“Tensor products” between metric spaces and Banach spaces

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Tensor products are normally used to linearize bilinear maps.

What sense could there possibly be in thinking about tensor products of a metric space with a Banach space?
In Banach space theory, tensor products are used for more than linearizing bilinear maps.

There are many different choices for a “reasonable” norm on $E \otimes F$.

Most importantly, there are deep connections between tensor norms and operator ideals.
Duality relations

It often happens that

$$(E \otimes_\alpha F)^* \equiv \mathcal{A}(E, F^*)$$

for some tensor norm $\alpha$ and some operator ideal $\mathcal{A}$.

Examples:

1. $(E \otimes_\pi F)^* \equiv L(E, F^*)$.
2. $(E \otimes_{d_p} F)^* \equiv \Pi_{p'}(E, F^*)$.
3. $(E \otimes_{w_2} F)^* \equiv \Gamma_2(E, F^*)$.

All of these examples have nonlinear counterparts.
Another type of result takes the form

\[ T \in \mathcal{A}(E, F) \iff \| T \otimes id_G : E \otimes_\alpha G \to F \otimes_\beta G \| < \infty \quad \forall G. \]

with \( \mathcal{A} \) an operator ideal and \( \alpha, \beta \) tensor norms.

Examples:

1. \( T \in \Pi_p(E, F) \iff \| T \otimes id_G : E \otimes_{d_p'} G \to F \otimes_{d_1} G \| < \infty \forall G. \)

2. \( T \in M_{q,p}(E, F) \iff \| T \otimes id_G : E \otimes_{d_p'} G \to F \otimes_{d_q'} G \| < \infty \forall G. \)

Again, these and other examples have nonlinear counterparts.
Duality results
A baby example for duality

Suppose we want to find a nonlinear version of
\[(E \otimes_{\pi} F)^* \equiv L(E, F^*).\]

In the nonlinear setting, Lipschitz maps play the role corresponding to that of linear bounded maps.

That means we want to find some sort of tensor product so that
\[(X \boxtimes_{\pi} F)^* \equiv \text{Lip}_0(X, F^*).\]

The easiest instance of this would be when \(F = \mathbb{R}\).
The Arens-Eells space

The Arens-Eells space of a metric space $X$ (denoted $\mathcal{AE}(X)$), also known as the free Lipschitz space of $X$ (denoted $\mathcal{F}(X)$) satisfies

$$\mathcal{F}(X)^* \equiv X^\# := \text{Lip}_0(X, \mathbb{R}) = \{f : X \to \mathbb{R} : \text{Lip}(f) < \infty, f(0) = 0\}.$$ 

It was introduced in [Arens/Eells 1956], and has been used in Banach space theory [Godefroy/Kalton 2003], [Kalton 2004].
A molecule on a metric space $X$ is a finitely supported $m : X \to \mathbb{R}$ such that
\[ \sum_{x \in X} m(x) = 0. \]

Note that the space of molecules is a vector space.

Those of the form $am_{xx'}$ where
\[ m_{xx'} := \chi_{\{x\}} - \chi_{\{x'\}} \]
with $a \in \mathbb{R}$ and $x, x' \in X$ are called atoms.

The Arens-Eells space of $X$ is the space of molecules with the norm
\[ \|m\|_{\mathcal{F}(X)} := \inf \left\{ \sum_{j=1}^{n} |a_j|d(x_j, x'_j) : m = \sum_{j=1}^{n} a_j m_{x_jx'_j} \right\}. \]
Properties of the Arens-Eells space

(a) $\|\cdot\|_{\mathcal{F}(X)}$ is a norm.

(b) $\delta : X \hookrightarrow \mathcal{F}(X)$ given by $\delta(x) = m_{x_0}$ is an isometric embedding.

(c) $\mathcal{F}(X)^* = \text{Lip}_0(X, \mathbb{R}) = X#$ via the duality pairing

$$\langle f, m \rangle = \sum_{x \in X} f(x)m(x)$$

(d) Whenever $T : X \to E$ is a Lipschitz map, there is a linear map $\hat{T} : \mathcal{F}(X) \to E$ such that $\|\hat{T}\| = \text{Lip}(T)$ and $\hat{T} \circ \delta = T$. 

\[ \begin{array}{ccc}
\mathcal{F}(X) & \xrightarrow{\delta} & X \\
\downarrow & & \downarrow \quad T \\
X & \xrightarrow{T} & E \\
\end{array} \]
Duality for $L(E, F)$

Theorem

$$(E \otimes_{\pi} F)^* = L(E, F^*).$$

Where for $w \in E \otimes F$

$$\|w\|_{\pi} = \inf \left\{ \sum_{j=1}^{n} \|u_j\| \cdot \|v_j\| : w = \sum_{j=1}^{n} u_j \otimes v_j \right\}$$

and the identification is given via trace duality, considering an element in $E \otimes F$ as a map $F^* \to E$. That is, for $w = \sum_{j=1}^{n} u_j \otimes v_j \in E \otimes F$ and $T : E \to F^*$,

$$\langle T, w \rangle = \text{tr}(w \circ T) = \sum_{j=1}^{n} \langle Tx_j, y_j \rangle.$$
### Definition (C, 2011)

Let $X$ be a metric space and $E$ a Banach space. An *$E$-valued molecule on $X$* is a function $m : X \to E$ such that 

$$\sum_{x \in X} m(x) = 0.$$ 

An *$E$-valued atom* is a function of the form $v m_{xx'}$ with $x, x' \in X$ and $v$ in $E$.

- Every *$E$-valued molecule on $X$* can be expressed as a sum of *$E$-valued atoms*. 

For an $E$-valued molecule $m$, let
\[ \|m\|_\pi := \inf \left\{ \sum_{j=1}^{n} \|v_j\| d(x_j, x'_j) : m = \sum_{j=1}^{n} v_j m_{x_j,x'_j} \right\}. \]

We will denote by $X \otimes_\pi E$ the space of $E$-valued molecules on $X$ with the projective norm. It is not hard to show that
\[ (X \otimes_\pi E)^* = \text{Lip}_0(X, E^*) \]
with the duality given by the pointwise pairing
\[ \langle T, m \rangle = \sum_{x \in X} \langle T(x), m(x) \rangle. \]

It was known that $\text{Lip}_0(X, E^*)$ is a dual space [J. Johnson, 1970], but as far as I know the approach via molecules is new.
“Products” of operators

Proposition (C, 2012)

Let $S : X \to Z$ be a Lipschitz map mapping 0 to 0, and $T : E \to F$ a bounded linear map. Then there is a unique operator $S \boxtimes T : X \boxtimes_\pi E \to Z \boxtimes_\pi F$ such that

$$(S \boxtimes T)(vm_{xy}) = (Tv)m(Sx)(Sy), \quad \text{for all } v \in E, x, y \in X.$$ 

Furthermore, $\|S \boxtimes_\pi T\| = \text{Lip}(S) \|T\|$. 


Recall that a linear operator $T : E \to F$ is a \textit{linear quotient} if it is surjective and

$$\|w\| = \inf \{ \|v\| : v \in E, \; Tv = w \} \text{ for every } w \in F.$$

On the other hand, a map $S : X \to Z$ is called a \textit{C-co-Lipschitz} if for every $x \in X$ and $r > 0$,

$$f(B(x, r)) \supseteq B(f(x), r/C).$$

A map that is Lipschitz, co-Lipschitz and surjective is a \textit{Lipschitz quotient}.

\textbf{Theorem (C, 2012)}

\textit{Let $S : X \to Z$ be a Lipschitz quotient with Lipschitz and co-Lipschitz constants equal to 1, and mapping 0 to 0, and let $T : E \to F$ be a linear quotient map. Then $S \boxprod_{\pi} T : (X \boxprod_{\pi} E) \to (Z \boxprod_{\pi} F)$ is also a linear quotient map.}
Example: $X = \text{a graph-theoretic tree}$

Recall

$$
\|m\|_{\pi} = \inf \left\{ \sum_{j=1}^{n} \|v_j\| d(x_j, x'_j) : m = \sum_{j=1}^{n} v_j m_{x_jx'_j} \right\}
$$

Note we can consider only representations where the pairs $(x_j, x'_j)$ are endpoints of edges. Since $X$ is a tree, every molecule has only one such representation so

$$
X \boxtimes_{\pi} E \equiv \ell_1^N(E)
$$

where $N = \# \text{ of edges of } X$.

I suspect a similar result should work for more general metric trees as in [Godard 2010].
A tensor norm $\alpha$ is called *reasonable if it satisfies*

(a) $\alpha(u \otimes v) \leq \|u\| \cdot \|v\|$ for every $u \in E, v \in F$.

(b) $\alpha^*(u^* \otimes v^*) \leq \|u^*\| \cdot \|v^*\|$ for every $u^* \in E^*, v^* \in F^*$.

Reasonable tensor norms are characterized by being between the projective and injective tensor norms: a tensor norm $\alpha$ is reasonable if and only if $\varepsilon \leq \alpha \leq \pi$, where

$$\|w\|_\varepsilon = \sup \left\{ \sum_{j=1}^{n} \langle u^*, u_j \rangle \langle v^*, v_j \rangle : w = \sum_{j=1}^{n} u_j \otimes v_j, u^* \in B_{E^*}, v^* \in B_{F^*} \right\}.$$
Reasonable molecular norms

A norm $\| \cdot \|$ on the space of $E$-valued molecules on a metric space $X$ is called *reasonable* if

(i) $\| v m_{xx'} \| \leq \| v \| d(x, x')$ for all $x, x' \in X$, $v \in E$.

(ii) $|\langle v^* \circ m, f \rangle| \leq \| v^* \| \text{Lip}(f) \| m \|$ for all $v^* \in E^*$, $m \in \mathcal{M}(X, E)$ and $f \in X^\#$.

Reasonable molecular norms are also characterized by being between the projective and injective norms: a molecular norm $\alpha$ is reasonable if and only if $\varepsilon \leq \alpha \leq \pi$, where

$$\| m \|_\varepsilon = \sup \left\{ \sum_{j=1}^{n} \left[ f(x_j) - f(x'_j) \right] v^*(v_j) : m = \sum_{j=1}^{n} v_j m_{x_j x'_j}, f \in B_X^#, v^* \in B_{E^*} \right\}.$$
The injective norm

The injective norm is also deserving of its name: it behaves well under injections.

However, it is not so interesting for us because it “forgets” about the metric space and only takes into account the structure of $\mathcal{F}(X)$. In fact,

$$X \boxtimes \varepsilon E \equiv \mathcal{F}(X) \otimes \varepsilon E.$$
$p$-summing operators

$E, F$ Banach spaces, $T : E \to F$ a linear map, $1 \leq p \leq \infty$.

$T$ is called \textit{$p$-summing} if there exists $C > 0$ such that for any $v_1, \ldots, v_n$ in $E$ we have

$$
\left[ \sum_{j=1}^{n} \|Tv_j\|^p \right]^{1/p} \leq C \sup_{\phi \in B_{E^*}} \left[ \sum_{j=1}^{n} |\phi(v_j)|^p \right]^{1/p}.
$$

The \textit{$p$-summing norm of $T$} is

$$
\pi_p(T) := \inf C.
$$

The space of \textit{$p$-summing operators from $E$ to $F$} is denoted

$$
\Pi_p(E, F).
$$
Chevet-Saphar norms

**Theorem (Saphar 1970)**

\[
(E \otimes_{d_p} F)^* = \Pi_{p'}(E, F^*).
\]

Where

**Definition (Chevet 1969, Saphar 1965, 1970)**

For \(1 \leq p \leq \infty\) and \(w \in E \otimes F\), define \(p'\) by \(\frac{1}{p} + \frac{1}{p'} = 1\) and

\[
\|w\|_{d_p} := \inf \left\{ \sup_{\phi \in B_{E^*}} \left[ \sum_{j=1}^{n} |\phi(u_j)|^{p'} \right]^{1/p'} \cdot \left[ \sum_{j=1}^{n} \|v_j\|^{p} \right]^{1/p} : w = \sum_{j=1}^{n} u_j \otimes v_j \right\}.
\]
**Definition (Pietsch, 1966)**

$E, F$ Banach spaces, $T : E \to F$ a linear map, $1 \leq p \leq \infty$.

$T$ is called $p$-summing if there exists $C > 0$ such that for any $v_1, \ldots, v_n$ in $E$ we have

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The $p$-summing norm of $T$ is

$$\pi_p(T) := \inf C.$$
Definition (Farmer/Johnson, 2009)

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\]

The $p$-summing norm of $T$ is

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\]
Lipschitz $p$-summing operators

\textbf{Definition (Farmer/Johnson, 2009)}

$X$, $Y$ metric spaces, $T : E \to F$ a linear map, $1 \leq p \leq \infty$.

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Definition (Farmer/Johnson, 2009)

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$T$ is called **Lipschitz $p$-summing** if there exists $C > 0$ such that for any $v_1, \ldots, v_n$ in $E$ we have

$$
\left[ \sum_{j=1}^{n} \left\| T v_j \right\|^p \right]^{1/p} \leq C \sup_{\phi \in B_{E^*}} \left[ \sum_{j=1}^{n} \left| \phi(v_j) \right|^p \right]^{1/p}
$$

The $p$-summing norm of $T$ is

$$
\pi_p(T) := \inf C.
$$
Definition (Farmer/Johnson, 2009)

$X, Y$ metric spaces, $T : X \to Y$ a Lipschitz map, $1 \leq p \leq \infty$.

$T$ is called Lipschitz $p$-summing if there exists $C > 0$ such that for any $v_1, \ldots, v_n$ in $E$ we have

$$\left[ \sum_{j=1}^{n} \|Tv_j\|^p \right]^{1/p} \leq C \sup_{\phi \in B_{E^*}} \left[ \sum_{j=1}^{n} |\phi(v_j)|^p \right]^{1/p}$$

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Lipschitz $p$-summing operators

**Definition (Farmer/Johnson, 2009)**

Let $X, Y$ be metric spaces, $T : X \to Y$ a Lipschitz map, $1 \leq p \leq \infty$. $T$ is called Lipschitz $p$-summing if there exists $C > 0$ such that for any $x_1, \ldots, x_n, x'_1, \ldots, x'_n$ in $X$ we have

$$\left[ \sum_{j=1}^{n} d(Tx_j, Tx'_j)^p \right]^{1/p} \leq C \sup_{\phi \in B_{E^*}} \left[ \sum_{j=1}^{n} |\phi(v_j)|^p \right]^{1/p}$$

The $p$-summing norm of $T$ is

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Lipschitz $p$-summing operators

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Lipschitz $p$-summing operators

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\[
\left[ \sum_{j=1}^{n} d(Tx_j, Tx'_j)^p \right]^{1/p} \leq C \sup_{f \in B_X} \left[ \sum_{j=1}^{n} \left| f(x_j) - f(x'_j) \right|^p \right]^{1/p}
\]

The $p$-summing norm of $T$ is

\[\pi_p(T) := \inf C.\]
Lipschitz $p$-summing operators

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$X, Y$ metric spaces, $T : X \to Y$ a Lipschitz map, $1 \leq p \leq \infty$.

$T$ is called Lipschitz $p$-summing if there exists $C > 0$ such that for any $x_1, \ldots, x_n, x'_1, \ldots, x'_n$ in $X$ we have

$$
\left[ \sum_{j=1}^{n} d(Tx_j, Tx'_j)^p \right]^{1/p} \leq C \sup_{f \in B_{X\#}} \left[ \sum_{j=1}^{n} |f(x_j) - f(x'_j)|^p \right]^{1/p}
$$

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Lipschitz $p$-summing operators

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$T$ is called Lipschitz $p$-summing if there exists $C > 0$ such that for any $x_1, \ldots, x_n, x'_1, \ldots, x'_n$ in $X$ we have

$$\left[ \sum_{j=1}^{n} d(Tx_j, Tx'_j)^p \right]^{1/p} \leq C \sup_{f \in B_X \#} \left[ \sum_{j=1}^{n} |f(x_j) - f(x'_j)|^p \right]^{1/p}$$

The Lipschitz $p$-summing norm of $T$ is

$$\pi_p^L(T) := \inf C.$$
Duality for Lipschitz $p$-summing operators

**Theorem (C 2011)**

$$(X \boxtimes_{d_{p}} F)^* = \Pi^L_p(X, F^*).$$

Where $\Pi^L_p$ denotes the Lipschitz $p$-summing operators of [Farmer/Johnson 2009] and

**Definition (C 2011)**

For an $E$-valued molecule $m$ on a metric space $X$,

$$\|m\|_{d_p} = \inf \left\{ \left( \sum_j \lambda_j^p \|v_j\|^p \right)^{1/p} \sup_{f \in B_{X^#}} \left( \lambda_j^{-p'} |f(x_j) - f(x'_j)|^{p'} \right)^{1/p'} : m = \sum_j v_j m_{x_j x'_j}, \lambda_j > 0 \right\}.$$
Define for a linear map $T : E \rightarrow F$

$$\gamma_2(T) := \inf \left\{ \|R\| \cdot \|S\| \right\}$$

where

and $H$ is a Hilbert space.

$\Gamma_2(E, F)$ will denote the space of all operators admitting such a factorization.
Duality for $\Gamma_2(E, F)$

**Theorem**

$$(E \otimes_{w_2} F)^* = \Gamma_2(E, F^*)$$

Where for $w \in E \otimes F$

$$\|u\|_{w_2} = \inf \left\{ \left( \sum_{j=1}^{n} \|u_j\|^2 \right)^{1/2} \left( \sum_{i=1}^{n} \|v_i\|^2 \right)^{1/2} : u = \sum_{ij} a_{ij} u_j \otimes v_i, \| (a_{ij}) : \ell_2^n \to \ell_2^n \| \leq 1 \right\}$$

and the identification is given again via trace duality.
Define for a Lipschitz map $T : X \to Y$

$$\gamma_2^{\text{Lip}}(T) := \inf \{ \text{Lip}(R) \cdot \text{Lip}(S) \}$$

where

and $Z$ is a subset of a Hilbert space.
Duality for $\Gamma^\text{Lip}_2$

The norm on molecules that gives the duality for $\Gamma^\text{Lip}_2$ is

\[
\|m\|_{w_2} = \inf \left\{ \left( \sum_{i=1}^{n} \|v_i\|^2 \right)^{1/2} \left( \sum_{j=1}^{m} d(x_j, x'_j)^2 \right)^{1/2} : m = \sum_{i=1}^{n} v_i m_{y_i y'_i}, \; m_{y_i y'_i} = \sum_{j=1}^{m} a_{ij} m_{x_j x'_j}, \; \|(a_{ij}) : \ell^n_2 \to \ell^m_2\| \leq 1 \right\}
\]
Tensoring with the identity
Representation theorems

Operator ideals satisfying certain technical properties can be characterized by theorems of the following form:

**Representation theorem**

A linear operator $T : E \rightarrow F$ belongs to the operator ideal $\mathcal{A}$ if and only if for every Banach space $G$, the map

$$T \otimes id_G : E \otimes_\alpha G \rightarrow F \otimes_\beta G$$

is continuous.

Here, $\alpha$ and $\beta$ are certain tensor norms.
A linear operator $T : E \to F$ is $p$-summing if and only if for every Banach space $G$ the map

$$T \otimes id_G : E \otimes_{d_p} G \to F \otimes_{\pi} G$$

is continuous.

Moreover, in this case

$$\pi_p(T) = \inf_G \|T \otimes id_G\|$$
Theorem (C, 2011)

TFAE:
(a) \( T : X \to Y \) is Lipschitz \( p \)-summing.
(b) For every Banach space \( E \) (or only \( E = Y^\# \)),

\[
\left\| T \boxtimes id_E : X \boxtimes_{d_p} E \to Y \boxtimes_\pi E \right\| < \infty
\]
$$(q, p)$$-mixing operators

Theorem

Let $T : E \to F$ be a linear map, $1 \leq p \leq q \leq \infty$. TFAE:

(a) $\exists C > 0$ such that for every $S : F \to G$,

$$\pi_p(S \circ T) \leq C \pi_q(S).$$

(b) For every Banach space $G$ (or only $G = \ell_{q'}$),

$$\|T \otimes id_G : E \otimes_{d_p} G \to F \otimes_{d_{q'}} G\| < \infty$$
Similarly

**Theorem (C, 2011)**

*Let $T : X \to Y$ be a Lipschitz map, $1 \leq p \leq q \leq \infty$. TFAE:*

(a) \( \exists C > 0 \) such that for every $S : Y \to Z$,

\[
\pi_L^p(S \circ T) \leq C \pi_L^q(S).
\]

(b) For every Banach space $E$ (or only $E = \ell_{q'}$),

\[
\left\| T \boxtimes \text{id}_E : X \boxtimes_{d_p} E \to Y \boxtimes_{d_{q'}} E \right\| < \infty
\]