How long will it take?

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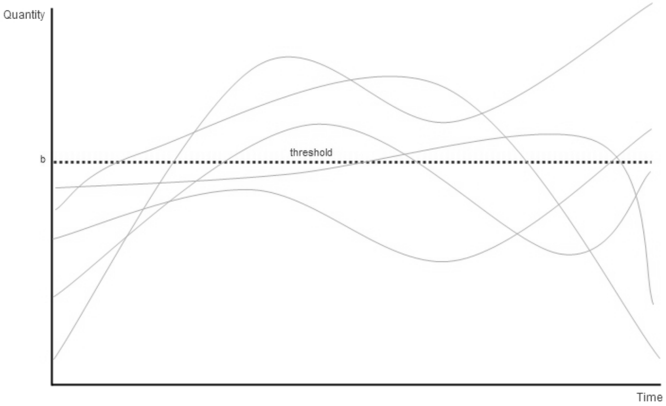
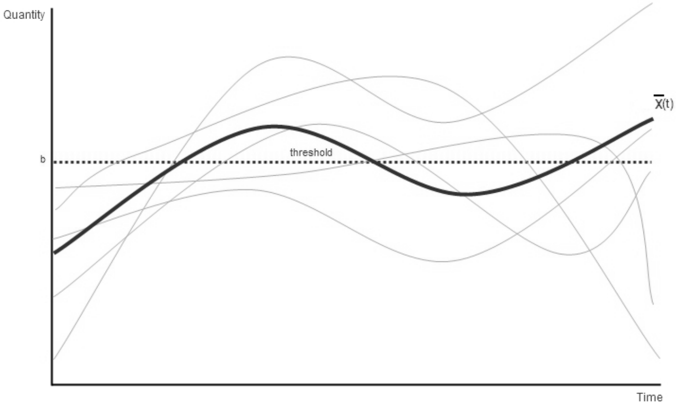
The study of the projected path of the climate system involves an assessment of the crossing of significant thresholds referred to as “impact threshold” or “tipping points”.

“Impact threshold” refers to “*any degree of change that can link the onset of a given ciritcal biophysical or socio-economeic impact to a particular climate state(s)*” (Pittock & Jones, 2000; Jones, 2001)

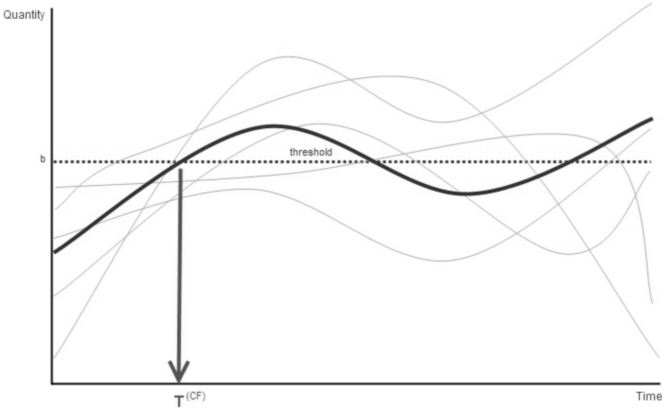
Traditional Forecast



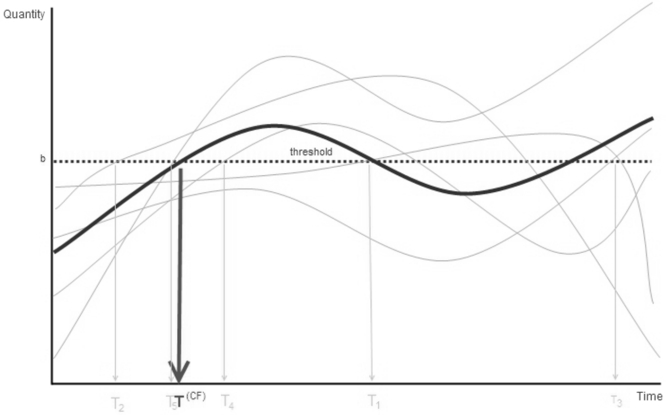
The Mean Path



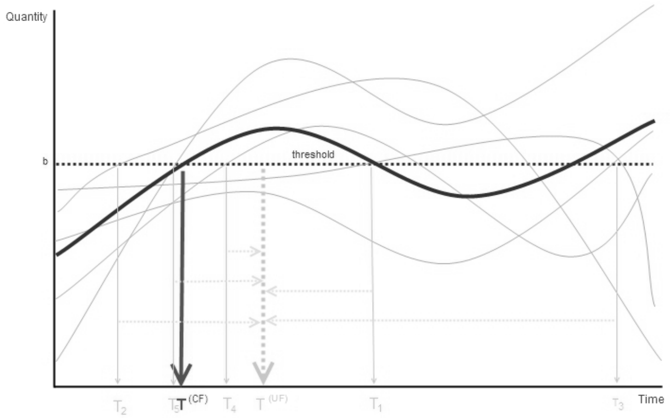
Traditional Estimator



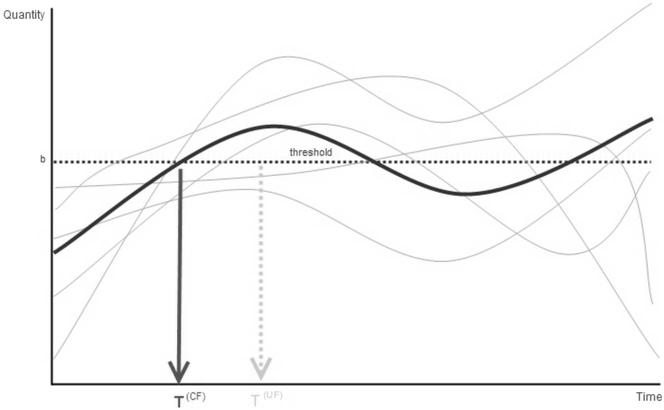
Can One Do a Better Job?



Consider the Mean of the Stopping Times T_i



Now we have two estimators: $T^{(CF)}, T^{(UF)}$



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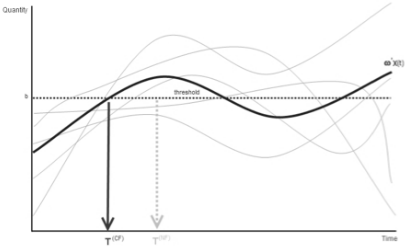

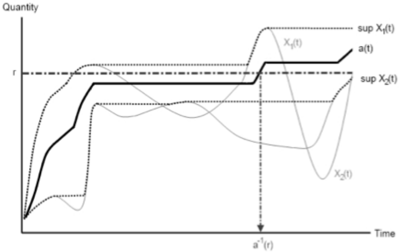

How Long will it Take?

A Projective Approach to Boundary Crossing

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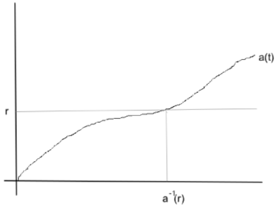
Outline

From Boundary Crossing of Non-random Functions to Boundary crossing of stochastic Processes

1. Motivation behind the estimator (Decoupling and Wald's equation)
2. General Results
3. An application motivated by cancer research.
4. An application of boundary crossing to the study of droughts

The main result consists of a natural extension of concepts of boundary crossing by nonrandom functions to the case of random processes.

To motivate the development, suppose that a problem of interest can be studied by using a continuous increasing function $a(t)$ with the time of interest t_r equal the first time $t \geq 0$ such that $a(t)$ reaches a fixed level $r > 0$.



Then, it is easy to see that

$$a(t_r) = r \quad \text{and} \quad t_r = a^{-1}(r).$$

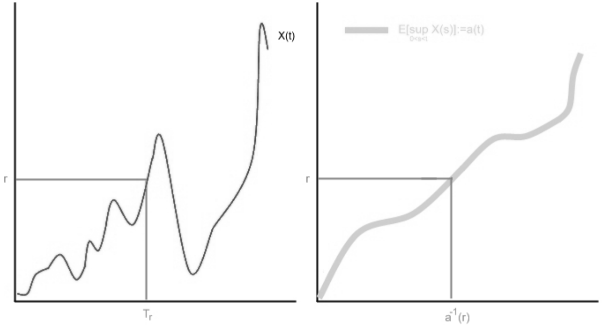


Figure: Illustration of $a(t) := E\left[\sup_{0\leq s\leq t} X(s)\right]$, $a^{-1}(r) = \inf\{s > 0 : a(s) > r\}$, $T_r = \inf\{s > 0 : X_s > r\}$.

Sample version of $a(t)$

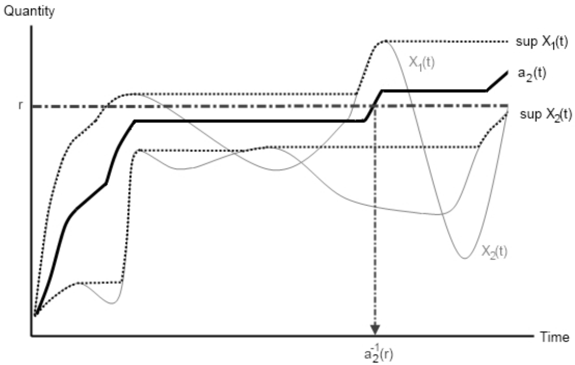


Figure: $a(t) := E\left[\sup_{0\leq s\leq t} X(s)\right]$, $a_n(t) := n^{-1}\sum_{i=1}^{n} \sup_{s\leq t} X_i(s)$
$a_n^{-1}(y) = \inf\{y > 0 : a_n(y) > r\}$ (a conservative estimator).

[THEOREM] (Wald 1944)

Let $\{X_i\}$ be iid random variables with finite mean, $S_n \triangleq \sum_{i=1}^{n} X_i$, and T a stopping time adapted to $\{X_i\}$. Then,

$$E[S_T] = E[T]E[X_1], \quad \text{(First Equation)},$$

whenever $E[T] < \infty$, $E|X_1| < \infty$.

If in addition $E[X_1] = 0$, $E[X_1^2] < \infty$, then

$$E[S_T^2] = E[T]E[X_1^2], \quad \text{(Second Equation)},$$

whenever $E[T] < \infty$.

Developed during World War II to deal with problems related to destructive sampling.

An application

Let $Pr\{X_i = 1\} = Pr\{X_i = -1\} = \frac{1}{2}$. Let $a, b > 0$ be integers.

$$T = \inf\{n : S_n = a \text{ or } -b\}.$$

Then, $E[T] = ab$.

Let $\{X_i\}$ and $\{\tilde{X}_i\}$ be two independent sequences of iid random variables with $\mathcal{L}(X_i) = \mathcal{L}(\tilde{X}_i)$ for $i \geq 1$. Set $\tilde{S}_n = \tilde{X}_1 + \tilde{X}_2 + \ldots \tilde{X}_n$.

Then,

$$E[S_T] = E[\tilde{S}_T] = E[T]E[\tilde{X}_1] = E[T]E[X_1],$$

whenever $E|X| < \infty$ and $E[T] < \infty$, and

$$E[S_T^2] = E[\tilde{S}_T^2] = E[T]E[\tilde{X}_1^2] = E[T]E[X_1^2],$$

whenever $E[T] < \infty$.

The above examples of "decoupling" equalities; see de la Peña and Giné (1999).

We recall a result of Klass concerning decoupling of processes with independent increments.

Theorem

Let $\{d_i\}_{i\geq 1}$ be a sequence of independent random random variables taking values in a Banach space, and consider $\{\tilde{d}_i\}_{i\geq 1}$, an independent copy of this sequence. Let $S_n = \sum_{i=1}^n d_i$, and similarly $\tilde{S}_n = \sum_{i=1}^n \tilde{d}_i$. Then for any $p > 0$, there exist two strictly positive constants, c_p and C_p, such that for any stopping time T with respect to S,

$$c_p E\left[\sup_{n\leq T} |\tilde{S}_n|^p\right] \leq E\left[\sup_{n\leq T} |S_n|^p\right] \leq C_p E\left[\sup_{n\leq T} |\tilde{S}_n|^p\right].$$

Moreover, C_p can be chosen equal to $20(18)^p$.

This result may be extended to continuous time processes with independent increments. In fact,

Theorem

Let $\{X_i\}_{i\geq 1}$ be a continuous stochastic processes taking values in a Banach space with independent increments. Let $\tilde{X}$ be an independent copy of X. We note

$$X_t^* = \sup_{0\leq s\leq t} \|X_s\| \quad \text{and} \quad \tilde{X}_t^* = \sup_{0\leq s\leq t} \|\tilde{X}_s\|.$$

Let $p > 0$. Suppose that for every $t > 0$, $E[X_t^*]^p < \infty$. There exist then two positive constants c_p and C_p, independent of X, such that for any stopping time T with respect to X,

$$c_p E\left[\tilde{X}_T^*\right]^p \leq E\left[X_T^*\right]^p \leq C_p E\left[\tilde{X}_T^*\right]^p.$$

Moreover, C_p can be chosen equal to $20(18)^p$; see de la Peña and Eisenbaum (1994).

A more general results follows...

Let $\{N_t = (B_t^{(1)}, B_t^{(2)}, \ldots, B_t^{(d)},), t \geq 0\}$, $1 \leq d \leq \infty$, be a vector of possibly dependent standard Brownian motions, with N_t taking values in a separable Banach space $(\mathcal{B}, \|\cdot\|)$.

Let T be a stopping time adpated to $\sigma(\{N_t\})$. Let $\{\widetilde{N}_t\}$ be an independent copy of $\{N_t\}$ with $\{\widetilde{N}_t\}$ independent of T as well.

Letting $\|N_t\|_1 = \sup_{i \leq d} |B_t^{(i)}|$ and for $p = 1$ in particular,

$$c_p E\left[\sup_{i \leq d} \left|\widetilde{B}_T^{(i)}\right|\right] \leq E\left[\sup_{t \leq T} \sup_{i \leq d} \left|B_t^{(i)}\right|\right] \leq C_p E\left[\sup_{i \leq d} \left|\widetilde{B}_T^{(i)}\right|\right].$$

The independence between T and $\widetilde{X}_t = (\widetilde{B}_t^{(1)}, \ldots, \widetilde{B}_t^{(d)})$ gives

$$E\left[\sup_{i \le d} \left|\widetilde{B}_T^{(i)}\right|\right] \le E\left[\sup_{i \le d} \left|\sqrt{T}\widetilde{B}_1^{(i)}\right|\right] = E\left[\sup_{i \le d} \left|\widetilde{B}_1^{(i)}\right|\right] = E\left[T^{1/2} E\|\widetilde{X}_1\|_1\right].$$

Using the above two results, we derive the following bounds on the average values of $T^{1/2}$.

$$E\left[T_{b,d}^{1/2}\right] \approx \frac{b^2}{E\left[\sup_{i \le d} |B_1^{(i)}|^2\right]}.$$

$$a(t) = E\left[\sup_{s\le t}\sup_{i\le d}\left\|B_s^{(i)}\right\|\right] = \sqrt{t}E\left[\sup_{s\le 1}\sup_{i\le d}\left\|B_s^{(i)}\right\|\right]$$

and

$$a^{-1}(r) = \frac{r^2}{E\left[\sup_{s\le 1}\sup_{i\le d}\left\|B_s^{(i)}\right\|\right]}$$

Theorem

For arbitrary processes with $a(t) = E[\sup_{s \leq t} X_s]$,

$$E\left[a(T_r)\right] \geq \frac{r}{2}.$$

Proof.

$$
\begin{aligned}
E[a(T_r)] &= \int_0^{\infty} \Pr\{a(T_r) > \xi\} d\xi = \int_0^{r} \Pr\{T_r > a^{-1}(\xi)\} d\xi \\
&= \int_0^{r} \Pr\{\sup_{s \le a^{-1}(\xi)} X_s \le r\} d\xi = \int_0^{r} [1 - \Pr\{\sup_{s \le a^{-1}(\xi)} X_s \ge r\}] d\xi \\
&\ge \int_0^{r} (1 - \frac{E[\sup_{s \le a^{-1}(\xi)} X_s]}{r}) d\xi = \int_0^{r} (1 - \frac{\xi}{r}) d\xi = \frac{r}{2}.
\end{aligned}
$$

□

Theorem

If, in addition, $a(\cdot)$ is concave and $X(t)$ is a continuous process with independent increments, we have

$$E[T_r] \leq a^{-1}(2r).$$

We get the right order of magnitude. Combining with the previous result, we yield

$$a^{-1}(\frac{r}{2}) \leq E[T_r] \leq a^{-1}(2r).$$

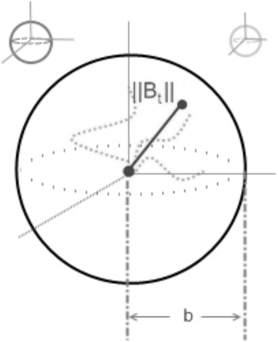
More general results including certain Markov processes and stochastic integrals $X_t = \int_0^t f(B_s, s) dB_s$ will be discussed later.

1. Example motivated by cancer research (a first step)

Let $\boldsymbol{p_i} = (p_i^{(1)}, p_i^{(2)}, p_i^{(3)}), i = 1, \ldots, m$, be a set of m suspected tumours in a human body.

At time $t = 0$, we start a three-dimension Brownian motion $\boldsymbol{B}_t$ with coordinates processes centered at each one of these points.

For each i a sphere of radius $r_i(t)$ is given. We are interested in getting quantitative information on how long it will take before the radius (volume) of the biggest sphere exceeds a fixed level (say r, $r > 0$) as the size of m varies.


$||B_t||$
b

Let $\mathcal{X} = \{\boldsymbol{X}_i(t) = (X_i^{(1)}(t), X_i^{(2)}(t), X_i^{(3)}(t))\}_{i=1,2,\dots,m}$ where, for each i,

$$\boldsymbol{X}_i(t) = \boldsymbol{p}_i + \boldsymbol{B}_i(t).$$

The square of the radius of the largest sphere is given by

$$\|\mathcal{B}(t)\|^2 = \sup_{j \le m} \sum_{i=1}^{3} (B_j^{(i)}(t))^2.$$

Let $T_{r,m} = \inf\{s > 0 : \|\mathcal{B}(s)\| = r\}$ be the stopping times of interest.

$$a(t) = E[\sup_{0 \le s \le t} \|\mathcal{B}(s)\|] = \sqrt{t} E[\sup_{0 \le s \le 1} \|\mathcal{B}(s)\|] \Longleftrightarrow a^{-1}(t) = \frac{t^2}{(E[\sup_{0 \le s \le 1} \|\mathcal{B}(s)\|])^2},$$

and, by the linearity of $a(t)$ and the previous result, we obtain

$$\frac{r^2}{2(E[\sup_{0 \le s \le 1} \|\mathcal{B}(s)\|])^2} \le E[T_r] \le \frac{2r^2}{(E[\sup_{0 \le s \le 1} \|\mathcal{B}(s)\|])^2}.$$

The above result can be further simplified by the extension of Lévy inequality. We therefore get

$$E\left[T_{r,d}\right] \approx \frac{r^2}{\left(E\left[\sup_{j\leq d}\sqrt{\chi^2_{3,j}}\right]\right)^2},$$

where $\chi^2_{3,j}$ denote independent chi-square random variables with three degrees of freedom.

One quantity of interest: the relative time between a patient with d_1 tumors and one with d_2 tumors.

$$\frac{E[T_{r,d_1}]}{E[T_{r,d_2}]} \approx \frac{\left(E\left[\sup_{j\leq d_2}\sqrt{\chi^2_{3,j}}\right]\right)^2}{\left(E\left[\sup_{j\leq d_1}\sqrt{\chi^2_{3,j}}\right]\right)^2}.$$

Sample version of $a(t)$

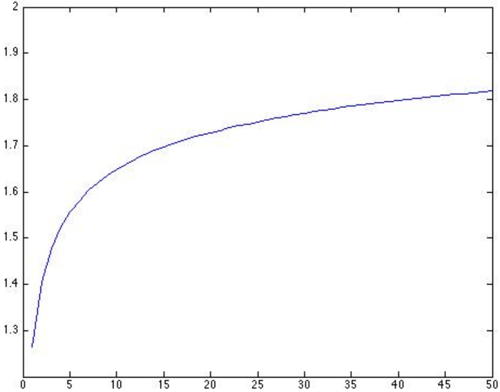


Figure: A plot of $\left(E\left[\sup_{j\le d}\sqrt{\chi^2_{3,j}}\right]\right)^2$, for $d = 1, \dots, 50$.

Question

How far can we take this? (Can we get a universal lower bound?)

Theorem

For $X(t) \geq 0$, *define* $a(t) = E\left[\sup_{s \leq t} X_s\right]$, $T_r = \inf\{s > 0 : X_s > r\}$,

$$E[T_r] \geq (1-\epsilon)a^{-1}(\epsilon r)$$

for any $0 < \epsilon < 1$. *In particular, when* $\epsilon = \frac{1}{2}$ *we have*

$$E[T_r] \geq \frac{1}{2}a^{-1}(\frac{r}{2}), r > 0.$$

See de la Peña and Yang (2004).

Example

Let $f_{a1,\dots,ak}(x) = 1, x \in \mathbb{R}^d$ describe a family of surfaces parametrized by $a_1, \dots, a_k$. Then the time it takes for the process X_t in $\mathbb{R}^d$ to cross the surface $f_{a1,\dots,ak}(x) = 1$ is given by $T_{a1,\dots,ak} = \inf\{t > 0 : f_{a1,\dots,ak}(x) > 1\}$.

We consider the process $\zeta_t = f_{a1,\dots,ak}(X_t)$ and $T_{a1,\dots,ak}(r) = \inf\{t > 0 : \zeta_t > 1\}$. For example, when X is real, for $a < 0 < b$, we may define

$$f_{a,b}(x) = \begin{cases} x/a, & x < 0 \\ x/b, & x \geq 0. \end{cases}$$

and hence $T_{a,b} = \inf\{t > 0 : X_t \notin [a,b]\} = \inf\{t > 0 : f_{a,b}(X_t) > 1\}$.

First consider $T_{a,b}(r) = \inf\{t > 0 : f_{a,b}(X_t) > r\}$. Then, if we let $a_{a,b}(t) = E\left[\sup_{0 \leq s \leq t} f_{a,b}(X_s)\right]$, we will have

$$\frac{1}{2} a_{a,b}^{-1}\left(\frac{1}{2}\right) \leq E\left[T_{a,b}\right].$$

The above results can be extended easily to $\boldsymbol{X} \in \mathbb{R}^d$. Examples of different bounds are shown as follows:

1. $T_r = \inf\{t \geq 0 : X_t \notin [l, u]\}, l < 0 < u,$
2. $T_r = \inf\{t \geq 0 : |X_{t,1} - X_{t,2}| > r\},$
3. $T_r = \inf\{t \geq 0 : X_t > \eta(X_t, r, t, Y)\}$, where $X_t \in \mathbb{R}, \eta$ is a positive function and Y is an arbitrary random variable.

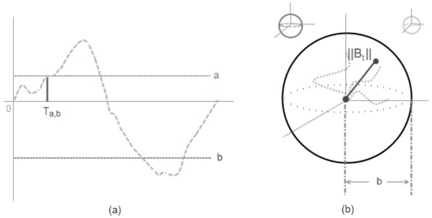


Figure: Illustrations of different bounds: (Left) $T_r = \inf\{t \geq 0 : X_t \notin [b, a]\}, b < 0 < a$, (Right) $T_r = \inf\{t \geq 0 : \|\boldsymbol{B}_t\| > b, \boldsymbol{B}_t \in \mathbb{R}^3\}$.

2.Drought in US Southwest and Mediterranean (Revisit)

The data used to carry out the demonstration of the proposed method are time series of Southwest (U.S.) and Mediterranean region precipitation, calculated from IPCC† Fourth Assessment (AR4) model simulations of the twentieth and twenty-first centuries.

Output from nineteen models is considered for each of the areas.

† IPCC here refers to Intergovernmental Panel on Climate Change. It is the leading body for the assessment of climate change, established by the United Nations Environment Programme (UNEP) and the World Meteorological Organization (WMO) to provide the world with a clear scientific view on the current state of climate change.

Climate Change

- The projections used to carry out the demonstration are time series of two subtropical regions:

  (a) the US southwest (125°W to 95°E; 25°N to 40°N) and
  (b) the Mediterranean (10°W to 50°E; 30°N to 45°N)

  calculated from IPCC Fourth Assessment (AR4) model simulations of the 20th and the 21st century (Randall et al. 2007; Meehl et al. 2007).

The models are forced* in the 20th century with the observed, time dependent greenhouse gas concentrations, anthropogenic aerosols, and volcanic areosols.

In the future simulations, the models are forced with forcing scenario A1B (see IPCC 2000) - “middle of the road” estimate.

*Natural changes in the components of earth’s climate system and their interactions are the cause of internal climate variability, or "internal forcings." Scientists generally define the five components of earth’s climate system to include Atmosphere, hydrosphere, cryosphere, lithosphere (restricted to the surface soils, rocks, and sediments), and biosphere.

Example 2: Drought in US Southwest and Mediterranean

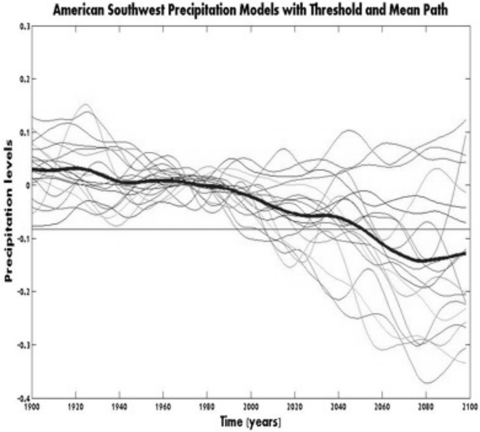
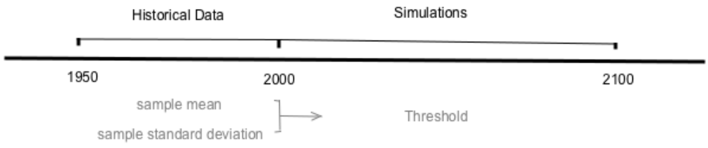


Figure: Time series of precipitation, averaged over the southwest U.S. (125°W to 95°W and 25°N to 40°N), from 19 models that participated in the IPCC Fourth Assessment (see Seager et al. for model information).

Determining the Threshold

- We use a threshold derived from 19 simulated paths to represent climate states. The models are forced with forcing scenario A1B (see IPCC 2000) - “middle of the road” estimate.
- Specifically, we define the threshold as being one standard deviation below the 21-year averaged rainfall that are sampled annually between 1950 and 2000.



- Assumptions:
  - each of the models provides a/an (exchangeable) realisation of the process under study and

- Define
$$T_{r,i} := \inf\{t \in [0,\tau] : X_i(t) > r\} \quad , i = 1, \ldots, n(= 19),$$
as the first hitting time of the ith simulated path X_i.

- Three forecasts:
  1. Mean of the first-hitting time: (unbiased, smallest MSE)

$$T_r^{(UF)} := \frac{1}{19}\sum_{i=1}^{19} T_{r,i}$$

  2. First-hitting time of the mean path:

$$T_r^{(CF)} := \inf\left\{t \in [0,\tau] : \frac{1}{19}\sum_{i=1}^{19} X_i(t) > r\right\}.$$

  3. $a_n^{-1}(r)$.

- For the Mediterranean region, $T_r^{(UF)}$ is the “statistically” preferred estimator of $E[T_r]$. Note that $E[T_r^{(UF)}] = E[T_r]$ is unbiased and has the minimum MSE.

However, what if not all the paths have finite hitting times?

$$T_r^{(UF)} = \frac{1}{19}\sum_{i=1}^{19} T_{r,i} = \infty.$$

One can either truncate the paths or take the path of the median crossing. However, this involves disposing of valuable information.

For the US Southwest region, only 16 out of the 19 models crossed the threshold before the end of the 21st century. Here one may exclude the three potential outliers. Alternatively, we use $a_n^{-1}(r)$.

Results

Details of the results are tabulated as follows:

	$T^{(UF)}$	$T^{(CF)}$	$a_n^{-1}(r)$
Mediterranean	2010.21	2040	2004
Southwest US	∞	2018	2002

According to climatological observation, the transition to a more arid climate in these two regions is already underway. In this context, the results of the new approach provide the best (conservative) estimate. among the three estimators studied of the actual shift to drier conditions.

Summary

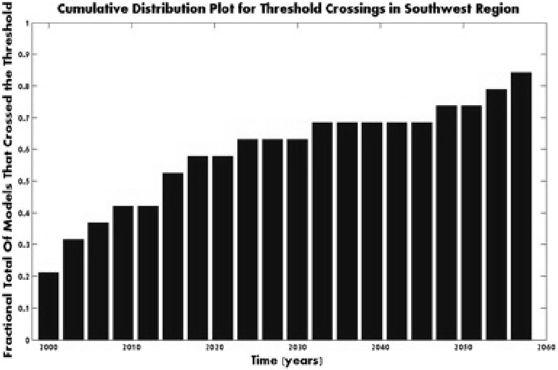
The sample average of the crossing times is an unbiased estimator of the population average crossing-time (as estimated using the available paths) and it is “statistically” preferred.

In the cases where some of the paths never cross the boundary, $a_n^{-1}(r)$ provides an alternative. With more assumptions, we also know that the quantity of interest is of certain order.

Under some conditions, we are able to calculate the order of magnitude of $E[T_r]$ on the basis of $a_n^{-1}(r)$.

Future work: Work on regional precipitation simulations from National Center for Atmospheric Research (NCAR).

Connection with Survival Analysis



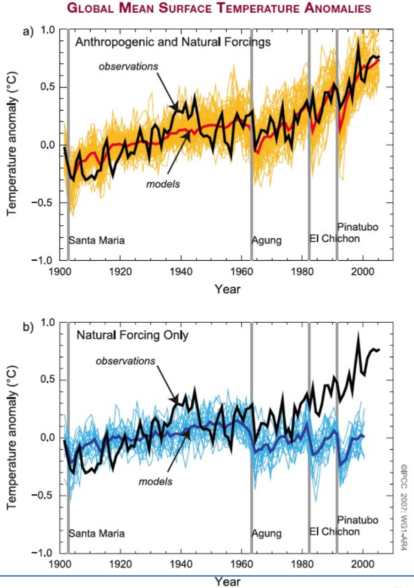


Figure TS.23. *(a) Global mean surface temperature anomalies relative to the period 1901 to 1950, as observed (black line) and as obtained from simulations with both anthropogenic and natural forcings. The thick red curve shows the multi-model ensemble mean and the thin yellow curves show the individual simulations. Vertical grey lines indicate the timing of major volcanic events. (b) As in (a), except that the simulated global mean temperature anomalies are for natural forcings only. The thick blue curve shows the multi-model ensemble mean and the thin lighter blue curves show individual simulations. Each simulation was sampled so that coverage corresponds to that of the observations. {Figure 9.5}*

Adapted from IPCC. (2007).
Climate Change 2007: The Physical Science Basis: Cambridge

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Proposition

Assuming, in addition that X_t *is a Markov process and that* $T_{kr} - T_{(k-1)r} \geq_{st} T_r$*, for* $k \geq 2$*,then*

$$E[a(T_r)] \leq 2r \qquad (1)$$

and

$$E[T_r] \leq a^{-1}(2r). \qquad (2)$$

Remark: Equation (2)*, coupled with (1.1) of de la Peña and Yang (2004), gives the right order of magnitude of the expected value of the first hitting time:*

$$\frac{1}{2}a^{-1}(\frac{r}{2}) \leq E[T_r] \leq a^{-1}(2r). \qquad (3)$$

Proof.

First notice that if X_t is a Markov process with continuous paths - irreducible state space $[0, A]$, $0 < A \leq \infty$. If $r < A$, then $Pr\{T_r < \infty\} = 1$. There is no restriction that the process is time homogeneous. Observe that, because of the continuous paths, we have

$$T_r = T_s + (T_r - T_s), s < r,$$

thus T_r is stochastically greater than T_s and $T_r - T_s$. Now

$$Pr\{T_r > x + y\} = Pr\{T_r > x\}Pr\{T_r - x > y | T_r > x\}$$

and

$$T_r - x | T_r > x, X(x) = y \sim T_r - T_y,$$

by the Markov property, and since $T_r - T_y \leq_{st} T_r$, for $0 \leq y < t$, $Pr\{T_r - x > y | T_r > x\} \leq Pr\{T_r > y\}$, so that

$$\bar{F}_{T_r}(x + y) \leq \bar{F}_{T_r}(x)\bar{F}_{T_r}(y),$$

which is the submultiplicative, or new better than used (NBU), property. For details about NBU property, see Barlow and Proschan (1975). □

Proof.

Define $\bar{G}(t) := \frac{1}{\mu}\int_t^\infty \bar{F}_{T_r}(x)dx$, where $\mu = E[T_r]$, the stationary renewal distribution corresponding to T_r, since

$$\begin{aligned}\bar{G}(t) &= \frac{\bar{F}_{T_r}(t)}{\mu}\int_t^\infty \frac{\bar{F}_{T_r}(x)}{\bar{F}_{T_r}(t)}dx \\ &\leq \frac{\bar{F}_{T_r}(t)}{\mu}\int_t^\infty \bar{F}_{T_r}(x-t)dx \\ &= \frac{\bar{F}_{T_r}(t)}{\mu}\mu = \bar{F}_{T_r}(t),\end{aligned}$$

it follows that $G \leq_{st} F_{T_r}$.
The stationary renewal distribution corresponding to F_{T_r} has $X_1^* \sim G$, and $\{X_i\}_{i\geq 1} \sim F_{T_r}$. It satisfies,

$$M^*(t) = E[N^*(t)] = E[\#\text{ of renewals in }[0,t]] = \frac{t}{\mu}.$$

□

Proof.

An ordinary renewal process has $X_1 \sim F$ with $F \geq_{st} G$, since F is NBU. It follows that

$$\begin{aligned} M(t) &= E[N(t)] \\ &= E[\#\text{ of renewals in } [0,t] \text{ for ordinary renewal process}] \\ &\leq E[N^*(t)] = \frac{t}{\mu}, \end{aligned}$$

and hence

$$E[M(T_r)] \leq E[\mu^{-1} T_r] = 1.$$

Under the assumption that $T_{kr} - T_{(k-1)r} \geq_{st} T_r$, then

$$\tilde{N}_t r \leq \max_{0 \leq s \leq t} X_s \leq (\tilde{N}_t + 1) r,$$

where $\tilde{N}_t = \max\{k : T_{kr} \leq t\}$, i.e. the number of renewals prior to time t. □

Proof.

It follows that

$$\begin{aligned} a(t) &\leq rE[N_t+1] \\ &= (M_t+1)r \\ &\leq \left(\frac{t}{\mu}+1\right)r \end{aligned} \tag{4}$$

and thus

$$E[a(T_r)] \leq \left(\frac{E[T_r]}{\mu}+1\right)r \leq 2r.$$

By plugging in, specifically, $t = E[T_r]$ into (4), we yield

$$E[T_r] \leq a^{-1}\left(\frac{E[T_r]}{\mu}+1\right)r = a^{-1}(2r), \tag{5}$$

which completes the proof. □