Randomly Weighted Self-Normalized Sums and Self-Normalized Lévy Processes

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High Dimensional Probability Workshop

Outline

1. Self-normalized sums
   - Motivation
   - Results
2. Self-normalized Lévy processes
   - Introduction
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Breiman 1965

Coin tossing $\longrightarrow$ random walk $S_1, S_2, \ldots$.
Put $Y_1, Y_2, \ldots$ the interarrival times between the zeros of $S_1, S_2, \ldots$.
$X, X_1, X_2, \ldots$ iid $\mathbf{P}\{X = 0\} = \frac{1}{2} = \mathbf{P}\{X = 1\}$.

$$T_n = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}$$

is the proportion of the time that the random walk spends in $[0, \infty)$.

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Arc-sine law

In this case:

$$\lim_{n\to\infty} \mathbf{P}\{T_n \le x\} = \frac{2}{\pi} \arcsin \sqrt{x}$$

In general

$Y, Y_1, Y_2, \ldots$ non-negative iid rv's with df G

$X, X_1, X_2, \ldots$ iid with df F, independent from $Y, Y_1, Y_2, \ldots$, $\mathbf{E}|X| < \infty$

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Remark

If $\mathbf{E}Y < \infty$, then

$$\frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i} = \frac{\frac{\sum_{i=1}^n X_i Y_i}{n}}{\frac{\sum_{i=1}^n Y_i}{n}} \xrightarrow{\text{a.s.}} \mathbf{E}X.$$

Theorem (Breiman, 1965)

If T_n converges in distribution for every F, and the limit is non-degenerate for at least one F, then $Y \in D(\alpha)$, for some $\alpha \in [0, 1)$.

Conjecture (Breiman)

If T_n has a non-degenerate limit for some F, then $Y \in D(\alpha)$ for some $\alpha \in [0, 1)$.

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$D(\alpha)$

Domain of attraction of an α-stable law:

$$Y \in D(\alpha) \Leftrightarrow 1 - G(x) = \frac{\ell(x)}{x^\alpha},$$

where ℓ is slowly varying.

$D(0)$

$Y \in D(0)$ if $1 - G(x)$ is slowly varying

in which case (Darling, 1952)

$$\frac{\max\{Y_i : i = 1, 2, \ldots, n\}}{\sum_{i=1}^n Y_i} \xrightarrow{\mathbf{P}} 1$$

and so

$$\frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i} \xrightarrow{\mathcal{D}} X$$

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$\mathbf{E}|X|^{2+\delta} < \infty$

Theorem (Mason & Zinn, 2005)

Assume that $\mathbf{E}|X|^{2+\delta} < \infty$. Then $T_n \to R$, where R is non-degenerate, iff $Y \in D(\alpha)$, $\alpha \in [0, 1)$.

Proof

$$T_n = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}$$

Assume that $\mathbf{E}X = 0$, then $\mathbf{E}T_n^2 = \mathbf{E}X^2 n \mathbf{E}\frac{Y_1^2}{(\sum_{i=1}^n Y_i)^2}$ y

$$\mathbf{E}\frac{Y_1^2 + \ldots + Y_n^2}{(Y_1 + \ldots + Y_n)^2} = n \int_0^\infty s\phi''(s)\phi^{n-1}(s)\mathrm{d}s,$$

where $\phi(s) = \mathbf{E}\mathrm{e}^{-sY}$.

$$n\mathbf{E}\frac{Y_1^2}{(\sum_{i=1}^n Y_i)^2} \to 1 - \alpha \ \Rightarrow Y \in D(\alpha)$$

(Fuchs & Joffe & Teugels 2002, Mason & Zinn 2005).

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Studentization

$X, X_1, X_2, \ldots$ iid

Conjecture (Logan & Mallows & Rice & Shepp, 1973)

$$\frac{\sum_{i=1}^n X_i}{\sqrt{\sum_{i=1}^n X_i^2}} \xrightarrow{\mathcal{D}} W,$$

where $\mathbf{P}\{|W| = 1\} < 1$*, iff* $X \in D(\alpha)$, $\alpha \in (0, 2]$*; if* $\alpha > 1$, $\mathbf{E}X = 0$*; if* $\alpha = 1$, $X \in D(Cauchy)$.

Results

Giné & Götze & Mason (1997): W is standard normal iff $X \in D(2)$ and $\mathbf{E}X = 0$

Chistyakov & Götze (2004): in general

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Notation

$\mathrm{id}(a, b, \nu)$ infinitely divisible distribution on $\mathbb{R}^d$ with characteristic exponent

$$\mathrm{i}u'b - \frac{1}{2}u'au + \int \left(\mathrm{e}^{\mathrm{i}u'x} - 1 - \mathrm{i}u'xI(|x| \leq 1)\right) \nu(\mathrm{d}x),$$

where $b \in \mathbb{R}^d$, $a \in \mathbb{R}^{d \times d}$ is a positive semidefinite matrix and ν is the Lévy measure.

Theorem

If along a subsequence $\{n'\}$

$$\frac{1}{a_{n'}} \sum_{i=1}^{n'} Y_i \xrightarrow{\mathcal{D}} W_2, \text{ as } n' \to \infty,$$

where $W_2 \sim \mathrm{id}(0, b, \Lambda)$, *then*

$$\left(\frac{\sum_{i=1}^{n'} X_i Y_i}{a_{n'}}, \frac{\sum_{i=1}^{n'} Y_i}{a_{n'}} \right) \xrightarrow{\mathcal{D}} (W_1, W_2), \ n' \to \infty,$$

where $(W_1, W_2) \sim \mathrm{id}(\mathbf{0}, \mathbf{b}, \Pi)$

$$\mathbf{b} = \begin{pmatrix} \left(b - \int_0^1 x \Lambda(\mathrm{d}x)\right) \mathbf{E}X + \int_{0<u^2+v^2\le 1} u \Pi(\mathrm{d}u, \mathrm{d}v) \\ \left(b - \int_0^1 x \Lambda(\mathrm{d}x)\right) + \int_{0<u^2+v^2\le 1} v \Pi(\mathrm{d}u, \mathrm{d}v) \end{pmatrix},$$

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Theorem

i.e. its characteristic function

$$\mathbf{E}e^{i(\theta_1 W_1 + \theta_2 W_2)} = \exp\Bigg\{ i(\theta_1 b_1 + \theta_2 b_2) + \int_0^\infty \int_{-\infty}^\infty \left(e^{i(\theta_1 x + \theta_2 y)} - 1 - (i\theta_1 x + i\theta_2 y)\,\mathbf{1}_{\{x^2+y^2 \le 1\}} \right) F(\mathrm{d}x/y)\,\Lambda(\mathrm{d}y) \Bigg\}.$$

Remark

$$(W_1, W_2) \stackrel{\mathcal{D}}{=} (a_1 + U, a_2 + V),$$

where $(a_1, a_2) = \left(\left(b - \int_0^1 x \Lambda(\mathrm{d}x) \right) \mathbf{E} X, b - \int_0^1 x \Lambda(\mathrm{d}x) \right)$

$$\mathbf{E} \mathrm{e}^{\mathrm{i}(\theta_1 U + \theta_2 V)} = \exp \left\{ \int_0^\infty \int_{-\infty}^\infty \left(\mathrm{e}^{\mathrm{i}(\theta_1 x + \theta_2 y)} - 1 \right) F(\mathrm{d}x/y) \Lambda(\mathrm{d}y) \right\}$$

Under the assumptions of the theorem

$$\left(\frac{\sum_{1 \leq i \leq n' t} X_i Y_i}{a_{n'}}, \frac{\sum_{1 \leq i \leq n' t} Y_i}{a_{n'}} \right)_{t>0} \xrightarrow{\mathcal{D}} (a_1 t + U_t, a_2 t + V_t)_{t>0} \; n' \to \infty,$$

where (U_t, V_t), $t \geq 0$, is the corresponding Lévy process.

Feller class

$\xi, \xi_1, \ldots$ iid with df F, $S_n = \sum_{i=1}^n \xi_i$. F is in the *Feller class*, if there exists A_n, B_n, such that every subsequence n' has a further subsequence n'', such that

$$\frac{S_{n''} - A_{n''}}{B_{n''}} \xrightarrow{\mathcal{D}} W,$$

where W is non-degenerate. F is in the *centered Feller class*, if we can choose $A_n \equiv 0$.

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Characterization

Theorem (Feller, 1966)

Y is in the Feller class, iff

$$\limsup_{x\to\infty} \frac{x^2 \mathbf{P}\{|Y| > x\}}{\mathbf{E}[Y^2 I(|Y| \le x)]} < \infty.$$

Theorem (Maller, 1979)

Y is in the centered Feller class, iff

$$\limsup_{x\to\infty} \frac{x^2 \mathbf{P}\{|Y| > x\} + x|\mathbf{E}[YI(|Y| \le x)]|}{\mathbf{E}[Y^2 I(|Y| \le x)]} < \infty.$$

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centered Feller

Proposition

Assume that Y is in the centered Feller class. Then any subsequential limit of

$$\left(\frac{W_{1,n}}{a_n}, \frac{W_{2,n}}{a_n} \right) := \left(\frac{\sum_{i=1}^n X_i Y_i}{a_n}, \frac{\sum_{i=1}^n Y_i}{a_n} \right),$$

has denstity on $\mathbb{R}^2$ (Griffin, 1986), which implies that

$$\frac{\sum_{i=1}^{n'} X_i Y_i}{\sum_{i=1}^{n'} Y_i} \xrightarrow{\mathcal{D}} \frac{W_1}{W_2}$$

has density on $\mathbb{R}$.

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has density on $\mathbb{R}$.

non-centered

$$\frac{1}{a_{n'}}\left\{\sum_{i=1}^{n'} Y_i - b_{n'}\right\} \xrightarrow{\mathcal{D}} W, \text{ as } n' \to \infty,$$

where W is non-degenerate, $b_{n'}/a_{n'} \to \infty$, then

$$\frac{1}{b_{n'}}\sum_{i=1}^{n'} Y_i \xrightarrow{\mathcal{P}} 1, \ n' \to \infty,$$

by the theorem

$$\frac{1}{b_{n'}}\sum_{i=1}^{n'} X_i Y_i \xrightarrow{\mathcal{P}} \mathbf{E}X, \ n' \to \infty,$$

and so

$$\sum_{i=1}^{n'} X_i Y_i \Big/ \sum_{i=1}^{n'} Y_i \xrightarrow{\mathcal{P}} \mathbf{E}X, \ n' \to \infty$$

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non-Feller

Y is not in the Feller class $\Leftrightarrow$ $\limsup_{x\to\infty} \frac{x^2 \mathbf{P}\{Y>x\}}{\mathbf{E} Y^2 I(Y\leq x)} = \infty$ and

$$\limsup_{x\to\infty} \frac{xE\left(YI(Y\leq x)\right)}{x^2 P\left\{Y > x\right\} + EY^2 I(Y \leq x)} < \infty$$

Assume that $\mathbf{E}|X| < \infty$ and $\mathbf{P}\{X = x_0\} > 0$. Then there exist a subsequence $\{n'\}$ such that

$$\lim_{\varepsilon\to 0} \liminf_{n'\to\infty} \mathbf{P}\left\{ \frac{\sum_{i=1}^{n'} X_i Y_i}{\sum_{i=1}^{n'} Y_i} \in (x_0 - \varepsilon, x_0 + \varepsilon) \right\} > 0.$$

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non-Feller II

$Y \in D(0)$ if $1 - G(x)$ is slowly varying $\Leftrightarrow$

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In this case (Darling, 1952)

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Outline

This is an ongoing joint work with Ross Maller.

Remark

$$(W_1, W_2) \stackrel{\mathcal{D}}{=} (a_1 + U, a_2 + V),$$

where $(a_1, a_2) = \left(\left(b - \int_0^1 x \Lambda(\mathrm{d}x) \right) \mathbf{E}X, b - \int_0^1 x \Lambda(\mathrm{d}x) \right)$

$$\mathbf{E}\mathrm{e}^{\mathrm{i}(\theta_1 U + \theta_2 V)} = \exp\left\{ \int_0^\infty \int_{-\infty}^\infty \left(\mathrm{e}^{\mathrm{i}(\theta_1 x + \theta_2 y)} - 1 \right) F(\mathrm{d}x/y) \Lambda(\mathrm{d}y) \right\}$$

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where (U_t, V_t), $t \ge 0$, is the corresponding Lévy process.

Definition

Consider the Lévy process (U_t, V_t), $t \geq 0$, with characteristic function

$$\mathbf{E} \exp\left(\mathrm{i}\theta_1 U_t + \mathrm{i}\theta_2 V_t\right) =$$

$$\exp\left(t \int_{(0,\infty)} \int_{-\infty}^{\infty} \left(e^{\mathrm{i}(\theta_1 u + \theta_2 v)} - 1 \right) F\left(\mathrm{d}u/v\right) \Lambda\left(\mathrm{d}v\right) \right)$$

where

$$\int_{-\infty}^{\infty} |x| \, F\left(\mathrm{d}x\right) < \infty$$

and Λ is a Lévy measure on $(0, \infty)$, such that $\int_0^1 y \mathrm{d}\Lambda(y) < \infty$.

$$\frac{U_t}{V_t} \xrightarrow{\mathcal{D}} ?, \;\; t \to 0 \;\text{ or }\; t \to \infty$$

Outline

Representation I

Let $\varpi_1, \varpi_2, \ldots$ be iid Exp(1) rv's, $S_i = \sum_{j=1}^{i} \varpi_j$. Independently from $\varpi_1, \varpi_2, \ldots$ let $X_1, X_2, \ldots$ be iid, with df F.

$$N(t) = \sum_{j=1}^{\infty} I\{S_j \leq t\}, \text{ Poisson process}$$

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Representation II

Theorem

For any fixed $t > 0$,

$$(U_t, V_t) \stackrel{\mathcal{D}}{=} \left(\sum_{i=1}^{\infty} X_i \varphi\left(\frac{S_i}{t}\right), \sum_{i=1}^{\infty} \varphi\left(\frac{S_i}{t}\right) \right).$$

$$T_t = \frac{U_t}{V_t} \stackrel{\mathcal{D}}{=} \frac{\sum_{i=1}^{\infty} X_i \varphi\left(\frac{S_i}{t}\right)}{\sum_{i=1}^{\infty} \varphi\left(\frac{S_i}{t}\right)}$$

$$\mathbf{E}\,|X|^{2+\delta} < \infty \Rightarrow \mathbf{E}\,|T_t|^{2+\delta} \le \mathbf{E}\,|X|^{2+\delta} < \infty.$$

Therefore $T_t \to T$ implies $\mathbf{E}T_t^2 \to \mathbf{E}T^2$.

$$\mathbf{E}T_t^2 = (\mathbf{E}X)^2 + \mathbf{Var}(X)\mathbf{E}R_t,$$

where

$$R_t = \frac{\sum_{i=1}^{\infty} \varphi^2\left(\frac{S_i}{t}\right)}{\left(\sum_{i=1}^{\infty} \varphi\left(\frac{S_i}{t}\right)\right)^2}.$$

$R_t \in [0,1] \Rightarrow \mathbf{E}R_t \to 1-\beta$

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Proposition

If

$$\mathbf{E} R_t \to 1 - \beta, \textit{ as } t \downarrow 0 \textit{ (as } t \to \infty)$$

then $\overline{\Lambda}(x)$ *is regularly varying with index* $-\beta$.

Recall

$\varpi_1, \varpi_2, \ldots$ iid Exp(1) rv's, $S_i = \sum_{j=1}^{i} \varpi_j$.

$$N(t) = \sum_{j=1}^{\infty} I\{S_j \leq t\}, \text{ Poisson process}$$

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Idea

$$R_t = \frac{\sum_{i=1}^{\infty} \varphi^2\left(\frac{S_i}{t}\right)}{\left(\sum_{i=1}^{\infty} \varphi\left(\frac{S_i}{t}\right)\right)^2} = \frac{\int_0^\infty \varphi^2(s)\,\mathrm{d}N(ts)}{\left(\int_0^\infty \varphi(s)\,\mathrm{d}N(ts)\right)^2}.$$

For $T > 0$

$$R_t(T) = \frac{\int_0^T \varphi^2(s)\,\mathrm{d}N(ts)}{\left(\int_0^T \varphi(s)\,\mathrm{d}N(ts)\right)^2}.$$

Given that $N(Tt) = n$

$$R_t(T) \stackrel{\mathcal{D}}{=} \frac{\sum_{i=1}^{n} \varphi^2(V_i)}{\left(\sum_{i=1}^{n} \varphi(V_i)\right)^2},$$

where $V_1, \ldots, V_n$ are iid Uniform$(0, T)$.

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$$\mathbf{E}\frac{\xi_1^2 + \ldots + \xi_n^2}{(\xi_1 + \ldots + \xi_n)^2} = n \int_0^\infty s\phi''(s)\phi^{n-1}(s)\mathrm{d}s,$$

where $\phi(s) = \mathbf{E}\mathrm{e}^{-s\xi}$.

Idea II

$$\mathbf{E}R_t(T) = t\int_0^\infty u\left(\int_0^T \varphi^2(s)\mathrm{e}^{-u\varphi(s)}\mathrm{d}s\right)\mathrm{e}^{-t\int_0^T(1-\mathrm{e}^{-u\varphi(s)})\mathrm{d}s}\,\mathrm{d}u$$

$$\mathbf{E}R_t(T) \to \mathbf{E}R_t$$

and so, as $T \to \infty$

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$$\mathbf{E} R_t = \int_0^\infty -tu\Phi''(u)\, \mathrm{e}^{-t\Phi(u)} \mathrm{d}u,$$

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Karamata's Tauberian Theorem (Maller & Mason, 2008)

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Theorem

Assume that for a subsequence t_k ($\to 0$ ó $\to \infty$) and for B_k

$$\frac{V_{t_k}}{B_k} \xrightarrow{\mathcal{D}} V$$

where $V \sim id(b, \Lambda_0)$, then

$$\left(\frac{U_{t_k}}{B_k}, \frac{V_{t_k}}{B_k} \right) \xrightarrow{\mathcal{D}} (U, V),$$

where $(U, V) \sim id(\mathbf{0}, \mathbf{c}, \Pi_0)$, with Lévy measure $\Pi_0 (\mathrm{d}x, \mathrm{d}y) = \mathrm{d}F\,(x/y)\, \Lambda_0\, (\mathrm{d}y)$ on $(0, \infty) \times \mathbb{R}$ y

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b\mathbf{E}X + \int_{0<u^2+v^2\le 1} u \mathrm{d}\Pi_0\,(u, v) \\ b + \int_{0<u^2+v^2\le 1} v \mathrm{d}\Pi_0\,(u, v) \end{pmatrix}.$$

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Theorem

Assume that $\mathbf{E}|X|^{2+\delta} < \infty$*. Then* U_t/V_t *has limit distribution as* $t \to 0$ *(*∞*), iff* $\overline{\Lambda}$ *is regularly varying at 0 (*∞*) with index* $-\beta \in (0, 1)$*. The df of the limit is*

$$\mathbf{P}\{U/V \le x\}$$
$$= \frac{1}{2} + \frac{1}{\pi\beta} \arctan \left[\frac{\int |u-x|^\beta \mathrm{sgn}(x-u) \mathrm{d}F(u)}{\int |u-x|^\beta \mathrm{d}F(u)} \tan \frac{\pi\beta}{2} \right].$$

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