Differential Schemes and Differential Cohomology

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1 Overview

From a historical perspective Algebraic Geometry was initially concerned with the study of zero sets of polynomials. It was realized from the outset that the subject was closely related to Number Theory, and work during the middle third of the last century, e.g. by Zariski and Weil, suggested that a reformulation of the subject, so as to incorporate such related fields, was in order. Such a reformulation was soon accomplished by Grothendieck, and subsequently refined by Deligne and others. It seems somewhat of an understatement to assert that Grothendieck's ideas revolutionized the way one views Algebraic Geometry.

Differential Algebraic Geometry began, somewhat more recently, as the study of zero sets of differential polynomials, e.g. if K denotes the field $\mathbf{R}(\cos t, \sin t)$ with derivation $' = d/dt$, then $(\cos t, \sin t) \in K^2$ is a zero of the differential polynomial $x_1^2 - x_1' x_2 - 1$ [11,12]. Although there have been significant attempts to reformulate the subject in the style of Grothendieck (e.g. see [4,5,6,10,12,13,15] and references therein), in particular so as to make use of those recently-developed techniques, these innovations have not been widely accepted. One reason for this, in the opinion of the organizers, is that many individuals who work in algebraic geometry and/or number theory are simply not familiar with differential algebra.

This meeting was conceived as an attempt to maintain a focus on, if not to alleviate, this problem. Several of the participants were selected specifically because they knew nothing about differential algebra, but were familiar with at least one of these two other areas.

2 Presentation Highlights

There were four talks. We give a summary of each, in the order presented.

1. Jim Freitag "Local Problems in Differential Algebra"

   The first part of the talk covered basic notions in differential algebraic geometry by extending the basics of algebraic geometry to the differential setting. Notions of dimension were then introduced, illustrated by several important examples and the end of the talk designed to indicate some of the power of model theory in differential algebraic geometry (e.g. see [14]).

2. Henri Gillet " Differential Algebra, Functors, and Arithmetic Geometry"

   The first part of the talk dealt with an application to deformations of a family of varieties that enables one to lift a derivation from the base space to the total space. This was done in characteristic 0 by means on an exponential map. The second part established an analogous result in characteristic p using formal group actions in place of derivations.

3. Andy Magid "Grothendieck Topology"

   The talk was an introduction, in outline form for a general audience, to Grothendieck topologies, abelian sheaves on such entities, and the Cech and derived functor cohomologies of the latter. The 0 and 1 Cech cohomology sets on non-abelian sheaves were also explained, as well as the connection to principal homogeneous spaces.

4. Ray Hoobler "Cohomology in a Differential Algebra Setting"

   The talk examined how a differential Azumaya algebra over an ordinary differential ring could be made locally isomorphic to a matrix ring with coordinate-wise differentiation. The basic idea was to use a Grothendieck topology to set up and locally solve a differential equation whose solutions provide a constant basis for the matrix ring. In this way the differential Azumaya algebras become local principal homogeneous spaces in the Grothendieck topology; well-known cohomological tools could then be used to identify the differential Brauer group with the ordinary Brauer group [7]. The partial case follows a similar pattern using results of Andre [1].

3 Outcome of the Workshop

There were several ongoing discussions outside of the main talks. They centered around questions such as:

- What kind of differential ring extension is analogous to a separable ring extension and can it be used to generate a useful Grothendieck topology?
- What do the "points" in the $\Delta-$flat topology look like?
- What kind of differential ring extensions have lifting properties analogous to Grothendieck's definition of a formally smooth extension?
- How would a differential algebraic space be defined?

Some of these issues continue to be discussed by email and may ultimately lead to full understanding and, possibly, publication.

4 List of Participants

Carlos Arreche (Graduate Center, City University of New York)
Mark Bauer (University of Calgary)
Richard Churchill (Graduate Center and Hunter College, City University of New York, and University of Calgary)
Robin Cockett (University of Calgary)
James Freitag (University of Illinois at Chicago)
Henri Gillet (University of Illinois at Chicago)
Ray Hoobler (Graduate Center and City College, City University of New York)
Rick Jardine (University of Western Ontario)
Lourdes Juan (Texas Tech University)
William Keigher (Rutgers University at Newark)
Andy Magid (University of Oklahoma)
R. Padmanabhan (University of Manitoba)
Camilo Sanabria (Universidad de los Andes [Bogatá])
Renate Scheidler (University of Calgary)
William Sit (City College of the City University of New York)
Yao Sun (Key Laboratory of Mathematics Mechanization, Academy of

Mathematics & Systems Science [Beijing])
Dingkang Wang (Academy of Mathematics & Systems Science [Beijing])
Matt Wechter (University of Illinois at Chicago)
Yang Zhang (University of Manitoba)

References

[1] Y. André, Différentielles non commutatives et théorie de Galois difféntielle ou aux différences, *Ann. Sci. École Norm. Sup.* (4) **34** (2001), no. 5, pp. 685-739.

[2] A. Buium, *Differential Function Fields and Moduli of Algebraic Varieties*, LNM 1226, Springer, New York, 1986.

[3] A. Buium, *Differential Algebraic Groups of Finite Dimension*, LNM 1506, Springer, New York, 1992.

[4] A. Buium, *Differential Algebra and Diophantine Geometry*, Hermann, Paris, 1994.

[5] G. Carrà Ferro, The ring of global sections of the structure sheaf on the differential spectrum, *Matematiche* (Catania) **33** (1978) no. 1, pp. 1-17 (1980) (Italian).

[6] H. Gillet, Differential Algebra - a scheme theory approach, in *Differential Algebra and Related Topics*, Proceedings of the International Workshop (Newark, November 2000), eds. L. Guo, P.J. Cassidy, W.F. Keigher & W.Y. Sit, World Scientific 2002, pp. 95-123.

[7] R. Hoobler, *The Differential Brauer Group* (revised version), posted on the website of the Kolchin Seminar on Differential Algebra, 12/09.

[8] L. Juan and A.R. Magid, Differential central simple algebras and Picard-Vessiot representations, *Proc. Amer. Math. Soc.* **136** (2008) no. 6, pp. 1911-1918.

[9] I. Kaplansky, *An Introduction to Differential Algebra*, Second Edition, Hermann, Paris, 1976.

[10] W.R. Keigher, Differential schemes and premodels of differential fields, *J. Algebra* **79** (1981), n. 2, pp. 37-50.

[11] E.R. Kolchin, *Differential Algebra and Algebraic Groups*, Academic Press, New York, 1973.

[12] J.J. Kovacic, Differential Schemes, in *Differential Algebra and Related Topics*, Proceedings of the International Workshop (Newark, November 2000), eds. L. Guo, P.J. Cassidy, W.F. Keigher & W.Y. Sit, World Scientific 2002, pp. 71-94.

[13] J.J. Kovacic, Global sections of diffspec, *J. Pure Appl. Algebra* **171** (2002), no. 2-3, pp. 265-288.

[14] T. Scanlon, Model theory and differential algebra, in *Differential Algebra and Related Topics*, Proceedings of the International Workshop (Newark, November 2000), eds. L. Guo, P.J. Cassidy, W.F. Keigher & W.Y. Sit, World Scientific 2002, pp. 125-50.

[15] D. Trushin, Global sections of structure sheaves of Keigher rings, *Proc. Amer. Math. Soc.* **140**, (2012), 505-511.