Gaussian Summation Processes and Weighted Summation Operators on Trees

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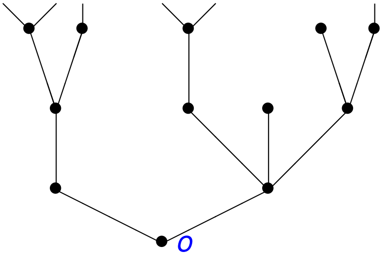
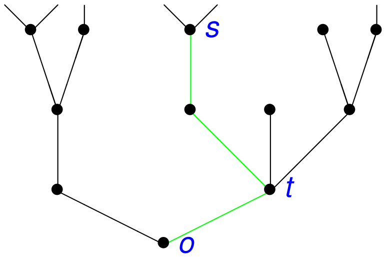
Outline

A tree



A tree T with a root o.

Partial order and levels



Partial order:

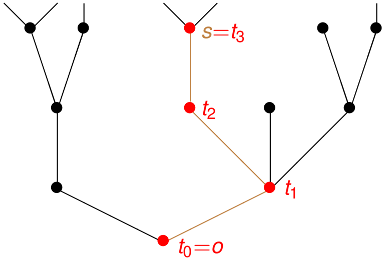
$$t \preceq s, \; s \succeq t.$$

Level number:

$$|s| := \#\{t : t \prec s\}.$$

Gaussian summation processes

We define a tree-indexed summation process $Y(s), s \in T$.



$$Y(s) = \sigma(s) \sum_{t \preceq s} \alpha(t)\, \xi(t)$$

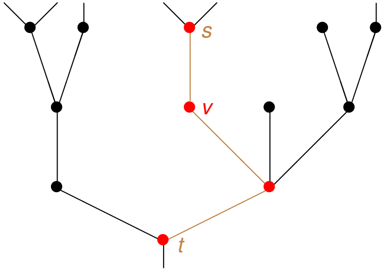
Here $\xi(\cdot)$ are independent $N(0,1)$ random variables sitting at the nodes. $\alpha(\cdot)$ - non-negative weight; $\sigma(\cdot)$ - non-negative, non-increasing weight. Used by X.Fernique (1976) in the studies of majorizing measure criteria ($\sigma = 1$). Applications in biology, chemistry, informatics, ...

Tree as a metric space

Typically, a Gaussian process Y is studied via Dudley distance on T,

$$d_Y(s,t)^2 := \mathbb{E}\,(Y(s) - Y(t))^2.$$

However, for summation process another distance is easier to handle,
We construct a distance $d(\cdot,\cdot)$ on T based on the weights α and σ.

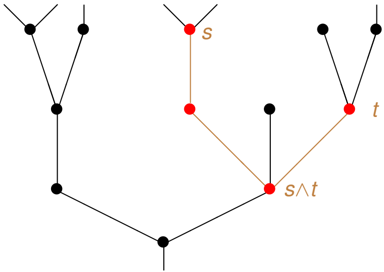


If $t \preceq s$,

$$d(s,t) := \max_{t \prec v \preceq s} \left(\sum_{t \prec u \preceq v} \alpha(u)^2 \right)^{1/2} \sigma(v).$$

Tree as a metric space (continued)

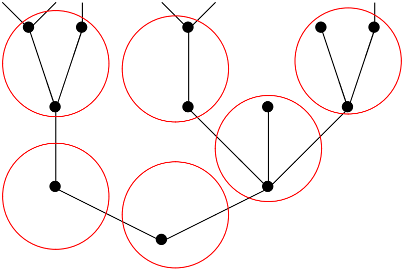
We construct a distance $d(\cdot,\cdot)$ on T based on the weights α and σ



If s and t are not comparable, then

$$d(s,t) := \max\{d(s, s\wedge t), d(t, s\wedge t)\}.$$

Covering numbers of (T, d)

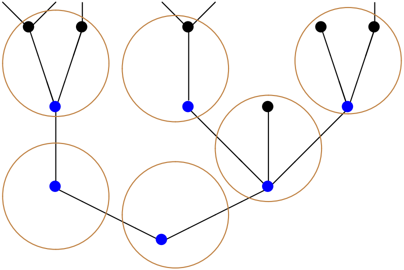


Covering numbers:

$$N(T, d, \varepsilon) := \inf\left\{ n \geq 1 : \exists\{t_j\}, T = \bigcup_{j=1}^{n} B_\varepsilon(t_j) \right\}$$

Order covering numbers of (T, d)

A new, more convenient but equivalent concept: order covering numbers



Order covering numbers:

$$\widetilde{N}(T,d,\varepsilon) := \inf\left\{ n \geq 1 : \exists\{t_j\}, T = \bigcup_{j=1}^{n} \widetilde{B}_\varepsilon(t_j) \right\}$$

where $\widetilde{B}_\varepsilon(t) := \{s \in T : d(s,t) \leq \varepsilon, s \succeq t\}$. Actually,

$$N(T,d,\varepsilon) \leq \widetilde{N}(T,d,\varepsilon) \leq N(T,d,\varepsilon/2).$$

Conditions for boundedness

Sufficient Conditions (Dudley)

Conditions

$$\int_0^\infty \sqrt{\ln N(T, d_Y, \varepsilon)}d\varepsilon < \infty$$

and

$$\int_0^\infty \sqrt{\ln N(T, d, \varepsilon)}d\varepsilon < \infty$$

are equivalent. Either of them yields $\sup_{t \in T} |Y(t)| < \infty \quad \text{a.s.}$

Necessary Conditions (Sudakov)

Conditions

$$\sup_{\varepsilon > 0} \varepsilon^2 \ln N(T, d_Y, \varepsilon) \qquad \text{and} \qquad \sup_{\varepsilon > 0} \varepsilon^2 \ln N(T, d, \varepsilon)$$

are equivalent. Either of them is necessary for $\sup_{t \in T} |Y(t)| < \infty \quad \text{a.s.}$

Boundedness: binary tree

Let T be a binary tree, and $\alpha(t) = \alpha(|t|)$, $\sigma(t) = \sigma(|t|)$.

Case $\alpha \downarrow$

Let $\alpha(\cdot)$ be non-increasing. Then $\sup_{t \in T} |Y(t)| < \infty \quad \text{a.s.}$ iff

$$\sup_n \sigma(n) \sum_{k=1}^{n} \alpha(k) < \infty.$$

Case $\alpha \uparrow$

Let $\alpha(\cdot)$ be non-decreasing. Then $\sup_{t \in T} |Y(t)| < \infty \quad \text{a.s.}$ iff

$$\sup_n \sigma(n) \sqrt{n} \left(\sum_{k=1}^{n} \alpha(k)^2 \right)^{1/2} < \infty.$$

Boundedness: binary tree (continued)

Let T be a binary tree, and $\alpha(t) = \alpha(|t|)$, $\sigma(t) = \sigma(|t|)$.

In many cases only the product $\alpha(\cdot)\sigma(\cdot)$ is important for the properties of Y

but

An example

The process

$$Y'(s) := (|s| + 1)^{-1} \sum_{t \preceq s} \xi(t)\,, \quad s \in T\,,$$

is a.s. bounded, while

$$Y''(s) := \sum_{t \preceq s} (|t| + 1)^{-1} \xi(t)\,, \quad s \in T\,,$$

is a.s. unbounded.

Problem setting

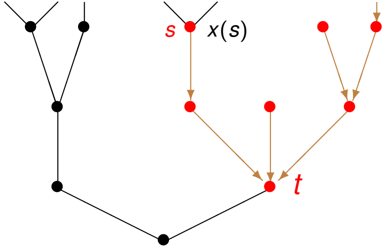
We study the asymptotic behavior of

$$\mathbb{P}\{\sup_{t \in T} |Y(t)| \leq \varepsilon\}, \qquad \text{as } \varepsilon \to 0.$$

This small deviation problem is known to be related with entropy behavior of some linear operators.

Let us describe these operators for summation process.

Weighted summation operator



Take weights $\alpha(\cdot), \sigma(\cdot)$ on T and let $V_{\alpha,\sigma} : \ell_1(T) \to \ell_2(T)$,

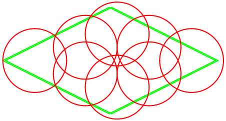
$$(V_{\alpha,\sigma}x)(t) := \alpha(t) \sum_{s \succeq t} \sigma(s)x(s), \quad t \in T,$$

Dual operator

$$(V^*_{\alpha,\sigma}x)(s) := \sigma(s) \sum_{t \preceq s} \alpha(t)x(t), \quad s \in T.$$

Dyadic entropy numbers

We measure compactness of operator $V := V_{\alpha,\sigma}$ by dyadic entropy numbers



$$e_n(V) := \inf \left\{ \varepsilon > 0 : \exists \{y_j\}, \{Vx : ||x||_1 \leq 1\} \subset \bigcup_{j=1}^{2^{n-1}} B_\varepsilon(y_j) \right\}$$

Main problem in operator language

Find the behavior of $e_n(V)$, as $n \to \infty$.

Main result

Small deviations, operator entropy, and the distance $d(\cdot,\cdot)$

Let $a > 0$, $b \geq 0$. If

$$N(T, d, \varepsilon) \approx \varepsilon^{-a} |\ln \varepsilon|^b, \qquad \text{as } \varepsilon \to 0,$$

then

$$e_n(V_{\alpha,\sigma}) \approx n^{-1/a-1/2} |\ln n|^b, \qquad \text{as } n \to \infty,$$

$$e_n(V^*_{\alpha,\sigma}) \approx n^{-1/a-1/2} |\ln n|^b, \qquad \text{as } n \to \infty,$$

$$-\ln \mathbb{P}\{\sup_{t \in T} |Y(t)| \leq \varepsilon\} \approx \varepsilon^{-a} |\ln \varepsilon|^b, \qquad \text{as } \varepsilon \to 0.$$

Arguments:

- New: evaluation of $e_n(V)$ via distance d.
- Duality theorem on entropy numbers: $e_n(V^*) \approx e_n(V)$.
- Kuelbs-Li theory relating $e_n(V^*_{\alpha,\sigma})$ to small deviations $\mathbb{P}\{\sup_T |Y| \leq \varepsilon\}$.

Bounds via weights and tree structure

Let $R(n) := \#\{t \in T : |t| = n\}$ denote the number of nodes of level n.

Small deviations for polynomial tree and polynomial weights

Assume that $R(n) \le c\, n^H$ and $\alpha(t)\sigma(t) \le c|t|^{-\gamma/2}$ with $\gamma > 1$, $H \ge 0$. Then

$$N(T, d, \varepsilon) \le Q(\varepsilon)$$

and

$$-\ln \mathbb{P}\{\sup_{t\in T} |Y(t)| \le \varepsilon\} \le Q(\varepsilon),$$

where

$$Q(\varepsilon) := C \begin{cases} \varepsilon^{-\frac{2H}{\gamma-1}}, & \gamma < H+1, \\ \varepsilon^{-2}|\ln \varepsilon|, & \gamma = H+1, \\ \varepsilon^{-\frac{2(H+1)}{\gamma}}, & \gamma > H+1. \end{cases}$$

Bounds via weights and tree structure (continued)

Small deviations for exponential tree and exponential weights

Let T be a binary tree and $\alpha(t)\sigma(t) \le c2^{-\gamma|t|}$ with $\gamma > 0$. Then

$$N(T, d, \varepsilon) \le Q(\varepsilon)$$

and

$$-\ln \mathbb{P}\{\sup_{t\in T} |Y(t)| \le \varepsilon\} \le Q(\varepsilon),$$

where

$$Q(\varepsilon) := C\, \varepsilon^{-\frac{1}{\gamma}}.$$

This example was studied in Aurzada and Lifshits (2008).

Bounds via weights and tree structure (continued)

Remark: there are extremely *interesting* and *difficult* examples for the case of exponentially growing trees and polynomially decreasing weights.

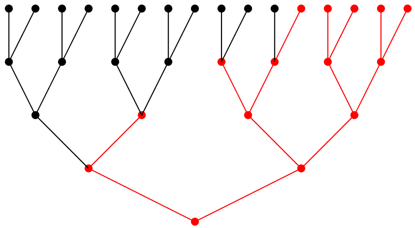
Here the study of $e_n(V_{\alpha,\sigma})$ is meaningful
but
the study of small deviations is meaningless since the summation process is unbounded,

$$\mathbb{P}\{\sup_{t\in T}|Y(t)| \le \varepsilon\} = 0, \qquad \forall\varepsilon > 0.$$

Therefore, our results for this case belong to operator theory.

Biased tree

One very interesting example is a biased tree: given an integer sequence $R(n)$ we take binary tree and for level n keep only $R(n)$ rightmost nodes.
Assuming $R(n+1) \leq 2R(n)$ we obtain a tree whose branches may die out very quickly, if $R(\cdot)$ is growing slowly.



Biased tree, $R(n) = n + 1$.

Example: bounds for biased trees

Let T be a biased tree associated with a size level sequence $R(n)$.

Small deviations for polynomial biased tree

Assume that $R(n) \approx n^H$, $\sigma(t) \equiv 1$, and $\alpha(t) = |t|^{-\gamma/2}$ with $\gamma > 1$, $H \geq 0$.
Then

$$N(T, d, \varepsilon) \approx Q(\varepsilon)$$

and

$$-\ln \mathbb{P}\{\sup_{t \in T} |Y(t)| \leq \varepsilon\} \approx Q(\varepsilon),$$

where

$$Q(\varepsilon) := C\, \varepsilon^{-\frac{2(H+1)}{\gamma}}.$$

This example is important for showing sharpness of our general estimates.

Published as

- Compactness properties of weighted summation operators on trees, STUDIA MATHEMATICA, 2011, 202, 17–47. www.arxiv.org/abs/1006.3867.
- Random Gaussian sums on trees, ELECTR. J. PROBAB., 2011, 16, 739–763. www.arxiv.org/abs/1012.2683.
- Bounds for entropy numbers for some critical operators, to appear in Trans. Amer. Math. Soc., 2010+. www.arxiv.org/abs/1002.1377.
- Compactness properties of weighted summation operators on trees – the critical case, 2010+, to appear in STUDIA MATHEMATICA. www.arxiv.org/abs/1009.2339.