# Renormalization in the model of Brownian motions in Poissonian potentials

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Our story starts from the book "Brownian motion, obstacles and random media" (A-S. Sznitman), where a Brownian particle $B_s$ moves in a space full of obstacles randomly located in the form of Poisson cloud $\omega(dx)$ with intensity $dx$.

Independence is assumed between the Brownian motion and the Poisson cloud. The notations "$\mathbb{E}_0$" and "$\mathbb{E}$" are for the expectations w.r.t. the Brownian motion and the Poisson field, respectively.

The central part of the book is to investigate the long term asymptotic behaviors of the trajectory of the particle which survives from being trapped by the obstacles.

The quenched Gibbs measure of the form

$$\mu_{t,\omega} = \frac{1}{Z_{t,\omega}} \exp\left\{ -\theta \int_0^t V(B_s)ds \right\} \mathbb{P}_0$$

and the annealed Gibbs measure of the form

$$\mu_t = \frac{1}{Z_t} \exp\left\{ -\theta \int_0^t V(B_s)ds \right\} \mathbb{P} \otimes \mathbb{P}_0$$

are introduced to generates the trajectory of the Brownian particle surviving from being trapped by the obstacles, where the random potential

$$V(x) = \int_{\mathbb{R}^d} K(y - x)\omega(dy) \quad x \in \mathbb{R}^d$$

represents the impact of the Poisson field at $x$.

In Sznitman's work, the shape function $K(x)$ is assumed to be non-negative bounded and locally supported.

By Newton's law of universal gravitation, on the other hand, a natural way to define the potential function for $d = 3$ is

$$V(x) = \int_{\mathbb{R}^3} \frac{\omega(dy)}{|y - x|^2}$$

which represents the net attraction at the location $x$ in a gravitational field generated by the Poissonian stars.

In general, we propose to take

$$K(x) = |x|^{-p}.$$

A disappointing fact is

$$\int_{\mathbb{R}^d} \frac{\omega(dy)}{|y-x|^p} \equiv \infty \quad \forall 0 < p \le d$$

which is indicated by the fact that

$$\begin{aligned} &\mathbb{E} \exp\left\{ -\int_{\mathbb{R}^d} \frac{\omega(dy)}{|y-x|^p} \right\} \\ &= \exp\left\{ -\int_{\mathbb{R}^d} \left(1 - \exp\left\{ -\frac{1}{|y-x|^p} \right\}\right) dx \right\} \\ &= \exp\left\{ -\int_{\mathbb{R}^d} \left(1 - \exp\left\{ -\frac{1}{|y|^p} \right\}\right) dx \right\} = \infty. \end{aligned}$$

The way we resolve this problem is renormalization.

# Renormalization

To show our idea, we retreat back to the case when the shape function $K(x)$ is bounded and locally supported. We have as

$$\mu_{t,\omega} \equiv \frac{1}{Z_{t,\omega}} \exp\left\{ -\theta \int_0^t V(B_s)ds \right\} \mathbb{P}_0$$
$$= \frac{1}{\overline{Z}_{t,\omega}} \exp\left\{ -\theta \int_0^t \overline{V}(B_s)ds \right\} \mathbb{P}_0$$

where

$$\overline{V}(x) = V(x) - \mathbb{E}\, V(x) \qquad x \in \mathbb{R}^d.$$

The key observation is that

$$\mathbb{E}\, V(x) = \int_{\mathbb{R}^d} K(y-x)dy = \int_{\mathbb{R}^d} K(y)dy = \text{constant}.$$

# Renormalization

Naturally, in the case $K(x) = |x|^{-p}$ we introduce

$$\overline{V}(x) = \int_{\mathbb{R}^d} \frac{1}{|y-x|^p}\big[\omega(dy) - dy\big] \qquad x \in \mathbb{R}^d$$

and consider the quenched Gibbs measure

$$\mu_{t,\omega} = \frac{1}{\overline{Z}_{t,\omega}} \exp\left\{ -\theta \int_0^t \overline{V}(B_s)ds \right\} \mathbb{P}_0$$

and the annealed Gibbs measure of the form

$$\mu_t = \frac{1}{\overline{Z}_t} \exp\left\{ -\theta \int_0^t \overline{V}(B_s)ds \right\} \mathbb{P} \otimes \mathbb{P}_0$$

instead, whenever the above establishments make sense.

# Renormalization

Based on the general theory of non-Gaussian random integrals (Rajput-Rosinski and others), the renormalized potential

$$\overline{V}(x) = \int_{\mathbb{R}^d} \frac{1}{|y-x|^p}\big[\omega(dy) - dy\big] \qquad x \in \mathbb{R}^d$$

is well-defined when $d/2 < p < d$.

# Parabolic Anderson models

The parabolic Anderson model formulated in term of the the following initial vale problem.

$$\begin{cases} \partial_t u(t,x) = \kappa \Delta u(t,x) + \xi(x) u(t,x) \\ u(0,x) = 1 \end{cases}$$

where $\xi(x)$ is a suitable random field.

# Parabolic Anderson models

The mathematical relevance to our subject is the Feynman-Kac representation

$$u(t,x) \stackrel{d}{=} u(t,0) = \mathbb{E}_0 \exp\left\{(2\kappa)^{-1}\int_0^{(2\kappa)^{-1}t} \xi(B_s)ds\right\}$$

and our choice on the birth-death rate

$$\xi(x) = \pm\overline{V}(x) = \pm\int_{\mathbb{R}^d} \frac{1}{|x-y|^p}\big[\omega(dy) - dy\big].$$

# Integrabilities

Our goal is to study the finiteness and the long-term asymptotics of the annealed and quenched exponential moments

$$\mathbb{E}_0 \otimes \mathbb{E} \exp\left\{ \pm\theta \int_0^t \overline{V}(B_s)ds \right\} \text{ and } \mathbb{E}_0 \exp\left\{ \pm\theta \int_0^t \overline{V}(B_s)ds \right\}$$

The moments with negative exponent are finite (demonstrated later). The first concrete question is the integrability of the exponential moments with positive exponents.

We first bring an unfortunate news

**Theorem (Chen-Kulik (2010))**

*For each* $\theta > 0$ *and* $t > 0$,

$$\mathbb{E} \otimes \mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\} = \infty.$$

# Integrabilities

Bad as it is, the above theorem does not say anything conclusive about the correspondent "quenched" integrability, which is the one needed for the parabolic Anderson model.

**Theorem (Chen-Kulik (2010))**

*Assume $d/2 < p < d$. For any $t > 0$ and $\theta > 0$*

$$\mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\} \begin{cases} < \infty & a.s. \quad \text{when } p < 2 \\ = \infty & a.s. \quad \text{when } p > 2. \end{cases}$$

# Integrabilities

The most delicate case is when $p = 2$ (necessarily $d = 3$ by the relation $d/2 < p < d$).

## Theorem (Chen-Rosinski)

*. Under $d = 3$ and $p = 2$,*

$$\mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\} \begin{cases} < \infty & a.s. \quad \text{when } \theta < \frac{1}{16} \\ = \infty & a.s. \quad \text{when } \theta > \frac{1}{16}. \end{cases}$$

**Remark.** This result is mathematically relevant to Hardy's inequality

$$\int_{\mathbb{R}^3} \frac{f^2(x)}{|x|^2} dx \le 4\|\nabla f\|_2^2 \quad f \in W^{1,2}(\mathbb{R}^3).$$

# Asymptotics

Having constructed the models, we are interested in the long term asymptotics for the annealed and quenched exponential moments

$$\mathbb{E} \otimes \mathbb{E}_0 \exp\left\{ -\theta \int_0^t \overline{V}(B_s)ds \right\} \text{ and } \mathbb{E}_0 \exp\left\{ -\theta \int_0^t \overline{V}(B_s)ds \right\}$$

under $p/2 < p < d$ and for the quenched exponential moments

$$\mathbb{E}_0 \exp\left\{ \theta \int_0^t \overline{V}(B_s)ds \right\}$$

when $d/2 < p < d$, $p \leq 2$.

# Annealed asymptotics

In the following, the classic results (where $K(x)$ is bounded and (or) locally supported and no renormalization is needed) and new results (where $K(\cdot) = |\cdot|^{-p}$ with renormalization) are listed together for comparison. Recall our notation

$$V(x) = \int_{\mathbb{R}^d} K(y-x)\omega(dy)$$

$$\overline{V}(x) = \int_{\mathbb{R}^d} \frac{1}{|y-x|^p}\big[\omega(dy) - dy\big]$$

# Annealed asymptotics

## Theorem (Donsker and Varadhan (1975))

*For any* $\theta > 0$

$$\lim_{t\to\infty} t^{-\frac{d}{d+2}} \log \mathbb{E} \exp\left\{ -\theta \int_0^t V(B_s)ds \right\} = -\frac{d+2}{2}\omega_d^{\frac{2}{d+2}} \left(\frac{2\lambda_d}{d}\right)^{\frac{d}{d+2}}$$

*where* $\omega_d$ *is the volume of the* $d$*-dimensional unit ball, and* $\lambda_d$ *is the principal eigenvalue of* $(1/2)\Delta$ *on the* $d$*-dimensional unit ball with zero boundary values.*

# Annealed asymptotics

## Theorem (Chen and Kulik (2010))

*Assume $d/2 < p < d$. For any $\theta > 0$*

$$\lim_{t\to\infty} t^{-d/p} \log \mathbb{E} \otimes \mathbb{E}_0 \exp\Big\{ -\theta \int_0^t \overline{V}(B_s)ds \Big\}$$
$$= \theta^{d/p} \frac{\omega_d p}{d-p} \Gamma\Big(\frac{2p-d}{p}\Big).$$

**Comparison.** Shape insensitivity vs shape dependence; sub-linearity vs sup-linearity; diffusivity vs non-diffusivity.

# Annealed asymptotics

**Proof.** By Jensen inequality,

$$
\begin{aligned}
&\mathbb{E} \otimes \mathbb{E}_0 \exp\left\{ -\theta \int_0^t \overline{V}(B_s)ds \right\} \\
&\le \frac{1}{t}\int_0^t \mathbb{E}_0 \otimes \mathbb{E} \exp\left\{ -\theta t \overline{V}(B_s)\right\} \\
&= \mathbb{E}_0 \otimes \mathbb{E} \exp\left\{ -\theta t \overline{V}(0)\right\} = \exp\left\{ \int_{\mathbb{R}^d} \varphi\Big(\frac{\theta t}{\vert x\vert^p}\Big) dx \right\} \\
&= \exp\left\{ (\theta t)^{d/p} \int_{\mathbb{R}^d} \varphi\Big(\frac{1}{\vert x\vert^p}\Big) dx \right\} \\
&= \exp\left\{ (\theta t)^{d/p} \frac{\omega_d p}{d-p}\Gamma\Big(\frac{2p-d}{p}\Big) \right\}
\end{aligned}
$$

where $\varphi(a) = e^{-a} - 1 + a$

# Annealed asymptotics

On the other hand,

$$\begin{aligned}
&\mathbb{E} \otimes \mathbb{E}_0 \exp\left\{-\theta\int_0^t \overline{V}(B_s)ds\right\} \\
&\geq \mathbb{E} \otimes \mathbb{E}_0 \exp\left\{-\theta\int_0^t \overline{V}(B_s)ds\right\}1_{\{\max_{s\le t}|B_s|\le\epsilon\}} \\
&\approx \mathbb{E} \exp\left\{-\theta\int_0^t \overline{V}(0)ds\right\}\mathbb{P}_0\left\{\max_{s\le t}|B_s|\le\epsilon\right\} \\
&\geq \exp\left\{(\theta t)^{d/p}\int_{\mathbb{R}^d}\varphi\Big(\frac{1}{|x|^p}\Big)dx\right\}\exp\left\{-c\epsilon^{-2}t\right\} \\
&\approx \exp\left\{(\theta t)^{d/p}\frac{\omega_d p}{d-p}\Gamma\Big(\frac{2p-d}{p}\Big)\right\}.
\end{aligned}$$

□

# Quenched asymptotics

## Theorem (Sznitman (1998))

$$\lim_{t\to\infty} \frac{(\log t)^{2/d}}{t} \log \mathbb{E}_0 \exp\Big\{ -\theta \int_0^t V(B_s)ds \Big\} = -\lambda_d \quad a.s. - \mathbb{P}$$

*where $\lambda_d$ is the principal eigenvalue of $(1/2)\Delta$ on the $d$-dimensional unit ball with zero boundary values.*

# Quenched asymptotics

## Theorem (Chen (2010))

*Under $d/2 < p < d$,*

$$\lim_{t\to\infty} t^{-1}(\log t)^{-\frac{d-p}{d}} \log \mathbb{E}_0 \exp\Big\{ -\int_0^t \overline{V}(B_s)ds\Big\}$$
$$= \frac{\theta d^2}{d-p}\Big(\frac{\omega_d}{d}\Gamma\Big(\frac{2p-d}{p}\Big)\Big)^{p/d} \quad a.s. - \mathbb{P}.$$

# Quenched asymptotics

## Theorem (Gärtner, König and Molchanov (2000))

$$\lim_{t\to\infty}\frac{\log\log t}{t\log t}\log\mathbb{E}_0\exp\left\{\theta\int_0^t V(B_s)ds\right\} = d\theta K(0) \quad a.s. - \mathbb{P}.$$

**Remark.** A condition posted in above theorem is

$$K(0) = \max_{x\in\mathbb{R}^d} K(x).$$

So the boundedness of $K(\cdot)$ is crucial for the above theorem.

# Quenched asymptotics

## Theorem (Chen (2010))

*Under* $d/2 < p < \min\{d, 2\}$,

$$\lim_{t\to\infty} \frac{1}{t}\Big(\frac{\log\log t}{\log t}\Big)^{\frac{2}{2-p}} \log \mathbb{E}_0 \exp\Big\{\theta \int_0^t \overline{V}(B_s)ds\Big\}$$
$$= \frac{1}{2}(2-p)^{\frac{4-p}{2-p}}\Big(\frac{d\theta\sigma_1(d,p)}{2+d-p}\Big)^{\frac{2}{2-p}} \quad a.s. - \mathbb{P}$$

*where* $\sigma_1(d,p)$ *is the best constant of the inequality*

$$\int_{\mathbb{R}^d} \frac{f^2(x)}{|x|^p}dx \le C\|f\|_2^{2-p}\|\nabla f\|_2^p \quad \forall f \in W^{1,2}(\mathbb{R}^d).$$

**Comparison.** Non-diffusive pattern vs diffusive pattern.

**Comment.** As a centered signed measure, $\omega(dx) - dx$ is highly non-symmetric at the exponential level. For the lower tail, it is the tail of $K(\cdot)$ at infinity that decides the asymptotic behavior. For the upper tail, it is the blow-up speed of $K(\cdot)$ at zero that decides the asymptotic behavior.

A natural question is to ask what happens when $p = 2$ (and therefore $d = 3$), in correspondent to the last theorem.

# Critical case $p = 2$ and $d = 3$

Let $l(t)$ be a slow-varying function on $[1, \infty)$

## Theorem (Chen-Rosinski)

*Under* $0 < \theta < 16^{-1}$,

$$\limsup_{t\to\infty} t^{-\frac{k+1}{k-1}} l(t)^{-\frac{2}{3(k-1)}} \log \mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\}$$

$$= \begin{cases} 0 & a.s. \text{ if } \int_1^\infty \frac{dt}{t\cdot l(t)} < \infty \\ \infty & a.s. \text{ if } \int_1^\infty \frac{dt}{t\cdot l(t)} = \infty \end{cases}$$

*where* $k = \left[(8\theta)^{-1}\right]$ *is the integer part of* $(8\theta)^{-1}$.

# Critical case $p = 2$ and $d = 3$

## Theorem (Chen-Rosinski)

*Under* $0 < \theta < 16^{-1}$,

$$\liminf_{t\to\infty} t^{-\frac{k+1}{k-1}} l(t)^{\frac{2}{3(k-1)}} \log \mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(B_s)ds\right\}$$

$$= \begin{cases} 0 & \textit{a.s. if } \displaystyle\int_1^\infty \frac{1}{t}\exp\big\{-c\cdot l(t)\big\}dt = \infty \textit{ for some } c > 0 \\ \infty & \textit{a.s. if } \displaystyle\int_1^\infty \frac{1}{t}\exp\big\{-c\cdot l(t)\big\}dt < \infty \textit{ for every } c > 0 \end{cases}$$

*where* $k = \left[(8\theta)^{-1}\right]$ *is the integer part of* $(8\theta)^{-1}$.

# Critical case $p = 2$ and $d = 3$

**Remark.** Our theorems in the critical setting present both $\theta$-depenence and $\theta$-independence. On the one hand, putting $\theta$ into different sub-intervals of the partition

$$\Big(0, \frac{1}{16}\Big) = \Big(\frac{1}{24}, \frac{1}{16}\Big) \cup \bigcup_{k=3}^{\infty} \Big(\frac{1}{8(k+1)}, \frac{1}{8k}\Big]$$

leads to dratically different asymptotic behaviors. On the other hand, moving $\theta$ around within the same sub-interval does not bring any change to the behavior of the system.

# Idea for quenched asymptotics

By Feynman-Kac formula,

$$\mathbb{E}_0 \exp\left\{ \pm\theta \int_0^t \overline{V}(B_s)ds \right\}$$

$$\approx \sup_{g\in\mathcal{F}_d([-t,t]^d)} \left\{ \pm\theta \int_{[-t,t]^d} \overline{V}(x)g^2(x)dx - \frac{1}{2}\int_{[-t,t]^d} |\nabla g(x)|^2 dx \right\}.$$

where

$$\mathcal{F}_d([-t,t]^d) = \{g \in {}^2([-t,t]^d);\ \|g\|_2 = 1,\ \nabla g \in {}^2([-t,t]^d)\}$$

So the problem is to estimate the random variation on the right hand side.

# Brownian motion in mobile medium

In the above discussion, the obstacles do not change with time. There are real needs to study the model with changing environment. Assume, for example, the obstacles are initially located according to the Poisson field $\omega(dx)$ and each of them moves independently as Brownian motion. Let $\omega_t(dx)$ be the distribution of the obstacles at the time $t$. It can be shown that for each $t > 0$, $\omega_t(dx) \overset{d}{=} \omega(dx)$. Therefore, the renormalized potential

$$\overline{V}(t,x) = \int_{\mathbb{R}^d} \frac{1}{|y-x|^p} \big[\omega_t(dy) - dy\big] \quad x \in \mathbb{R}^d$$

is defined if $d/2 < p < d$.

# Brownian motion in mobile medium

We are interested in the long term behaviors of the moments

$$\mathbb{E}_0 \otimes \mathbb{E} \exp\left\{ \pm\theta \int_0^t \overline{V}(s, B_s)ds \right\}, \quad \mathbb{E}_0 \exp\left\{ \pm\theta \int_0^t \overline{V}(s, B_s)ds \right\}$$

One might think that the situation would be the same as the time-independent case. It turns out not so. Indeed, there are some substantial differences in behavior and new challenges due to time-inhomogeinity.

# Brownian motion in mobile medium

## Theorem (Chen-Xiong)

*Under $d/2 < p < d$,*

$$\log \mathbb{E} \otimes \mathbb{E}_0 \exp\left\{ -\theta \int_0^t \overline{V}(s, B_s)ds \right\}$$

$$\sim \begin{cases} C_1(\theta) t^{d/p} & \text{if } p < 2 \\ C_2(\theta) t^{3/2} & \text{if } p = 2, d = 3 \\ C_3(\theta) t^{\frac{4+d-2p}{2}} & \text{if } 2 < p < \frac{d+2}{2} \\ C_4(\theta) t \log t & \text{if } p = \frac{d+2}{2} > 2 \\ C_5(\theta) t & \text{if } p > \max\left\{2, \frac{d+2}{2}\right\}. \end{cases}$$

# Brownian motion in mobile medium

**Theorem (Chen-Xiong)**

*Under $d/2 < p < \min\{d, 2\}$ (That's right, double large deviation!)*

$$\lim_{t\to\infty} \frac{1}{t} \log\log \mathbb{E} \otimes \mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(s, B_s)ds\right\} = C_6(\theta).$$

# Brownian motion in mobile medium

## Theorem (Chen-Xiong)

*In the case $p = 2$ (and therefore $d = 3$), for every $t > 0$*

$$\mathbb{E} \otimes \mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(s, B_s)ds\right\} \begin{cases} < \infty & \text{if } \theta < \frac{1}{8} \\ = \infty & \text{a.s. if } \theta < \frac{1}{8} \end{cases}$$

*Further, under $\theta < 1/8$ (single logarithm again!)*

$$\lim_{t\to\infty} t^{-3/2} \log \mathbb{E} \otimes \mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(s, B_s)ds\right\} = C_7(\theta).$$

# Brownian motion in mobile medium

## Theorem (Chen-Xiong)

*In the case $p > 2$, for any $t > 0$,*

$$\mathbb{E} \otimes \mathbb{E}_0 \exp\left\{\theta \int_0^t \overline{V}(s, B_s)ds\right\} = \infty$$

The study of the quenched moment

$$\mathbb{E}_0 \exp\left\{ \pm\theta \int_0^t \overline{V}(s, B_s)ds\right\}$$

remains open.

Thank you!