then

$$\phi_X(\varepsilon) = -\log \mathbb{P}\left[\sup_{0\le t\le 1} |X_t| \le \varepsilon\right] \preceq \varepsilon^{-1/\gamma}.$$

Example: Brownian motion X is γ-times "differentiable" (Hölder), $\gamma < \frac{1}{2}$.

$$-\log \mathbb{P}\left[\sup_{0\le t\le 1} |X_t| \le \varepsilon\right] \approx \varepsilon^{-2} = \varepsilon^{-\frac{1}{1/2}}.$$

Main result

Define fractional differentiation: Let $\gamma > 0$ (recall $X_0 = 0$)

$$X_t^{(\gamma)} = x(t) \qquad \text{if} \qquad X_t = \int_0^t (t-s)^{\gamma-1} x(t) \mathrm{d}t.$$

Theorem

Let $(X_t)_{t\in[0,1]}$ be a centred Gaussian process and $\gamma > 1/2$. If $X^{(\gamma)}$ exists and $X^{(\gamma)} \in L_2[0,1]$ then

$$\phi_X(\varepsilon) = -\log \mathbb{P}\left[\sup_{0\le t\le 1} |X_t| \le \varepsilon\right] \preceq \varepsilon^{-1/\gamma}.$$

Example: Brownian motion X is γ-times "differentiable" (Hölder), $\gamma < \frac{1}{2}$.

$$-\log \mathbb{P}\left[\sup_{0\le t\le 1} |X_t| \le \varepsilon\right] \approx \varepsilon^{-2} = \varepsilon^{-\frac{1}{1/2}}.$$

What happens for different norms?

Main result: Different norms

Let us *not* recall from Lifshits/Simon'05 the notion of $||.||$ being a

"translation invariant β-self-similar p-pseudo-additive functional semi-norm in the wide sense w.r.t. the Schauder system"

Included: L_p norms ($\beta = -1/p$, $p = p$), Hölder norms ($\beta = \eta$, $p = \infty$), p-variation norm ($\beta = 0$, $p = p$), certain Besov and Sobolev norms

Main result: Different norms

Let us *not* recall from Lifshits/Simon'05 the notion of $||.||$ being a

"translation invariant β-self-similar p-pseudo-additive functional semi-norm in the wide sense w.r.t. the Schauder system"

Included: L_p norms ($\beta = -1/p$, $p = p$), Hölder norms ($\beta = \eta$, $p = \infty$), p-variation norm ($\beta = 0$, $p = p$), certain Besov and Sobolev norms

Theorem

Let $(X_t)_{t\in[0,1]}$ be a centred Gaussian process and $||.||$ such a norm not defined above, and $\gamma > \beta + 1/p + 1/2$. If $X^{(\gamma)}$ exists and $X^{(\gamma)} \in L_2[0,1]$ then

$$-\log \mathbb{P}\left[||X|| \le \varepsilon\right] \preceq \varepsilon^{-1/(\gamma-\beta-1/p)}.$$

Some applications of the main result

In combination with Li'99 (weak decorrelation inequality) one obtains:

Theorem

Let Y, R be a centred Gaussian processes (not nec. indep.) and set $X_t = Y_t + R_t$.

Some applications of the main result

In combination with Li'99 (weak decorrelation inequality) one obtains:

Theorem

Let Y, R be a centred Gaussian processes (not nec. indep.) and set $X_t = Y_t + R_t$. If

$$-\log \mathbb{P}\left[\sup_{0\le t\le 1} |Y_t| \le \varepsilon\right] \approx \varepsilon^{-r}$$

and $R^{(\gamma)} \in L_2[0,1]$ for some $\gamma > \max(1/r, 1/2)$

Some applications of the main result

In combination with Li'99 (weak decorrelation inequality) one obtains:

Theorem

Let Y, R be a centred Gaussian processes (not nec. indep.) and set $X_t = Y_t + R_t$. If

$$-\log \mathbb{P}\left[\sup_{0\le t\le 1} |Y_t| \le \varepsilon\right] \approx \varepsilon^{-r}$$

and $R^{(\gamma)} \in L_2[0,1]$ for some $\gamma > \max(1/r, 1/2)$ then

$$-\log \mathbb{P}\left[\sup_{0\le t\le 1} |X_t| \le \varepsilon\right] \approx \varepsilon^{-r}.$$

Some applications of the main result

In combination with Li'99 (weak decorrelation inequality) one obtains:

Theorem

Let Y, R be a centred Gaussian processes (not nec. indep.) and set $X_t = Y_t + R_t$. If

$$-\log \mathbb{P}\left[\sup_{0\le t\le 1} |Y_t| \le \varepsilon\right] \approx \varepsilon^{-r}$$

and $R^{(\gamma)} \in L_2[0,1]$ for some $\gamma > \max(1/r, 1/2)$ then

$$-\log \mathbb{P}\left[\sup_{0\le t\le 1} |X_t| \le \varepsilon\right] \approx \varepsilon^{-r}.$$

I.e. smoother remainder terms R do not matter; in particular if $R \in \mathcal{C}^\infty$.

Some applications of the main result

In combination with Li'99 (weak decorrelation inequality) one obtains:

Theorem

Let Y, R be a centred Gaussian processes (not nec. indep.) and set $X_t = Y_t + R_t$. If

$$-\log \mathbb{P}\left[\sup_{0\le t\le 1} |Y_t| \le \varepsilon\right] \approx \varepsilon^{-r}$$

and $R^{(\gamma)} \in L_2[0,1]$ for some $\gamma > \max(1/r, 1/2)$ then

$$-\log \mathbb{P}\left[\sup_{0\le t\le 1} |X_t| \le \varepsilon\right] \approx \varepsilon^{-r}.$$

I.e. smoother remainder terms R do not matter; in particular if $R \in \mathcal{C}^\infty$.
Standard example: FBM vs. RL process

$$c_H B_t^H = \int_0^t (t-s)^{H-1/2} \mathrm{d}B(s) + \int_{-\infty}^0 ((t-s)^{H-1/2} - (-s)^{H-1/2}) \mathrm{d}B(s).$$

Proof of the main result

Proof of the main result

Proposition (Chen/Li'03)

Let X, Y be a centred Gaussian r.v. in the Banach space E; $\mathcal{H}$ be the RKHS of Y. Then for all $\varepsilon, \lambda > 0$:

$$\mathbb{P}\left[||X||_E \leq \varepsilon\right] \geq \mathbb{P}\left[||Y||_E \leq \lambda\varepsilon\right] \cdot \mathbb{E}e^{-\frac{\lambda^2}{2}||X||_{\mathcal{H}}^2}.$$

Proof of the main result

Proposition (Chen/Li'03)

Let X, Y be a centred Gaussian r.v. in the Banach space E; $\mathcal{H}$ be the RKHS of Y. Then for all $\varepsilon, \lambda > 0$:

$$\mathbb{P}\left[||X||_E \leq \varepsilon\right] \geq \mathbb{P}\left[||Y||_E \leq \lambda\varepsilon\right] \cdot \mathbb{E}e^{-\frac{\lambda^2}{2}||X||_{\mathcal{H}}^2}.$$

Say, $E = L_\infty[0,1]$, choosing Y the Riemann-Liouville process,

$$Y_t = \int_0^t (t-s)^{\gamma-1/2-1} \mathrm{d}B(s),$$

yields that

$$||X||_{\mathcal{H}} = ||X^{(\gamma)}||_{L_2[0,1]};$$

Proof of the main result

Proposition (Chen/Li'03)

Let X, Y be a centred Gaussian r.v. in the Banach space E; $\mathcal{H}$ be the RKHS of Y. Then for all $\varepsilon, \lambda > 0$:

$$\mathbb{P}\left[||X||_E \leq \varepsilon\right] \geq \mathbb{P}\left[||Y||_E \leq \lambda\varepsilon\right] \cdot \mathbb{E}e^{-\frac{\lambda^2}{2}||X||_{\mathcal{H}}^2}.$$

Say, $E = L_\infty[0,1]$, choosing Y the Riemann-Liouville process,

$$Y_t = \int_0^t (t-s)^{\gamma-1/2-1} \mathrm{d}B(s),$$

yields that

$$||X||_{\mathcal{H}} = ||X^{(\gamma)}||_{L_2[0,1]};$$

and thus

$$\mathbb{P}\left[||X||_E \leq \varepsilon\right] \geq \mathbb{P}\left[||Y||_E \leq \lambda\varepsilon\right] \cdot \mathbb{E}e^{-\frac{\lambda^2}{2}||X^{(\gamma)}||_2^2} \geq e^{-c(\varepsilon\lambda)^{-\frac{1}{\gamma-1/2}}} \cdot q_K e^{-\frac{\lambda^2}{2}K^2}$$

having used Lifshits/Simon'05; optimizing λ, the result follows.

Similar applications

Proposition (Chen/Li'03)

Let X be a centred Gaussian r.v. in the Banach space E; Y be a $(\gamma - 1/2)$-R-L-process. Then for all $\varepsilon, \lambda > 0$:

$$\mathbb{P}\left[||X||_E \le \varepsilon\right] \ge \mathbb{P}\left[||Y||_E \le \lambda\varepsilon\right] \cdot \mathbb{E}e^{-\frac{\lambda^2}{2}||X^{(\gamma)}||_2^2}.$$

Relation between:

the SD prob. of $||X^{(\gamma)}||_2$,
i.e. of the γ-th derivative
w.r.t. the L_2 norm,

and

the SD prob. of X
w.r.t. the norm $||.||_E$

Outline

The entropy method

Let X be a centred Gaussian random variable with values in the dual Banach space $(E', ||.||)$: i.e.

$$\langle X, g \rangle \qquad \forall g \in E.$$

The entropy method

Let X be a centred Gaussian random variable with values in the dual Banach space $(E', ||.||)$: i.e.

$$\langle X, g \rangle \qquad \forall g \in E.$$

There is a linear operator $u : E \to L_2[0,1]$ belonging to X such that

$$\mathbb{E}e^{i\langle X,g \rangle} = \exp\left(-||u(g)||^2/2\right), \qquad g \in E.$$

Note: $u'(L_2[0,1])$ is the RKHS of X

The entropy method

On the one hand, we consider the small deviation function:

$$\phi_X(\varepsilon) = -\log \mathbb{P}\left[||X||_{E'} \leq \varepsilon\right] \left(= -\log \mathbb{P}\left[\sup_{0\leq t\leq 1} |X_t| \leq \varepsilon\right]\right)$$

The entropy method

On the one hand, we consider the small deviation function:

$$\phi_X(\varepsilon) = -\log \mathbb{P}\left[||X||_{E'} \leq \varepsilon\right] \left(= -\log \mathbb{P}\left[\sup_{0\leq t\leq 1} |X_t| \leq \varepsilon\right]\right)$$

On the other hand, the entropy numbers of u:

$$e_n(u) := \inf\{\varepsilon > 0 \mid \exists\ \varepsilon\text{-net of } 2^{n-1} \text{ points of } u(B_E) \text{ in } L_2[0,1]\},$$

where B_E is the unit ball in E (inverse of covering numbers).

The entropy method

Theorem (Kuelbs/Li'93, Li/Linde'99, A./Ibragimov/Lifshits/van Zanten'08)

For $r > 0$ and $\delta \in \mathbb{R}$:

$$\phi_X(\varepsilon) \preceq \varepsilon^{-r} |\log \varepsilon|^{\delta} \quad \Leftrightarrow \quad e_n(u) \preceq n^{-1/2-1/r} (\log n)^{\delta/r}$$

$$\phi_X(\varepsilon) \succeq \varepsilon^{-r} |\log \varepsilon|^{\delta} \quad \Leftrightarrow \quad e_n(u) \succeq n^{-1/2-1/r} (\log n)^{\delta/r}$$

where the first $\Leftarrow$ requires $\phi_X(\varepsilon) \preceq \phi(2\varepsilon)$.
Further, for $\delta > 0$ and $\kappa > 0$,

$$\phi_X(\varepsilon) \lesssim \kappa |\log \varepsilon|^{\delta} \quad \Leftrightarrow \quad -\log e_n(u) \gtrsim \kappa^{-1/\delta} n^{1/\delta}$$

$$\phi_X(\varepsilon) \gtrsim \kappa |\log \varepsilon|^{\delta} \quad \Leftrightarrow \quad -\log e_n(u) \lesssim \kappa^{-1/\delta} n^{1/\delta}.$$

small deviations $\leftrightarrow$ entropy numbers
(probabilistic) (functional analytic)

The entropy method

Theorem (Kuelbs/Li'93, Li/Linde'99, A./Ibragimov/Lifshits/van Zanten'08)

For $r > 0$ and $\delta \in \mathbb{R}$:

$$\phi_X(\varepsilon) \preceq \varepsilon^{-r} |\log \varepsilon|^\delta \quad \Leftrightarrow \quad e_n(u) \preceq n^{-1/2-1/r} (\log n)^{\delta/r}$$

$$\phi_X(\varepsilon) \succeq \varepsilon^{-r} |\log \varepsilon|^\delta \quad \Leftrightarrow \quad e_n(u) \succeq n^{-1/2-1/r} (\log n)^{\delta/r}$$

where the first $\Leftarrow$ requires $\phi_X(\varepsilon) \preceq \phi(2\varepsilon)$.
Further, for $\delta > 0$ and $\kappa > 0$,

$$\phi_X(\varepsilon) \lesssim \kappa |\log \varepsilon|^\delta \quad \Leftrightarrow \quad -\log e_n(u) \gtrsim \kappa^{-1/\delta} n^{1/\delta}$$

$$\phi_X(\varepsilon) \gtrsim \kappa |\log \varepsilon|^\delta \quad \Leftrightarrow \quad -\log e_n(u) \lesssim \kappa^{-1/\delta} n^{1/\delta}.$$

small deviations $\leftrightarrow$ entropy numbers
(probabilistic) (functional analytic)

The entropy method: recent results

several recent results using the above Thm in the case of slowly varying ϕ_X (exp. decreasing $e_n(u)$, slowly var. covering numbers)

- A./Ibragimov/Lifshits/van Zanten'08: spectral measure

$$\mathrm{d}F(u) = e^{-|u|^\nu}\mathrm{d}u, \qquad \tilde{F}(u) = \sum_{k\in\mathbb{Z}} e^{-|k|^\nu}\delta_{2\pi k}$$

- Kühn'11: L_2 and L_∞ case

$$\mathbb{E}X_tX_s = e^{-\sigma^2||t-s||^2}, \qquad t,s\in\mathbb{R}^d$$

- A./Gao/Kühn/Li/Shao'11+: L_2 and L_∞ case

$$\mathbb{E}X_tX_s = \frac{2^{2\beta+1}(ts)^\alpha}{(t+s)^{2\beta+1}}, \qquad t,s>0$$

- Karol'/Nazarov'11+: rather general spectral measure, $\mathbb{R}^d$ indexed, L_2 case

$$\mathrm{d}F(u) = e^{-G(u)}\mathrm{d}u$$

Relation of the main theorem to the entropy method

- with the fractional integration operator I_γ

$$X_t = I_\gamma(X^{(\gamma)})_t := \int_0^t (t-s)^{\gamma-1} X_s^{(\gamma)} ds$$

Relation of the main theorem to the entropy method

- with the fractional integration operator I_γ

$$X_t = I_\gamma(X^{(\gamma)})_t := \int_0^t (t-s)^{\gamma-1} X_s^{(\gamma)} ds$$

- u_γ – generating operator of $X^{(\gamma)}$; $u_\gamma \circ I'_\gamma = u$ generates X

Relation of the main theorem to the entropy method

- with the fractional integration operator I_γ

$$X_t = I_\gamma(X^{(\gamma)})_t := \int_0^t (t-s)^{\gamma-1} X_s^{(\gamma)} ds$$

- u_γ – generating operator of $X^{(\gamma)}$; $u_\gamma \circ I'_\gamma = u$ generates X
- The multiplicativity property of the entropy numbers yields:

$$e_n(u) = e_n(u_\gamma \circ I'_\gamma) \leq ||u_\gamma|| \cdot e_n(I_\gamma)$$

Relation of the main theorem to the entropy method

- with the fractional integration operator I_γ

$$X_t = I_\gamma(X^{(\gamma)})_t := \int_0^t (t-s)^{\gamma-1} X_s^{(\gamma)} ds$$

- u_γ – generating operator of $X^{(\gamma)}$; $u_\gamma \circ I'_\gamma = u$ generates X
- The multiplicativity property of the entropy numbers yields:

$$e_n(u) = e_n(u_\gamma \circ I'_\gamma) \leq ||u_\gamma|| \cdot e_n(I_\gamma)$$

- Now, $||u_\gamma||$ is finite by assumption and as is well-studied:

$$e_n(I_\gamma) \preceq n^{-1/2-\gamma}$$

Relation of the main theorem to the entropy method

- with the fractional integration operator I_γ

$$X_t = I_\gamma(X^{(\gamma)})_t := \int_0^t (t-s)^{\gamma-1} X_s^{(\gamma)} ds$$

- u_γ – generating operator of $X^{(\gamma)}$; $u_\gamma \circ I'_\gamma = u$ generates X
- The multiplicativity property of the entropy numbers yields:

$$e_n(u) = e_n(u_\gamma \circ I'_\gamma) \leq ||u_\gamma|| \cdot e_n(I_\gamma)$$

- Now, $||u_\gamma||$ is finite by assumption and as is well-studied:

$$e_n(I_\gamma) \preceq n^{-1/2-\gamma}$$

- The relation between e_n and ϕ_X gives

$$\phi_X(\varepsilon) \preceq \varepsilon^{-1/\gamma},$$

i.e. the main result.

Relation to entropy method: remarks, open questions

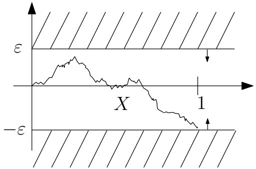
- Main result can also be proved via the entropy method. However, there is a purely probabilistic proof.
- The probabilistic proof can be extended to the setup of non-Gaussian *stable* processes.
- How about a reverse result? E.g. if for any $\delta > 0$

$$\lim_{\varepsilon \to 0} \varepsilon^{\delta} \left(-\log \mathbb{P} \left[\sup_{0 \le t \le 1} |X_t| \le \varepsilon \right] \right) = 0.$$

then (ex. a modif. of) $X \in \mathcal{C}^{\infty}$?

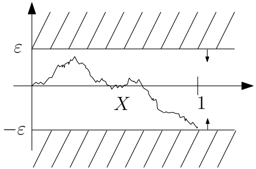
Summary

- the smoother the process, the slower the increase of the small deviation function (quantified)



Summary

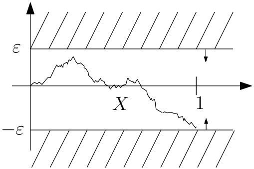
- the smoother the process, the slower the increase of the small deviation function (quantified)



- parallels the entropy method: small deviations vs. entropy numbers of the related operator

Summary

- the smoother the process, the slower the increase of the small deviation function (quantified)



- parallels the entropy method: small deviations vs. entropy numbers of the related operator
- entropy tools: here multiplicativity property vs.
probabilistic tools: here Chen/Li'99 inequality

Thank you for your attention!

Frank Aurzada
Technische Universität Berlin
page.math.tu-berlin.de/∼aurzada/