Novel Approaches to the Finite Simple Groups

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22. April–29. April 2012

1 Motivation

It is now accepted that the classification of finite simple groups (CFSG) is complete. Our purpose is in constructions which we re-examine in the light of observations that suggest that a substantial amount of insight is to be gained if we pursue novel connections to other, apparently disparate, areas of mathematics. This may lead to a natural home for the 26 sporadic simple groups which today are considered the complement of the Lie-Chevalley groups.

2 Overview of the Field

In order to understand the importance of the new ideas and conjectures linked with Monstrous Moonshine (MM) that we set out to develop, we have to understand its history, see [15, 19, 20]. Although the Monster is the most spectacular of the sporadic groups, it is just one of 26 groups under our wing. We discern three major evolutionary periods linked with the Monster group, the early and foundational one [5, 6, 7, 9, 10, 11, 17, 18, 22, 23], the time when the fruits of previous labour could be picked, culminating in Borcherds' Fields medal rewarded proof of the conjecture [2], and finally the post Moonshine one, where it is becoming evident that we are not dealing with one particular isolated phenomenon but rather with a (possible) new class of objects of which MM is the most prominent representative so far, and its new mathematical connections [16]. An electronic collection of relevant articles for the subject has been posted at: http://www.fields.utoronto.ca/˜jplazas/MoonshineRoadmap.html

3 Recent Developments and Open Problems

During the focused research group various parts of this picture became clearer. In particular the following lines of work connect our approach with some of the topics discussed from the perspective of other fields:

We give a list of open problems:

1. What is the relation between $M, 2B, 3F'_{24}$ and $E8$, $E7$, and $E6$ respectively?
2. What does adjacency signify when the affine Dynkin nodes are identified with moonshine functions?
3. What is the significance of $27AB$, two classes which restrict to the same MM function?
4. The (column) rank of the MM functions is 163 $(= 194 - 22 - 9)$. $h(\sqrt{-163}) = 1$. Is this significant?

5. Characterise the sign pattern of replicable functions when q-coefficients are replaced by their signs in $\{0, \pm 1\}$.
6. From Conway-Norton we note there are 360 cusps arising from the genus zero Riemann surfaces attached to the (classes of) cyclic subgroups of M. Are these identifiable with the 120 tritangent planes of $W(E8)$?

4 Presentation Highlights

Prof. John McKay from Concordia University initiated the meeting by reporting on recent developments in the field, presenting a series of open problems. Then he formulated several conjectures and explained how they would give rise to major new research directions (see the latter parts of this report). As part of the preparation of the meeting, some of McKay's ideas and problems were circulated amongst the participants before the meeting, and some of the subsequent talks referred directly to this program.

Prof. Matilde Marcolli from Caltech gave two presentations entitled: "Multiplicative genera for noncommutative manifolds?", http://www.its.caltech.edu/~matilde/NCManifoldsModular.pdf
and "From CFT to CFT", http://www.its.caltech.edu/~matilde/CFTtoCFT.pdf

In his first lecture, Prof. Doran, from the University of Alberta, explained "the connection between $K3$ surfaces of high Picard rank and (moonshine) modular functions, starting from the "mirror moonshine conjecture" of Lian-Yau, through more recent results on modular parametrisation in several variables. Special attention was paid to the master equation for modular parametrisation, a differential equation whose algebraic solutions are the classical modular equations for modular curves."

In his second lecture, Prof. Doran described "recent work relating these same $K3$ surfaces of high Picard rank to compactified fibres in Landau-Ginzburg models mirror to smooth Fano threefolds. The modular curves related to moonshine are closely linked to Fano threefolds of Picard rank one."

Prof. Jack Morava from Johns Hopkins University spoke on the elliptic cohomology approach, in particular as presented by N. Ganter [16], to generalised Moonshine and Hecke operators [21].

Dr. Jorge Plazas, from the Fields Institute in Toronto, gave a talk entitled "Non-commutative spaces as a vessel for moonshine phenomena", which he summarised as: "The study of finite simple groups has led recently to considerations touching upon a wide spectrum of areas. The underlying richness of the theory is clear from fascinating and puzzling phenomena at the core of which lies monstrous moonshine.

Many of the prevalent structures arising in this context find a natural common framework in non-commutative geometry. Non-commutative spaces of Q-lattices and modular Hecke algebras encode various of these structures. We need to understand the connection between these spaces and finite simple groups."

5 Scientific Progress Made

In order to best utilise the time in Banff, the organisers collected relevant articles and related questions for the workshop and sent them to the participants beforehand. In the style of a French "groupe de travail", they were asked to prepare a didactic presentation connecting the reading material with their personal mathematical expertise and perspective and to comment on the linked questions.

As an effect of the early preparation of all attendees, several conjectures previously made, in particular by J. McKay, could in fact be turned into concrete research plans during the time in Banff, one example being the application of non-commutative geometry and the Bost-Connes system to Monstrous Moonshine and replicability.

Also, it became clear that Faber polynomials, which are also linked to replicability [1, 8, 14, 19, 20, 22], have a connection with Witt vectors, λ-rings and their associated Adams operations and Hopf algebras [12, 13]. These new results strongly support earlier ideas made about the role Q-lattices should have for moonshine, as replicability can be understood in terms of Adams operations arising in the context of the structure of λ-rings. On the other hand, the understanding of λ-ring structures associated to non-commutative spaces of Q-lattices is fundamental [3].

According to ideas of Marcolli and Plazas, the action of the monster Lie algebra on the above spaces might be feasible once the position of the Virasoro algebra in relation to the Hopf algebra of codimension one foliations of Connes and Moscovici [4] is better understood.

By using the Hopf algebra of codimension one foliations they intend to express differential identities for principal moduli in a manner reminiscent of related work in mirror symmetry, which could be connected with moduli of $K3$ surfaces.

6 Outcome of the Meeting

The first success of the meeting was to bring together different mathematicians whose research interests are connected with the sporadic groups, and in particular the Monster, working separately, in order to create an initial network, which other people are invited to join, and to set-up a shared knowledge base. The list of participants included the organisers, J. McKay and R. Friedrich, further in alphabetic order, Ch. Doran, L. Hesselholt, M. Marcolli, J. Morava, M. Laca and J. Plazas. R. Donagi could not attend but was in contact.

Second, as a result of the gathering, three concrete research plans emerged, which are now further pursued. Namely:

Elliptic cohomology, integrable systems and replicability J. Morava, N. Ganter, L. Hesselholt

Univalent functions, Sato-Segal-Wilson Grassmannians and conformal field theory R. Friedrich, J. McKay

Non-commutative geometry and Bost-Connes systems Ch. Doran, M. Marcolli and J. Plazas.

During the BIRS workshop, Ch. Doran, J. Plazas and M. Marcoli began a collaboration on the application of methods of A. Connes et al. to the master equation and its generalisations.

Finally, during the week a thematically related article was completed by R. Friedrich and J. McKay [13].

Disclaimer

This report is partially based on the written résumés of the participants, with their permission granted to use and modify their texts by the organisers.

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