Schubert calculus for equivariant algebraic cobordism

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- Equivariant algebraic cobordism
- Borel presentation for equivariant cobordism of flag varieties
- Classical Schubert calculus
- Schubert calculus in cobordism

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Equivariant cohomology theories

Notation

k — field of characteristic zero,
G — linear algebraic group over k.

Motivation

Extend to the algebraic setting equivariant cohomology theories defined using the classifying space BG.

Classical equivariant cohomology

$$H^*_G(X) := H^*(X \times^G EG)$$

Remark

Note that $X \times^G EG$ is a fiber bundle over BG with the fiber X.

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Equivariant Chow groups

(Totaro, Edidin–Graham)

Define $CH^*_G(X)$ using approximations of the universal G-bundle $EG \to BG$ by algebraic fiber bundles $EG_i \to BG_i$:

$$CH^i_G(X) := CH^i(X \times^G EG_i).$$

Construction

Let V be a representation of G such that G acts freely on an open subvariety $U \subset V$, the quotient U/G is quasiprojective and $\mathrm{codim}(V \setminus U) > i$. Take $U \to U/G$ as $EG_i \to BG_i$.

Remark

Under the above assumptions, $CH^i(X \times^G U)$ does not depend on the choice of V.

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Algebraic cobordism

Notation

X/k — algebraic variety,
$\Omega^*(X)$ — algebraic cobordism ring of X.

- (Levine–Morel) Construction of the universal oriented cohomology theory $\Omega^*(-)$;
- (Levine–Pandharipande) Presentation of $\Omega^n(X)$ by generators (=projective morphisms $[Y \to X]$ of relative codimension n) and relations (=*double point relations*).

Example

$\Omega^*(pt) = \mathbb{L}$ — Lazard ring;

$$\mathbb{L} \simeq \mathbb{Z}[a_1, a_2, \ldots],$$

where $\deg(a_i) = -i$.

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Remark

In contrast with Chow groups, $\Omega^i(X \times^G U)$ might depend on the choice of V.

Solution

Use the filtration $\Omega^i(X) = F^0\Omega^i(X) \supset F^1\Omega^i(X) \supset \ldots$, where $F^j\Omega^i(X)$ is spanned by the classes $[\pi : Y \to X]$ such that $\operatorname{codim}(\pi(Y)) \geq j$.

- (Deshpande) $\Omega^*_G(X)$ for smooth X
- (Krishna, Heller–Malagón-López) $\Omega^*_G(X)$ for all X

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Observation

$$\Omega^i_G(X)_j := \frac{\Omega^i\left(X \overset{G}{\times} EG_j\right)}{F^j\Omega^i\left(X \overset{G}{\times} EG_j\right)}$$

does not depend on the choice of $EG_j \to BG_j$.

Definition

$$\Omega^i_G(X) = \varprojlim_j \Omega^i_G(X)_j$$

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Equivariant algebraic cobordism

Example

$G = T$ — split torus, Λ_T — the character lattice of T

$$\Omega^i_T(pt) := \varprojlim_j \left(\mathrm{Sym}^{<j}(\Lambda_T) \otimes \mathbb{L}\right)^i.$$

Remark

If we fix a basis $\chi_1, \ldots, \chi_n$ in Λ_T and put $x_i := c_1^T(L_{\chi_i})$ then

$$\Omega^*_T(pt) \simeq \mathbb{L}^{gr}[[x_1, \ldots, x_n]],$$

where $\mathbb{L}^{gr}[[x_1, \ldots, x_n]]$ is the *graded power series ring*.

Relation with BT

$$\Omega^*_T(pt) \simeq MU^*(BT).$$

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Flag varieties

Notation

G — connected reductive group with a maximal torus T split over k
$B \subset G$ — Borel subgroup containing T

Definition

$X = G/B$ is the *variety of complete flags*

Example $G = GL_n(k)$

X is the variety of complete flags in k^n:

$$X = \{\{0\} = V^0 \subset V^1 \subset \ldots \subset V^{n-1} \subset V^n = k^n | \dim V^i = i\}$$

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Borel presentation

Picard group of X

Each character χ of T gives rise to the G-equivariant line bundle $\mathcal{L}_\chi := G \overset{B}{\times} L_\chi$ on X. This gives the isomorphism

$$\mathrm{Pic}(X) \simeq \Lambda_T.$$

Fact

$CH^*(X) \otimes \mathbb{Q}$ (but not always $CH^*(X)$) is generated multiplicatively by $\mathrm{Pic}(X)$.

Torsion index

The *torsion index* of G is defined as the smallest positive integer t_G such that $t_G[pt]$ belongs to the subring of $CH^*(X)$ generated by $\mathrm{Pic}(X)$. For instance, $t_{GL_n} = 1$.

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Borel presentation for equivariant cobordism

Theorem (K.-Kishna, 2011)

Put $S := \Omega^*_T(pt)$. After inverting t_G

$$\Omega^*_T(G/B) \simeq S \otimes_{S^W} S,$$

where $S^W \subset S$ is the subring of the Weyl group invariants.

Remark

The isomorphism $S \otimes_{S^W} S \to \Omega^*_T(G/B)$ is given by

$$c_1^T(L_\chi) \otimes c_1^T(L_{\chi'}) \to c_1^T(L_\chi) \cdot c_1^T(\mathcal{L}_\chi).$$

Example $G = GL_n(k)$

$\Omega^*_T(G/B) \simeq \mathbb{L}^{gr}[[x_1, \ldots, x_n; t_1, \ldots, t_n]]/(s_i(x_1, \ldots, x_n) - s_i(t_1, \ldots, t_n), i = 1, \ldots, n)$.

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Borel presentation for usual cobordism

Corollary

After inverting t_G
$$\Omega^*(G/B) \simeq S \otimes_{S^W} \mathbb{L}.$$

Remark

This corollary is similar to the result of Calmés–Petrov–Zainoulline (2009), who described $\Omega^*(G/B)$ in terms of the completion of S with respect to its augmentation ideal.

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Schubert calculus

Definition
Let $W = N(T)/T$ denote the Weyl group of G. For each element $w \in W$, the *Schubert variety* $X_w \subset X$ is

$$X_w = \overline{BwB}.$$

Definition
The *Schubert cycle* $[X_w]$ is the class of X_w in $CH^*(X)$.

Fact
Schubert cycles $[X_w]$ for all $w \in W$ form a basis in $CH^*(X)$.

Central question
How to multiply $[X_w]$?

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Tool: divided difference operators

Definition

Let $\alpha_1, \ldots, \alpha_n$ be simple roots of G. *Divided difference operator* δ_i (for the simple root α_i) acts on $Sym(\Lambda_T)$ as follows:

$$\delta_i : f \mapsto \frac{f - s_i(f)}{c_1(\mathcal{L}_{\alpha_i})}.$$

Example $G = GL_n$

δ_i acts on $\mathbb{Z}[x_1, \ldots, x_n]$ as follows:

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Applications of divided difference operators

Theorem (Bernstein–Gelfand–Gelfand, Demazure, 1973)

Let $w = s_{i_1} \dots s_{i_\ell}$ be a reduced expression. In the Borel presentation,

$$[X_w] = \delta_{i_\ell} \dots \delta_{i_1} [X_{id}],$$

where $[X_{id}]$ is the class of a point.

Remark

For GL_n,

$$[X_{id}] = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

If $t_G \neq 1$, there is no denominator-free formula.

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Geometric meaning of divided difference operators

Gysin morphism

Let P_i be a minimal parabolic subgroup, and $p_i : G/B \to G/P_i$ the natural projection. Then the action of δ_i on $CH^*(G/B, \mathbb{Z})$ coincides with the action of $p_i^* \circ p_{i*}$:

$$\delta_i : CH^*(G/B, \mathbb{Z}) \xrightarrow{p_{i*}} CH^*(G/P_i, \mathbb{Z}) \xrightarrow{p_i^*} CH^*(G/B, \mathbb{Z}).$$

Example

If $G = GL_n$, then G/P_i is obtained by forgetting the i-th space in a flag.

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Generalizations of divided difference operators

Generalized cohomology theories

Let A^* be an oriented cohomology theory. Define *generalized divided difference operator* δ_i^A as the composition

$$\delta_i^A : A^*(G/B, \mathbb{Z}) \xrightarrow{{p_i^A}_*} A^*(G/P_i, \mathbb{Z}) \xrightarrow{p_i^{*A}} A^*(G/B, \mathbb{Z}).$$

Examples

- classical cohomology H^* or Chow ring CH^*
- K-theory K_0^*
- complex cobordism MU^* or algebraic cobordism Ω^*

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Generalized cohomology theories

Let A^* be an oriented cohomology theory. Define *generalized divided difference operator* δ_i^A as the composition

$$\delta_i^A : A^*(G/B, \mathbb{Z}) \xrightarrow{p^A_{i\,*}} A^*(G/P_i, \mathbb{Z}) \xrightarrow{p_i^{*A}} A^*(G/B, \mathbb{Z}).$$

Examples

- classical cohomology H^* or Chow ring CH^*
- K-theory K_0^*
- complex cobordism MU^* or algebraic cobordism Ω^*

Generalizations of divided difference operators

Question

Is there an algebraic formula for δ_i^A?

Formal group law

There exists a formal power series $F_A(x, y) = x + y + \dots$ with coefficients in A^0 such that

$$F(c_1^A(L), c_1^A(M)) = c_1^A(L \otimes M)$$

in $A^*(X)$ for any pair of line bundles L and M on a variety X.

Examples

CH^* $F(x, y) = x + y$

K_0^* $F(x, y) = x + y - xy$

Ω^* $F(x, y) = x + y - [\mathbb{P}^1]xy + ([\mathbb{P}^1]^2 - [\mathbb{P}^2])x^2y + \dots$

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Generalizations of divided difference operators

Theorem (follows from Quillen–Vishik formula)

$$\delta_i^A = (1+s_i)\frac{1}{c_1^A(\mathcal{L}_{\alpha_i})}$$

Example $G = GL_n$

$$\delta_i^A = (1+s_i)\frac{1}{x_i -_A x_{i+1}}$$

- If $A = CH$, then $\delta_i^A = \delta_i$.
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Schubert calculus for cobordism

Question

What are analogs of Schubert cycles in cobordism?

Remark

In general, Schubert varieties are not smooth.

Bott–Samelson varieties

For each sequence $(s_{i_1}, \ldots, s_{i_\ell})$ of simple reflections one can construct by successive $\mathbb{P}^1$-fibrations a smooth variety R_I of dimension ℓ together with a morphism $\pi_I : R_I \to X$. If $w = s_{i_1} \ldots s_{i_\ell}$ is a reduced decomposition then R_I is a resolution of singularities for $X_w = \pi_I(R_I)$.

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Results

Formulas for Bott–Samelson classes via divided difference operators. Algorithms for multiplying Bott-Samelson classes in the Borel presentation.

MU^* Bressler–Evens, 1992

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Example

$G = GL_3$

$$[R_{212}] = 1 + ([\mathbb{P}^1]^2 - [\mathbb{P}^2])x_1^2; \quad [R_{121}] = 1 + ([\mathbb{P}^1]^2 - [\mathbb{P}^2])x_1x_2;$$

$$[R_{12}] = -x_1 - [\mathbb{P}^1]x_1^2; \quad [R_{21}] = x_3 = -x_1 - x_2;$$

$$[R_1] = x_1x_2; \quad [R_2] = x_1^2;$$

$$[R_e] = -x_1^2x_2.$$

Schubert calculus for cobordism

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- "Positivity" of structure constants?
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