Projections of Probability Distributions: A Measure-theoretic Dvoretzky Theorem

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General phenomenon: if $X \in \mathbb{R}^d$ is a random vector and d is large, then (under some conditions on $\mathcal{L}(X)$), for a large measure of $\theta \in \mathbb{S}^{d-1}$, $\langle X, \theta \rangle$ is approximately Gaussian.

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Many authors have observed and contributed to the understanding of this phenomenon. In particular:

Theorem (Bobkov)

Suppose that X satisfies $\mathbb{E}X_iX_j = \delta_{ij}$ *and*

$$\mathbb{P}\left[\left|\frac{|X|}{\sqrt{d}} - 1\right| > \epsilon_d\right] \leq \epsilon_d.$$

Then

$$\sigma_{d-1}\left\{\theta \middle| d_\infty\left(\langle \theta, X\rangle, Z\right) \geq 4\epsilon_d + \delta\right\} \leq 4d^{3/8}e^{-cd\delta^4}.$$

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If so, how can k grow with d? Logarithmically? Polynomially?

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For θ in the Stiefel manifold $\mathfrak{W}_{d,k}$, let X_θ denote the projection of X onto the span of θ. Fix $\delta \in (0, 2)$, and let $k = \delta \frac{\log(d)}{\log(\log(d))}$. Then there is a $c > 0$ depending only on δ, L and L' such that for $\epsilon = \frac{2}{[\log(d)]^c}$, there is a subset $\mathfrak{T} \subseteq \mathfrak{W}_{d,k}$ with $\mathbb{P}_{d,k}[\mathfrak{T}^c] \leq Ce^{-c' d\epsilon^2}$, such that for all $\theta \in \mathfrak{T}$,

$$d_{BL}(X_\theta, \sigma Z) \leq C'\epsilon.$$

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$dim(E) = c\frac{\log(d)}{\log(\log(d))}$.
Define $f : E \to \mathbb{R}$ by $f(x) := (1 - d(x, \pi_E(S)))_+$. Then
$\|f\|_{BL} \leq 1$ and

$$\int f d\mu_{\pi_E(S)} = 1$$

but

$$\int f d\gamma_E \xrightarrow{d\to\infty} 0.$$

That is, for this choice of k, $d_{BL}(X_\theta, \sigma Z) \approx 1$ for all choices of
$\theta \in \mathfrak{W}_{d,k}$.

The example shows that $k_c = \frac{2\log(d)}{\log(\log(d))}$ is a sharp cut-off such that if X is a random vector in $\mathbb{R}^d$ satisfying some natural conditions on $\mathcal{L}(X)$, then most k-dimensional margins of X are approximately Gaussian for $k < k_c$ and this need not be true for $k > k_c$.

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Dvoretzky's theorem: Let $\|\cdot\|$ be any norm on $\mathbb{R}^d$ such that the maximum volume ellipsoid in its unit ball is a dilate of the sphere. Let $\epsilon > 0$ be fixed. Then there is some rescaling of $\|\cdot\|$ and a constant $C(\epsilon)$ such that if $k \leq C(\epsilon)\log(d)$ and if E is a random subspace of $\mathbb{R}^d$ of dimension k, then with probability tending to 1,

$$|v| \leq \|v\| \leq (1+\epsilon)|v|$$

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That is, if $k \leq C(\epsilon)\log(d)$, then most k-dimensional subspaces of the normed space $(\mathbb{R}^d, \|\cdot\|)$ look very similar to k-dimensional Euclidean space $(\mathbb{R}^k, |\cdot|)$.

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This is analogous to the difference between the main theorem and a result of Klartag, showing that if the random vector X has a log-concave distribution, then most projections are close to Gaussian for $k = d^{\epsilon}$ for a specific value of ϵ.

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  The bounded-Lipschitz distance is interpreted as the supremum of a stochastic process indexed by test functions. Concentration of measure on the Stiefel manifold implies that this process has subgaussian increments, allowing the expected supremum to be estimated via entropy methods.
- The bounded-Lipschitz distance $d_{BL}(X_\theta, X_\Theta)$ is tightly concentrated near its mean.
  This also follows from concentration of measure on the Stiefel manifold.

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Here, we take $W = \langle X, \Theta\rangle$, where $\Theta \in \mathfrak{W}_{d,k}$ is uniform and independent of X.

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where U is an independently chosen random orthogonal matrix and $R_{1,2}(\epsilon)$ rotates by ϵ in the span of the first two basis elements.

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The theorem on the last slide can be applied, and the result is that

$$d_{BL}(X_\Theta, \sigma Z) \leq \frac{C\sigma\sqrt{k}}{\sqrt{d}}.$$

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There are constants C, c (independent of d, k) such that if $F : \mathfrak{W}_{d,k} \to \mathbb{R}$ is Lipschitz with Lipschitz constant L,

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It's straightforward to show that $F(\theta) := d_{BL}(X_\theta, \sigma Z)$ is Lipschitz with constant $\sqrt{L'}$; this is the whole content of step 3.

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We need to estimate

$$\mathbb{E}_\theta d_{BL}(X_\theta, X_\Theta) = \mathbb{E}\left(\sup_{\|f\|_{BL}\le 1}\left|\mathbb{E}\left[f(X_\theta)\middle|\theta\right] - \mathbb{E}f(X_\Theta)\right|\right).$$

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If the stochastic process $\{X_f\}_{\|f\|_{BL}\le 1}$ is defined by

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Applying measure concentration to $F(\theta) := \mathbb{E}\left[(f-g)(X_\theta)\middle|\theta\right]$ shows that the process has the property:

$$\mathbb{P}\Big[\left|X_f - X_g\right| > \epsilon\Big] \le Ce^{-\frac{cd\epsilon^2}{\|f-g\|_{BL}^2}}.$$

Theorem (Dudley)

If a stochastic process $\{X_t\}_{t\in T}$ satisfies the a sub-Gaussian increment condition

$$\mathbb{P}\left[\left|X_t - X_s\right| > \epsilon\right] \le Ce^{-\frac{\epsilon^2}{2\delta^2(s,t)}} \qquad \forall\epsilon > 0,$$

then

$$\mathbb{E}\sup_{t\in T} X_t \le C\int_0^\infty \sqrt{\log N(T,\delta,\epsilon)}d\epsilon,$$

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Recall that our process satisfies

$$\mathbb{P}\left[\left|X_f - X_g\right| > \epsilon\right] \leq Ce^{-\frac{cd\epsilon^2}{\|f-g\|_{BL}^2}}.$$

The question, then, is: if $BL_1^k := \left\{ f : \mathbb{R}^k \to \mathbb{R} \middle| \|f\|_{BL} \leq 1 \right\}$, what is $N\left(BL_1^k, \frac{\|\cdot\|_{BL}}{\sqrt{d}}, \epsilon\right)$?

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Bad news: $N\left(BL_1^k, \frac{\|\cdot\|_{BL}}{\sqrt{d}}, \epsilon\right) = \infty$.

But not to worry: approximating Lipschitz functions by piecewise affine functions and using volumetric estimates in the resulting finite-dimensional normed space of approximating functions does the job, and ultimately we get (with the simplification $B = 1$)

$$\mathbb{E}_\theta d_{BL}(X_\theta, X_\Theta) \leq C \frac{k + \log(d)}{k^{\frac{2}{3}} d^{\frac{2}{3k+4}}}.$$

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Choosing $k = \frac{\delta \log(d)}{\log(\log(d))}$ and $\epsilon = \frac{2}{\log(d)^c}$ (for a particular c which depends on δ) finishes the proof.

Thank you.